Uniform Orderings for Generalised Colouring Numbers and Graph Classes with Bounded Expansion

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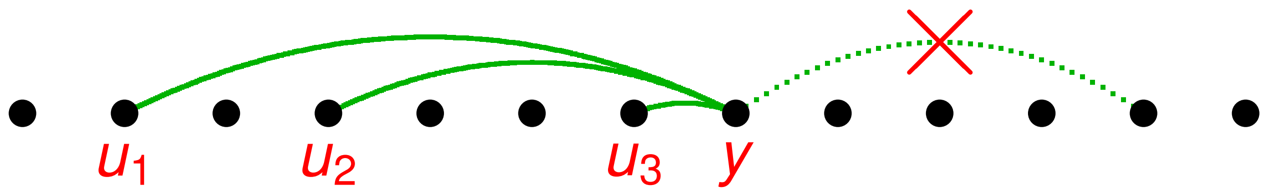
The normal colouring number

- - let $\boldsymbol{L}$ be a linear ordering of the vertices of a graph $\boldsymbol{G}$
  - for a vertex $y \in V(G)$,
    let $\boldsymbol{S}(\boldsymbol{G}, \boldsymbol{L}, \boldsymbol{y})$ be the neighbours u of y with $u <_L y$



  - and set $\boldsymbol{S}[\boldsymbol{G}, \boldsymbol{L}, \boldsymbol{y}] = S(G, L, y) \cup \{y\}$

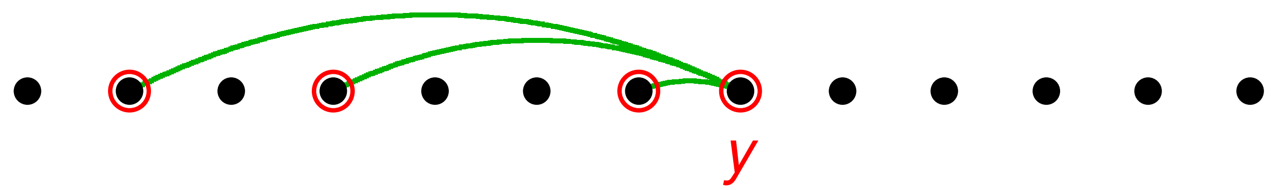
- then the **colouring number** $\mathbf{col}(\boldsymbol{G})$ is defined as

$$\mathrm{col}(G) = \min_{L} \max_{y \in V(G)} \big|S[G, L, y]\big|$$

Generalising the colouring number

- the set $S[G, L, y]$ can also be defined as

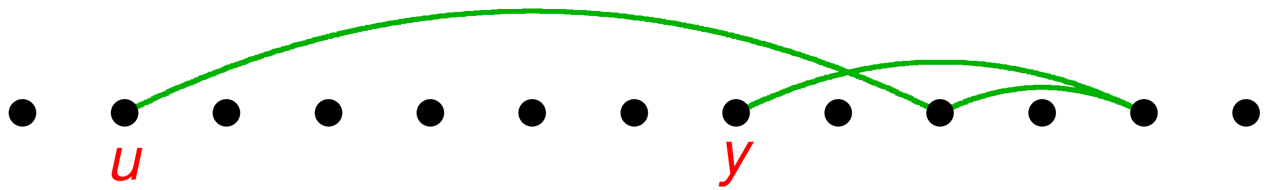
  "the set of vertices $u \leq_L y$
  for which there is an uy-path of length at most 1"



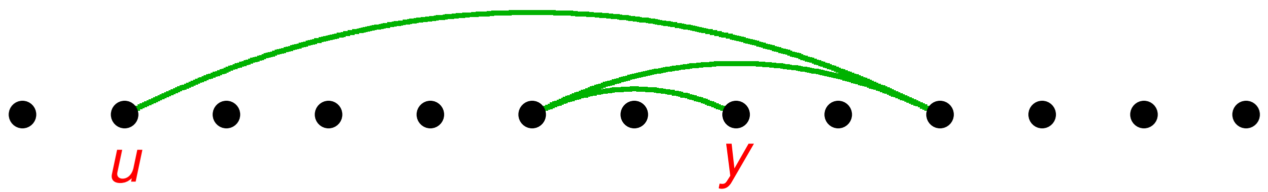
- what would happen if we allow longer paths ?

Generalising the colouring number

- what would happen if we allow longer paths ?
- for $u \leq_L y$:
  - a **strong uy-path** has all internal vertices larger than $\underline{y}$

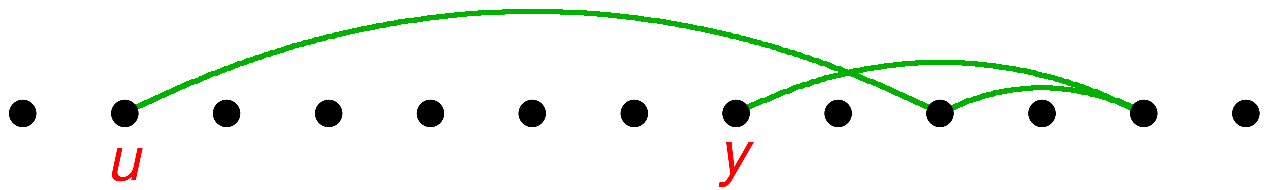


  - a **weak uy-path** has all internal vertices larger than $\underline{u}$



Strong generalised colouring numbers

- a **strong *uy*-path** has all internal vertices larger than $\underline{y}$

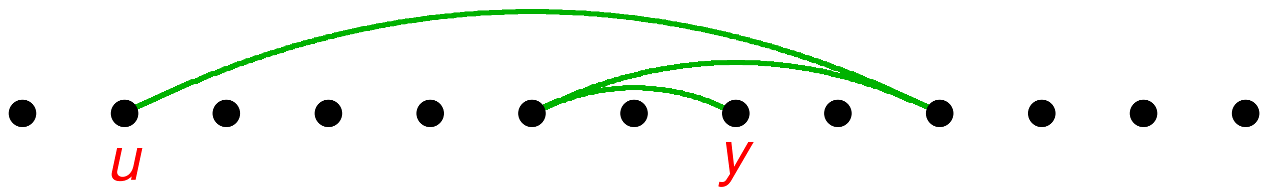


- let $\boldsymbol{S_r[G,L,y]}$ be the set of vertices $u \leq_L y$ for which there exists a strong uy-path with length at most r

- then define the **strong *r*-colouring number $\mathrm{scol}_r(G)$** by
  - $$\mathrm{scol}_r(G,L) = \max_{y \in V(G)} \big|S_r[G,L,y]\big|$$
  - $$\mathrm{scol}_r(G) = \min_L \; \mathrm{scol}_r(G,L)$$

Weak generalised colouring numbers

- - a **weak *uy*-path** has all internal vertices larger than $\underline{u}$



- let $\boldsymbol{W_r}[\boldsymbol{G}, \boldsymbol{L}, \boldsymbol{y}]$ be the set of vertices $u \leq_L y$ for which there exists a weak uy-path with length at most r

- then define the **weak *r*-colouring number** $\mathbf{wcol}_{\boldsymbol{r}}(\boldsymbol{G})$ by
  - $$\mathrm{wcol}_r(G, L) = \max_{y \in V(G)} \big| W_r[G, L, y] \big|$$
  - $$\mathrm{wcol}_r(G) = \min_{L} \ \mathrm{wcol}_r(G, L)$$

Some facts about generalised colouring numbers

- studied in some form (in particular $r = 2$) since early 1990's
- introduced in this form by Kierstead & Yang, 2003
- by definition: $\mathrm{scol}_1(G) = \mathrm{wcol}_1(G) = \mathrm{col}(G)$
-
  - obviously: $\mathrm{scol}_r(G) \le \mathrm{wcol}_r(G)$
  - but also: $\mathrm{wcol}_r(G) \le \big(\mathrm{scol}_r(G)\big)^r$

    (Proof: every weak path of length at most r is formed of at most r strong paths of length at most r.)

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  - but also: $\mathrm{wcol}_r(G) \leq \big(\mathrm{scol}_r(G)\big)^r$
- $\mathrm{scol}_1(G) \leq \mathrm{scol}_2(G) \leq \ldots \leq \mathrm{scol}_\infty(G) = \text{tree-width}(G) + 1$
- $\mathrm{wcol}_1(G) \leq \mathrm{wcol}_2(G) \leq \ldots \leq \mathrm{wcol}_\infty(G) = \text{tree-depth}(G)$

A structural application

- classes of graphs $\mathcal{G}$ with **bounded expansion** were introduced by Nešetřil & Ossona de Mendez in terms of "densities of shallow minors"

- **equivalent Definition** (Zhu, 2009)

  a class of graphs $\mathcal{G}$ has **bounded expansion**:

  - there exists a function $c : \mathbb{N} \to \mathbb{R}$ such that for every $G \in \mathcal{G}$ and every r we have $\mathrm{scol}_r(G) \leq c(r)$

- we can use the weak colouring numbers $\mathrm{wcol}_r(G)$ as well

An algorithmic aspect

Theorem (Courcelle, 1990)

- let $\mathcal{G}$ be a class of graphs with bounded tree-width
  - then every graph property that can be described using monadic second-order logic is decidable for $G \in \mathcal{G}$ in linear time

Theorem (Dvořák, Král' & Thomas, 2010)

- let $\mathcal{G}$ be a class of graphs with bounded expansion
  - then every graph property that can be described using first-order logic is decidable for $G \in \mathcal{G}$ in linear time

Orderings

- for every r,
  $\mathrm{scol}_r(G)$ is defined using some "good" ordering L of $V(G)$

Question

- can we use the same ordering L for different r ?

NO

- for every different r, r' and function $f(x)$,
  there exists a graph G such that for any ordering L of $V(G)$:
  - $\mathrm{scol}_r(G,L) = \mathrm{scol}_r(G) \implies \mathrm{scol}_{r'}(G,L) \geq f(\mathrm{scol}_{r'}(G))$
  - $\mathrm{scol}_{r'}(G,L) = \mathrm{scol}_{r'}(G) \implies \mathrm{scol}_r(G,L) \geq f(\mathrm{scol}_r(G))$

Nevertheless, uniform orderings are possible

Theorem (vdH & Kierstead, 2018+)

- for every graph G, there exists an ordering L^* of $V(G)$, such that for all r we have

$$\mathrm{scol}_r(G, L^*) \;\le\; (2^r + 1) \cdot \big(\mathrm{scol}_{2r}(G)\big)^{4r}$$

Corollary

- a class of graphs $\mathcal{G}$ has bounded expansion if and only if
  - there exists a function $c' : \mathbb{N} \to \mathbb{R}$ such that for every $G \in \mathcal{G}$ there exists an ordering L^* of $V(G)$, such that for every r we have $\mathrm{scol}_r(G, L^*) \le c'(r)$

Ideas of the proof

- the crucial idea of the proof goes back to a proof in the original work of Kierstead & Yang (2003) that introduced generalised colouring numbers

- the main part of that paper actually deals with a game variant of those numbers

The game colouring number

- two players, Alice and Bob, play on the vertex set $V(G)$ of a graph G
  - Alice and Bob create an ordering L' of $V(G)$, by alternatingly choosing the next vertex
  - Alice wants to end up with an ordering L' such that $\mathrm{scol}_r(G, L')$ is small (for some given r)

Theorem (Kierstead & Yang, 2003)

- whatever Bob does,
  Alice can guarantee the final ordering L' to satisfy:

$$\mathrm{scol}_r(G, L') \le 3\big(\mathrm{scol}_{2r}(G)\big)^{4r}$$

A first common ordering

- suppose Bob has some specific ordering in mind as well

that directly leads to:

Corollary

- let G_1, G_2 be two graphs on the same vertex set V and let r_1, r_2 be two natural numbers
  - then there exists an ordering L^* of V such that

$$\mathrm{scol}_{r_1}(G_1, L^*) \;\leq\; 3\big(\mathrm{scol}_{2r_1}(G_1)\big)^{4r_1}$$

and

$$\mathrm{scol}_{r_2}(G_2, L^*) \;\leq\; 3\big(\mathrm{scol}_{2r_2}(G_2)\big)^{4r_2}$$

Next step: a common ordering for many graphs

Theorem (vdH & Kierstead, 2018+)

- let $G_1, \ldots, G_k$ be a collection of graphs on the same set V and let $r_1, \ldots, r_k$ be natural numbers
  - then there exists an ordering L^* of V such that
    for $i = 1, \ldots, k$: $\operatorname{scol}_{r_i}(G_i, L^*) \le (k+1)\big(\operatorname{scol}_{2r_i}(G_i)\big)^{4r_i}$

Corollary

- for every graph G and natural number k
  - there exists an ordering L^* of $V(G)$ such that
    for $r = 1, \ldots, k$: $\operatorname{scol}_r(G, L^*) \le (k+1)\big(\operatorname{scol}_{2r}(G)\big)^{4r}$

The most general, "weighted", version

<u>**Theorem**</u> (vdH & Kierstead, 2018+)

- let $G_1, \ldots, G_k$ be a collection of graphs on the same set V, let $r_1, \ldots, r_k$ be natural numbers, and let $a_1, \ldots, a_k$ be natural numbers
  - set $A = a_1 + \cdots + a_k$
  - then there exists an ordering L^* of V such that

    for all $i = 1, \ldots, k$:

$$\mathrm{scol}_{r_i}(G_i, L^*) \;\leq\; \Big(\frac{A}{a_i} + 1\Big) \cdot \big(\mathrm{scol}_{2r_i}(G_i)\big)^{4r_i}$$

How to use this general, "weighted", version

- $$\mathrm{scol}_{r_i}(G_i, L^*) \;\le\; \left(\frac{A}{a_i} + 1\right) \cdot \left(\mathrm{scol}_{2r_i}(G_i)\right)^{4r_i}$$
- now set $k = \lfloor \log_2 |V| \rfloor$
- and for $i = 1, \ldots, k$, set $a_i = 2^{k-i}$
  - then: $A = a_1 + \cdots + a_k = 2^k - 1 \le 2^k$, so $\dfrac{A}{a_i} \le 2^i$
- next, for $i = 1, \ldots, k$ take $G_i = G$ and $r_i = i$, and we get:
  $$\mathrm{scol}_i(G, L^*) \;\le\; (2^i + 1)\left(\mathrm{scol}_{2i}(G)\right)^{4i}$$
- for $i > k$ we have $2^i + 1 > |V|$, so nothing to prove

Algorithmic aspects

- there exists an ordering L^* of V such that for all $i = 1, \ldots, k$:

$$\mathrm{scol}_{r_i}(G_i, L^*) \le \left(\frac{A}{a_i} + 1\right) \cdot \left(\mathrm{scol}_{2r_i}(G_i)\right)^{4r_i}$$

- if orderings L_i with $\mathrm{scol}_{2r_i}(G_i, L_i) = \mathrm{scol}_{2r_i}(G_i)$ are given, then L^* can be found in time polynomial in $|V|$ and A
- unfortunately, finding $\mathrm{scol}_r(G)$ is NP-hard for $r \ge 3$ (Grohe et al., 2015)
- but using results of Dvořák (2013), we can find in polynomial time an ordering L'_i such that $\mathrm{scol}_{2r_i}(G_i, L'_i)$ "approximates" $\mathrm{scol}_{2r_i}(G_i)$

Finding uniform ordering for bounded expansion classes

Corollary

- let $\mathcal{G}$ be a class with bounded expansion
  - then there exists a function $c' : \mathbb{N} \to \mathbb{R}$ and a polynomial time algorithm
  - that finds for every $G \in \mathcal{G}$:
    - an ordering L^* of $V(G)$
    - such that for every r: $\mathrm{scol}_r(G, L^*) \leq c'(r)$

Thanks for your attention!