Improper Colourings inspired by Hadwiger's Conjecture

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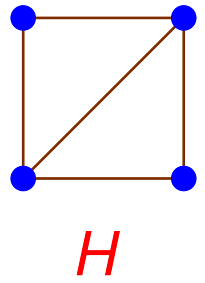
LSE

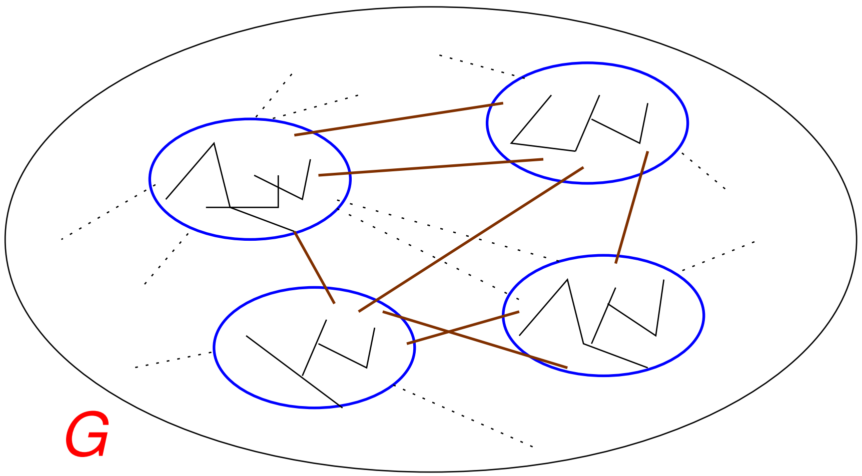
Graph minors

- a graph H is a **minor** of a graph G if:

  for $V(H) = \{v_1, \ldots, v_k\}$, there exist connected, disjoint subgraphs $H_1, \ldots, H_k$ of G such that:

  - if $v_i v_j \in E(H)$, then there is at least one edge in G between H_i and H_j





Graph colouring

- a **colouring** of a graph means colouring the vertices
- **proper colouring**: adjacent vertices have different colours

- recurring question in graph theory:

  what structural properties of a graph

  - allow proper colourings with few colours ?
  - force all proper colourings to use many colours ?

Hadwiger's Conjecture

Conjecture (Hadwiger, 1943)

- graph G needs at least t colours for a proper colouring

  $\Longrightarrow$ G has the complete graph K_t as a minor

the contrapositive is probably more intuitive:

- G has no K_t-minor

  $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$-colouring

Hadwiger's Conjecture – what is known

- no K_t -minor $\stackrel{?}{\Longrightarrow}$ G has a proper $(t-1)$ -colouring

known

- nothing to prove for $t = 1, 2$
- easy for $t = 3$
  (no K_3 -minor $\rightarrow$ no cycles $\rightarrow$ forest $\rightarrow$ 2 -colourable)
- not too hard for $t = 4$ (Hadwiger, 1943; Dirac, 1952)
- case $t = 5$ is equivalent to the Four Colour Theorem
  (Wagner, 1937)
- true for $t = 6$ (Robertson, Seymour & Thomas, 1993)

How many colours do we need ?

Theorem (Kostochka, 1984; Thomason, 1984)

- G has no K_t-minor
  $\Longrightarrow$ G has a vertex with degree at most $c\,t\sqrt{\log t}$

Corollary

- G has no K_t-minor
  $\Longrightarrow$ G has a proper colouring with $c\,t\sqrt{\log t}$ colours

Improper colourings

- what if we weaken the requirement on the colouring?
  - in a proper colouring:

    the collection of vertices with the same colour
    is just a collection of isolated vertices

  - we could be happy with:

    the collection of vertices with the same colour
    is just a subgraph with a "simple" structure

- **monochromatic subgraph**:
  subgraph formed by vertices with the same colour

Improper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $t-1$ colours such that each monochromatic subgraph has degree at most $c'\,t^2\log t$

- the bound $t-1$ on the number of colours is best possible:
  - for each d, there exists a graph G such that
    - G has no K_t-minor
    - every $(t-2)$-colouring of G has a monochromatic subgraph with degree at least d

Improper colourings – small monochromatic degree

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- G has no K_t-minor $\Longrightarrow$

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- the dependence on no K_t-minor is best possible:

Theorem (Ossona de Mendez, Oum & Wood, 2016)

- H a graph on t vertices, different from K_t :
  - G has no H-minor $\Longrightarrow$

    G can be coloured with $t-2$ colours such that each monochromatic subgraph has degree at most $f(t)$

no K_t-minor $\stackrel{?}{\Longrightarrow}$ proper $(t-1)$-colouring

Improper colourings – small monochromatic degree

Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

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Theorem (vdH & Wood, 2017)

- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $t-1$ colours such that each monochromatic subgraph has degree at most $t-2$

Improper colourings – small monochromatic components

Theorem (Kawarabayashi & Mohar, 2007)

- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $\lceil 15\frac{1}{2}\,t \rceil$ colours such that each monochromatic components has at most $f_1(t)$ vertices

- improved to
  - $\lceil 3\frac{1}{2}\,t - 1\frac{1}{2} \rceil$ colours; $f_2(t)$ vertices (Wood, 2010; incl.)
  - $3(t-1)$ colours; $f_3(t)$ vertices (Liu & Oum, 2015)
  - $2(t-1)$ colours; $f_4(t)$ vertices (Norin, 2015; unpubl.)

- all use Robertson & Seymour Graph Minor Structure Thm., or worse . . .

Improper colourings – small monochromatic components

Theorem (vdH & Wood, 2017)

- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $2(t-1)$ colours such that each monochromatic component has at most $\lceil \frac{1}{2}(t-2) \rceil$ vertices

note

- G has no K_t-minor $\Longrightarrow$

  at least $t-1$ colours are needed to guarantee monochromatic components of bounded order

  (same examples as for small monochromatic degree)

no K_t-minor $\stackrel{?}{\Longrightarrow}$ proper $(t-1)$-colouring

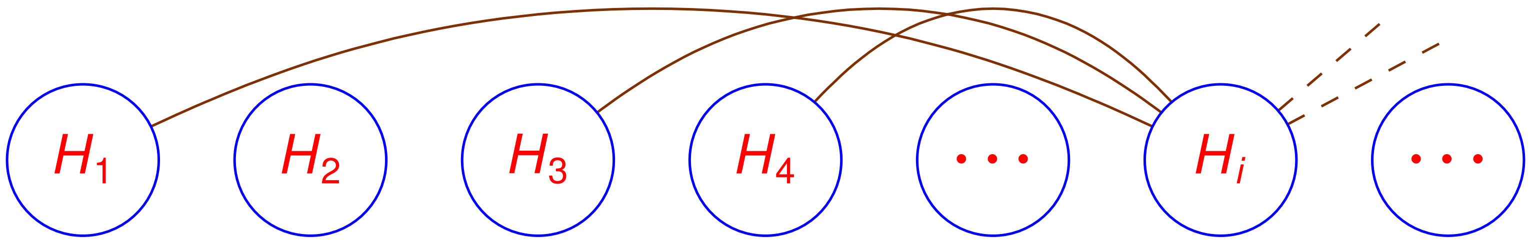
A simple decomposition theorem for K_t-minor-free graphs

Theorem (vdH & Wood, 2017)

- G has no K_t-minor $\Longrightarrow$

  G has a partition into subgraphs $H_1, \ldots, H_\ell$ such that

  - global structure: each H_i is adjacent to at most $t-2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$



A simple decomposition theorem for K_t-minor-free graphs

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  - global structure: each H_i is adjacent to at most $t-2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$
  - local structure:
    - each H_i has maximum degree at most $t-2$
    - each H_i can be coloured with 2 colours such that each monochromatic component of H_i has at most $\lceil \frac{1}{2}(t-2) \rceil$ vertices

Our main results on improper colourings

Theorem (vdH & Wood, 2017)

- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $t-1$ colours such that each monochromatic subgraph has degree at most $t-2$

Theorem (vdH & Wood, 2017)

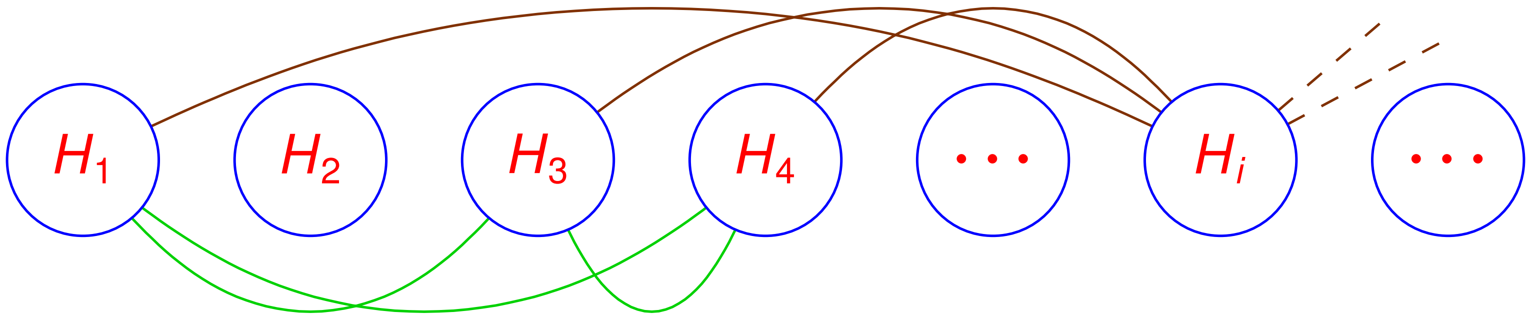
- G has no K_t-minor $\Longrightarrow$

  G can be coloured with $2(t-1)$ colours such that each monochromatic component has at most $\left\lceil \frac{1}{2}(t-2) \right\rceil$ vertices

no K_t-minor $\overset{?}{\Longrightarrow}$ proper $(t-1)$-colouring

The global structure we actually prove

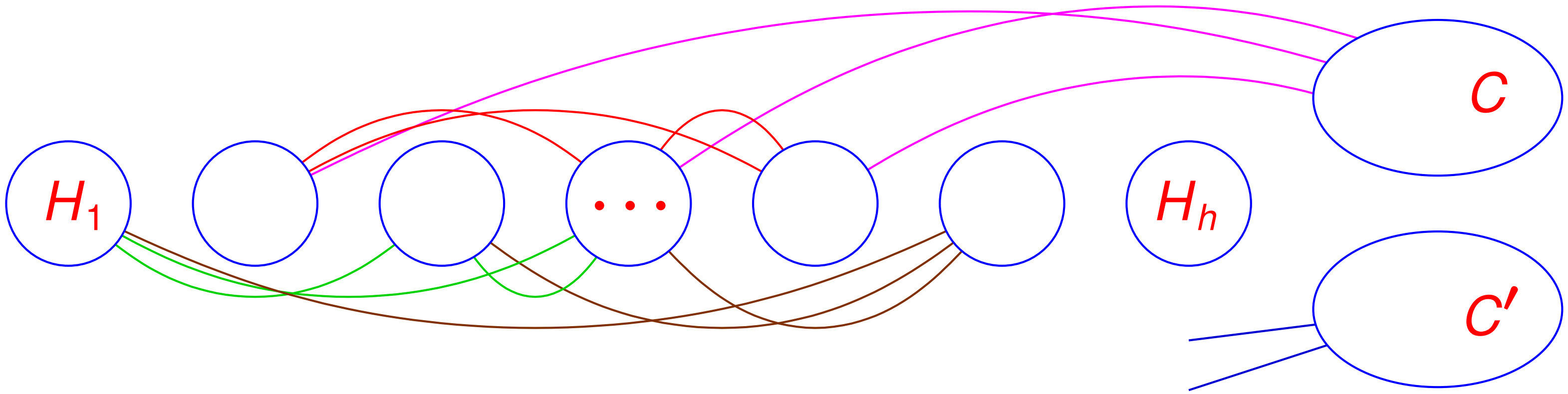
- G any graph $\Longrightarrow$
  we can construct (in many ways) a partition of G into induced subgraphs $H_1, \ldots, H_\ell$ such that:
  - each H_i is connected
  - each H_i is adjacent to k subgraphs $H_{i_1}, \ldots, H_{i_k}$ the earlier subgraphs $H_1, \ldots, H_{i-1}$
  - for each H_i, the adjacent subgraphs $H_{i_1}, \ldots, H_{i_k}$ are pairwise adjacent as well



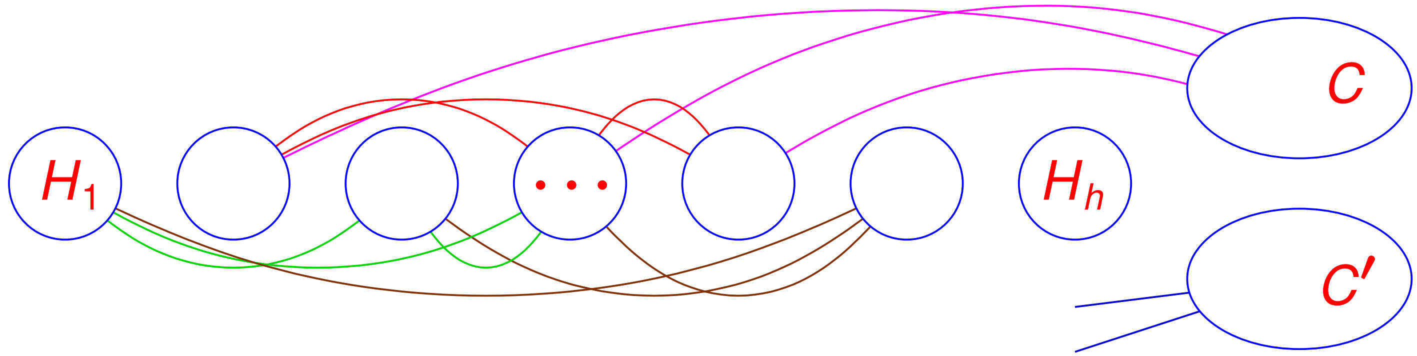
The global structure – proof

- we will construct the H_i one by one such that once $H_1, \ldots, H_h$ is constructed:
  - each H_i, $i \leq h$, satisfies the requirements
  - each component C of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$ satisfies:

    if C is adjacent to $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, then $H_{i_1}, \ldots, H_{i_k}$ are pairwise adjacent as well



The global structure – proof

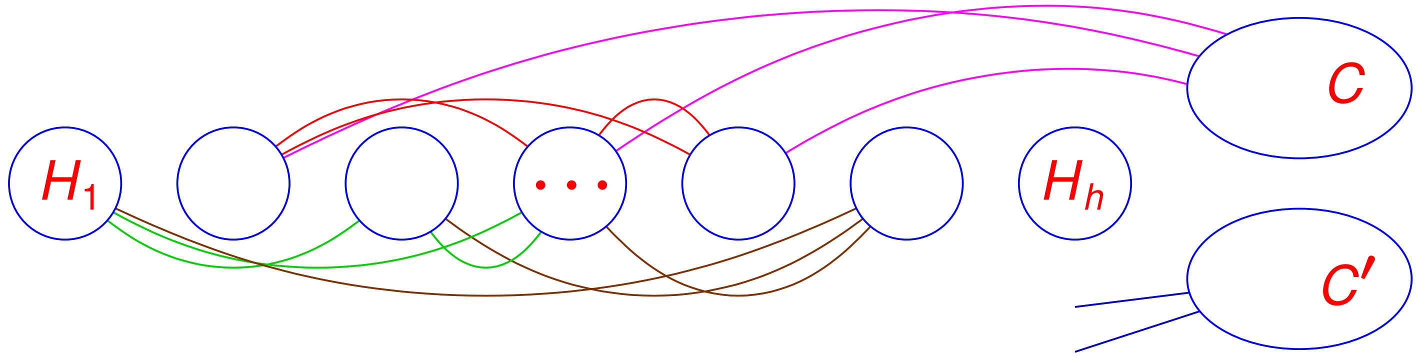


- start with H_1 any connected, induced subgraph of G

  (good choice for later: $V(H_1) = \{v\}$ for some $v \in V(G)$)

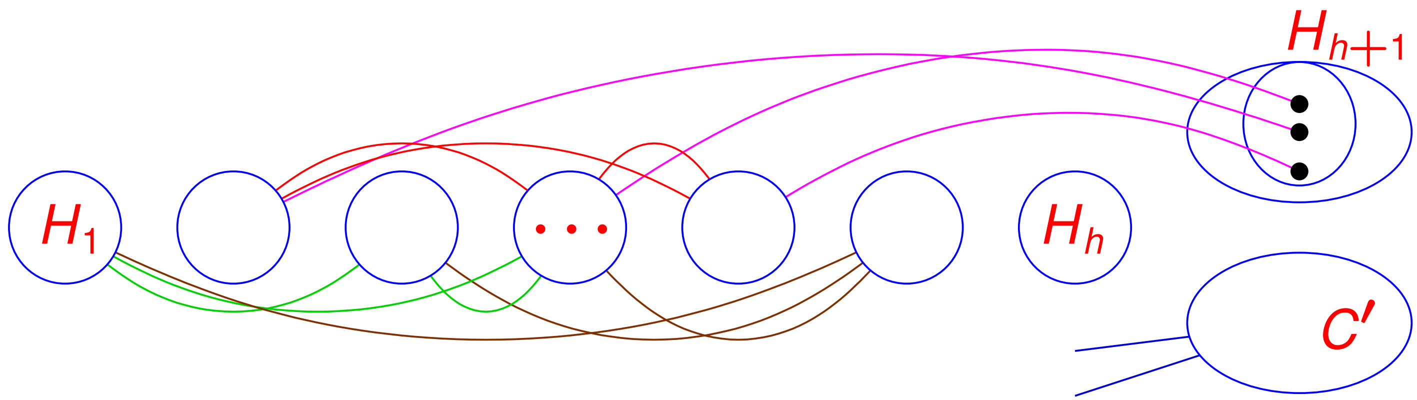
  - all requirements are trivially satisfied

The global structure – proof



- suppose $H_1, \ldots, H_h$ are already constructed and C is some component of $G - (V(H_1) \cup \cdots \cup V(H_h))$
  - so C is adjacent to $H_{i_1}, \ldots, H_{i_k}$, which are also pairwise adjacent
- for each H_{i_ℓ}, choose $a_{i_\ell} \in V(C)$ adjacent to H_{i_ℓ}
- now choose H_{h+1} a connected, induced subgraph of C containing all $a_{i_1}, \ldots, a_{i_k}$

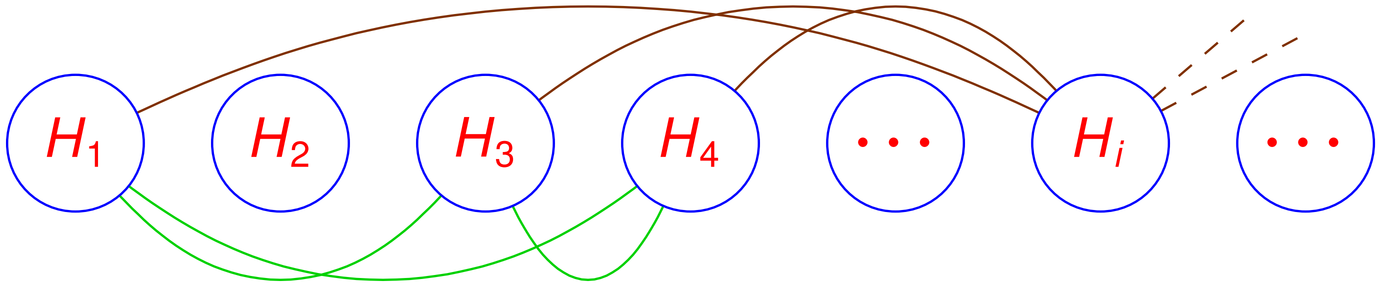
The global structure – proof



- H_{h+1} is still adjacent to $H_{i_1}, \dots, H_{i_k}$
- a component of $G - \big(V(H_1) \cup \dots \cup V(H_h) \cup V(H_{h+1})\big)$
  - is either a component of $G - \big(V(H_1) \cup \dots \cup V(H_h)\big)$ and hence still satisfies the requirements
  - or it is a component of $C - V(H_{h+1})$, hence it is adjacent to H_{h+1} and some of $H_{i_1}, \dots, H_{i_k}$, which are all pairwise adjacent

The global structure for K_t-minor-free graphs

- we can construct (in many ways) a partition of any graph G into induced, connected subgraphs $H_1, \ldots, H_\ell$ such that:
  - each H_i is adjacent to k subgraphs $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, which are pairwise adjacent as well



- G has no K_t-minor $\Longrightarrow$

  for each H_i we must have $k \le t-2$

The local structure – inside the H_i

- so how can we choose the H_i so that they satisfy:
  - small degree and
  - 2-colourable with small monochromatic components

- each H_i was chosen as some induced subgraph of some connected subgraph C, such that:
  - H_i is connected
  - $V(H_i)$ contains some set $A = \{a_{i_1}, \ldots, a_{i_k}\}$

- idea:
  choose H_i the smallest subgraph with those properties

The local structure – inside the H_i

Lemma

- C a connected graph, $A \subseteq V(C)$
  H a minimal, induced, connected subgraph of C,
  such that $V(H)$ contains all of A

- then H satisfies:
  - every vertex in H has degree at most $|A|$ in H
  - every vertex not in A is a cut-vertex of H
    - easy corollary: there is a 2-colouring of H with monochromatic components of size at most $\left\lceil \frac{1}{2}|A| \right\rceil$

Our decomposition theorem again

Theorem (vdH & Wood, 2017)

- G has no K_t-minor $\Longrightarrow$

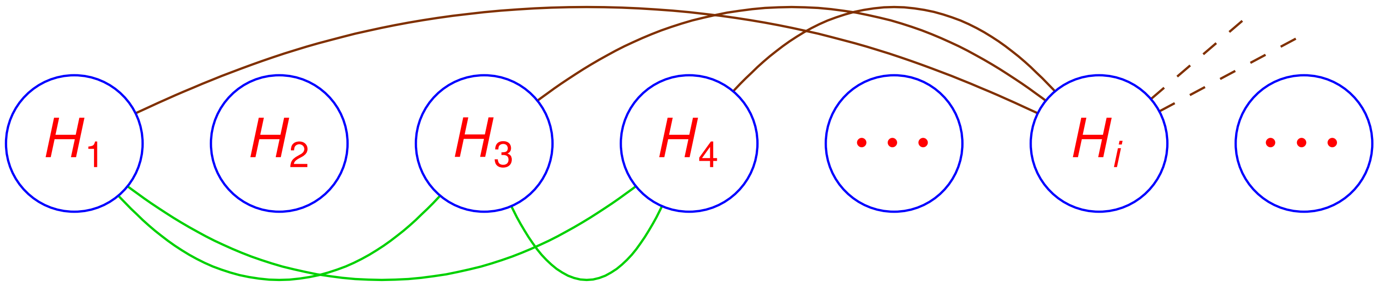
  G has a partition into subgraphs $H_1, \ldots, H_\ell$ such that

  - global structure: each H_i is adjacent to at most $t-2$ of the earlier subgraphs $H_1, \ldots, H_{i-1}$
  - local structure:
    - each H_i has maximum degree at most $t-2$
    - each H_i can be coloured with 2 colours such that each monochromatic component of H_i has at most $\lceil \frac{1}{2}(t-2) \rceil$ vertices
  - some more properties

With only a little bit extra work . . .

Theorem

- G has no K_t-minor $\Longrightarrow$ G has a partition into connected subgraphs $H_1, \ldots, H_\ell$ such that
  - contracting all H_i to single vertices gives a chordal graph with treewidth at most $t-2$
  - each H_i has treewidth at most $t-3$



A similar result for $K_{3,s}$-minor-free graphs

- we can prove a similar decomposition theorem for $K_{3,s}$-minor-free graphs

Corollary

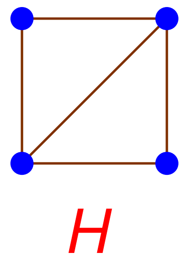
- G has no $K_{3,s}$-minor $\Longrightarrow$
  - G can be coloured with 3 colours such that each monochromatic subgraph has degree at most $4s$
  - G can be coloured with 6 colours such that each monochromatic component has at most $2s$ vertices

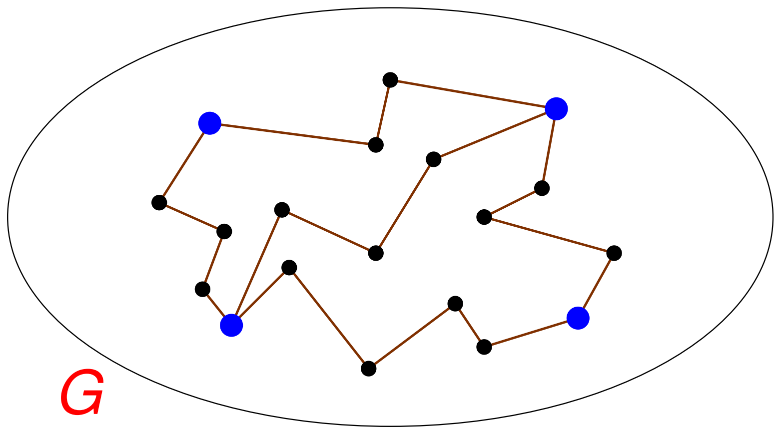
- the bound of 6 colours for small monochromatic components is probably not best-possible — could be 4

Variants – Topological minors

- a graph H is a **topological minor** of a graph G if:

  for $V(H) = \{v_1, \dots, v_k\}$,
  there are different $u_1, \dots, u_k$ in $V(G)$ such that:
  - if $v_i v_j \in E(H)$, then there is a u_i,u_j-path P_{ij} in G
  - apart from their end vertices, the paths P_{ij} are disjoint





- H a topological minor of G $\quad\Longrightarrow\quad$ H a minor of G

Hajós' Conjecture

Conjecture (Hajós, 1961)

- G has no K_t -topological-minor

  $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$ -colouring

- true for $t \leq 4$ (Dirac, 1952)
- false for $t \geq 7$ (Catlin, 1979)

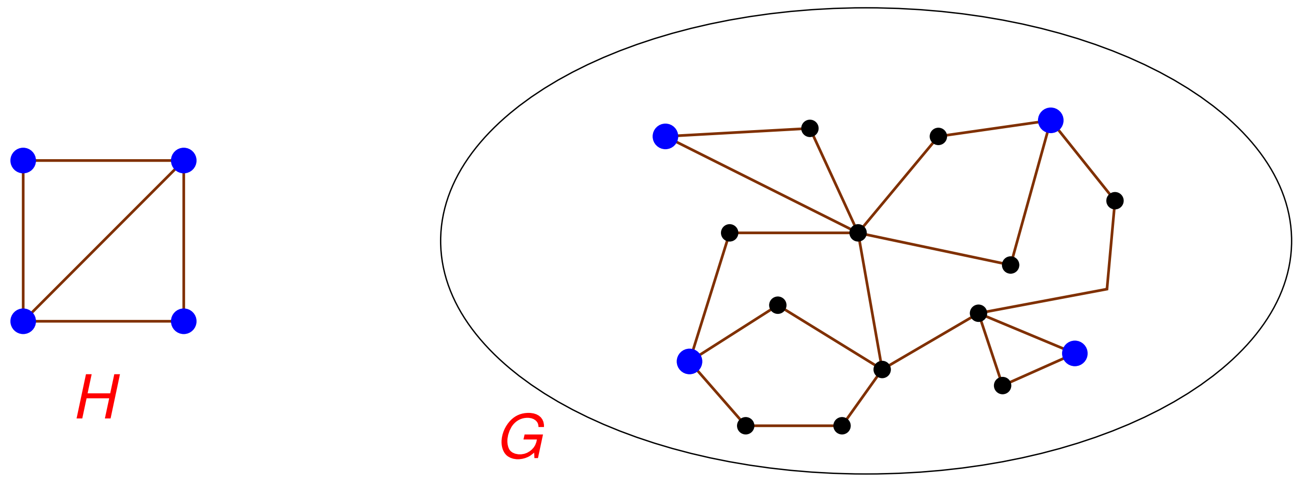
Theorem (Edwards, Kang, Kim, Oum & Seymour, 2015)

- G has no K_t -topological-minor $\Longrightarrow$ G can be coloured with $t-1$ colours such that each monochromatic subgraph has degree at most $c\,t^4$

Variants – Immersions

- a graph H is an **immersion** in a graph G if:

  for $V(H) = \{v_1, \ldots, v_k\}$,
  there are different $u_1, \ldots, u_k$ in $V(G)$ such that:
  - if $v_i v_j \in E(H)$, then there is a u_i,u_j-path P_{ij} in G
  - the paths P_{ij} are edge-disjoint



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  - if $v_i v_j \in E(H)$, then there is a u_i,u_j-path P_{ij} in G
  - the paths P_{ij} are edge-disjoint

- H is a topological minor of G $\Longrightarrow$
  - H is also a minor and an immersion in G
  - but in general not the other way round

- in general there is no relation between minor and immersion

Another conjecture

Conjecture (Lescure & Meyniel, 1989; Abu-Khzam & Langston, 2003)

- G does not contain a K_t-immersion

  $\overset{?}{\Longrightarrow}$ G has a proper $(t-1)$-colouring

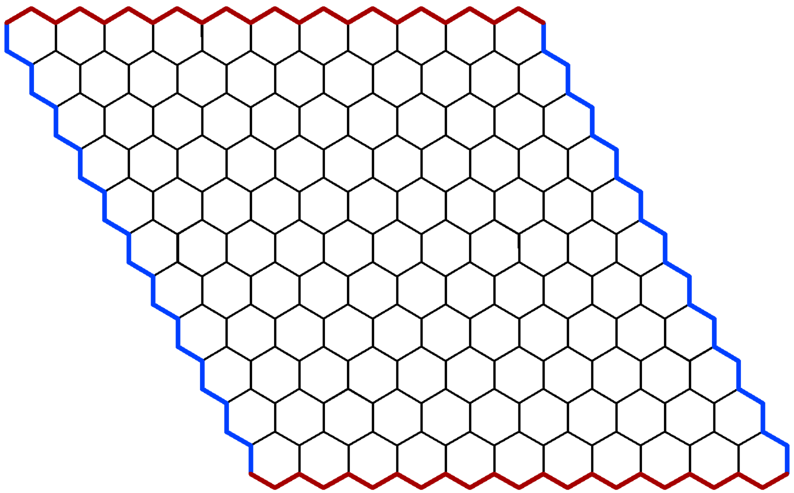
Theorem (vdH & Wood, 2017)

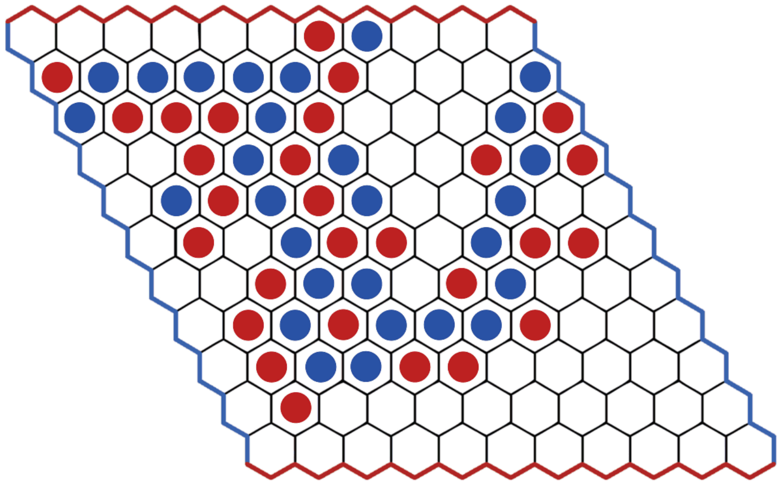
- G does not contain a K_t-immersion $\Longrightarrow$

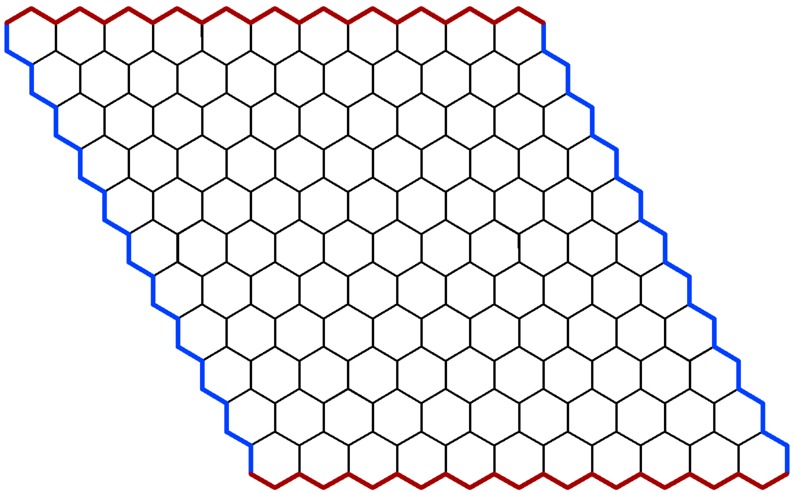
  G can be coloured with 2 colours such that each monochromatic subgraph has degree at most $f(t)$

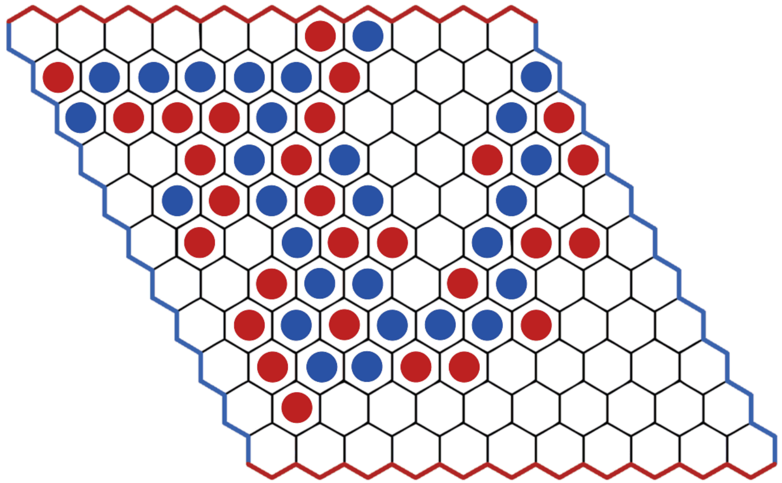
- to guarantee monochromatic components of bounded size, we need at least 3 colours (for $t \geq 8$)

Hex and 2-colouring









That's all folks! – Thanks for listening.