# “Simple” Decompositions of Graphs without a Complete Minor

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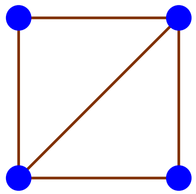
## *Graph minors*

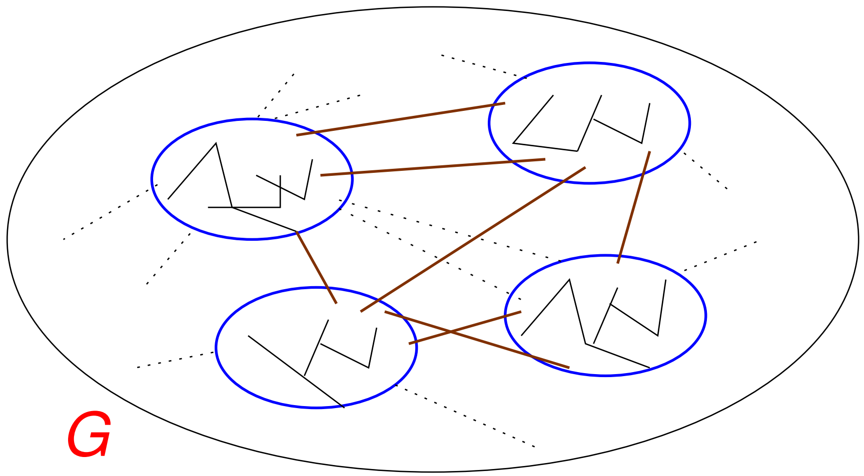
- a graph $H$ is a **minor** of a graph $G$ if:

  for $V(H) = \{v_1, \ldots, v_k\}$, there exist
  connected, disjoint subgraphs $H_1, \ldots, H_k$ of $G$ such that:

  - if $v_i v_j \in E(H)$, then there is
    at least one edge in $G$ between $H_i$ and $H_j$



$H$



## *Graph colouring*

- a **colouring** of a graph means colouring the vertices
- **proper colouring**: adjacent vertices have different colours
- recurring question in graph theory:

  what structural properties of a graph

  - allow proper colourings with few colours ?
  - force all proper colourings to use many colours ?

# *Hadwiger's Conjecture*

**Conjecture** ( Hadwiger, 1943 )

- $G$ has no complete graph $K_t$ as a minor

  $\stackrel{?}{\Longrightarrow}$ $G$ has a proper colouring with $(t-1)$ colours

## *Chordal decomposition of a graph*

**Definition** (Reed & Seymour, 1998)

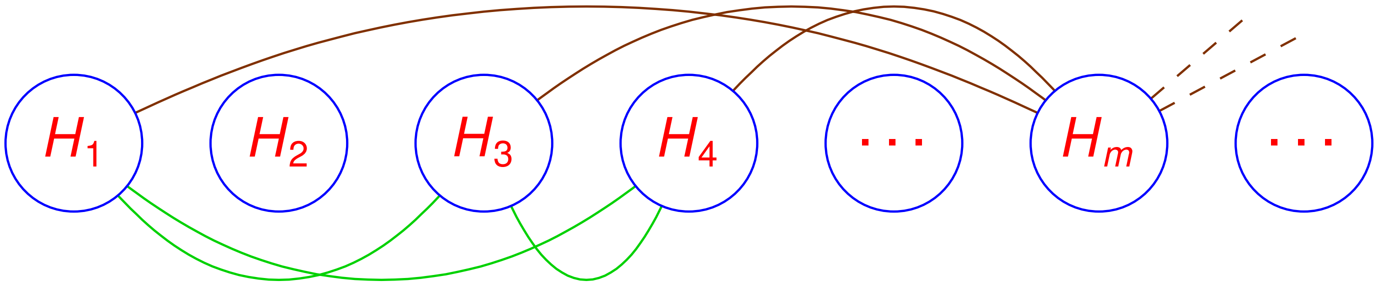
- a **chordal decomposition** of a graph $G$

  is a partition into induced subgraphs $H_1, \ldots, H_\ell$ such that

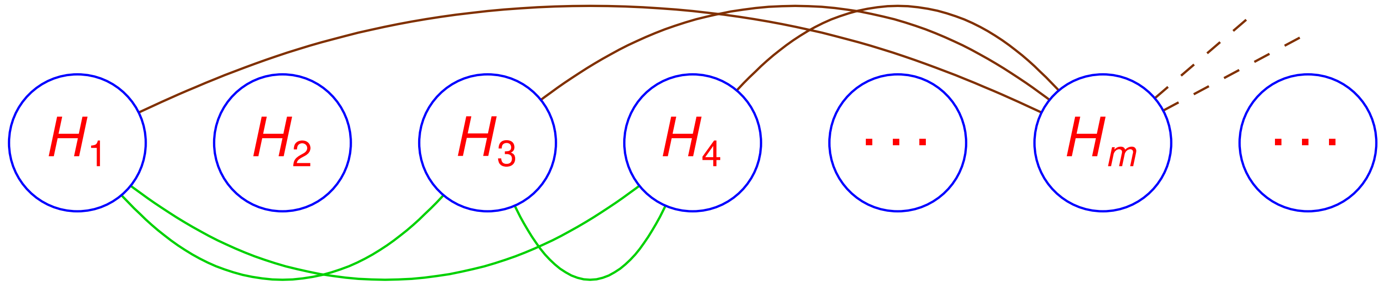
  - each subgraph $H_m$ is connected
  - contracting all $H_m$-s to single vertices
    gives a chordal graph

# *Chordal decomposition of a graph – writing it out*

- a **chordal decomposition** of a graph $G$ is a partition into a sequence of induced subgraphs $\mathcal{H} = (H_1, \ldots, H_\ell)$ such that
  - each subgraph $H_m$ is connected
  - for each $H_m$, its adjacent subgraphs $H_{m_1}, \ldots, H_{m_k}$ from the "earlier" subgraphs $H_1, \ldots, H_{m-1}$ are pairwise adjacent as well



## *Chordal decompositions*




- so if we define the **quotient graph** $\boldsymbol{Q(\mathcal{H})}$ as the graph obtained by contracting all $H_m$-s to single vertices, then
  - $G$ has no $K_t$-minor

    $\Longrightarrow$ $Q(\mathcal{H})$ is chordal with clique size at most $t-1$

## *Chordal decompositions are nice*

- $G$ has no $K_t$-minor

  $\Longrightarrow$ $Q(\mathcal{H})$ is chordal with clique size at most $t-1$

- looks like it could be "useful", since it gives:
  - $Q(\mathcal{H})$ has a proper colouring with at most $t-1$ colours
  - $Q(\mathcal{H})$ has treewidth at most $t-2$
  - lots of decision problems on $Q(\mathcal{H})$ can be solved in polynomial time
  - etc., etc.

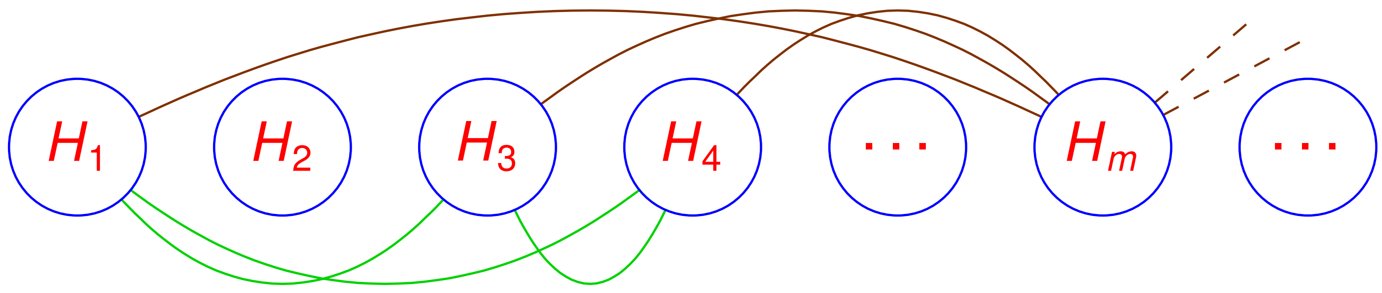
## *Chordal decompositions – do they exist ?*

**yes**:

- take $H_1, \ldots, H_\ell$ the components of $G$
- take some vertex $v \in V(G)$ and set
  - $V(H_1) = \{v\}$,
  - $H_2, \ldots, H_\ell$ the components of $G - v$
- main message of the talk:
  - there are many ways to construct chordal decompositions
  - and clever choices can guarantee the $H_m$-s to have a "simple" structure as well

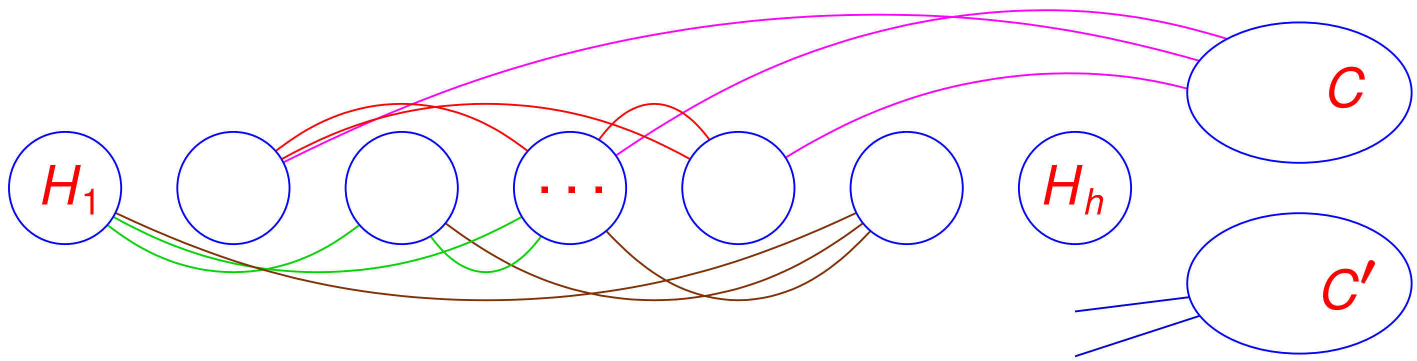
## *What we want to show*

- $G$ any graph $\Longrightarrow$
  we can construct (in many ways) a partition of $G$
  into induced subgraphs $H_1, \ldots, H_\ell$ such that:
  - each $H_m$ is connected
  - for each $H_m$, its adjacent subgraphs $H_{m_1}, \ldots, H_{m_k}$
    from the "earlier" subgraphs $H_1, \ldots, H_{m-1}$
    are pairwise adjacent as well

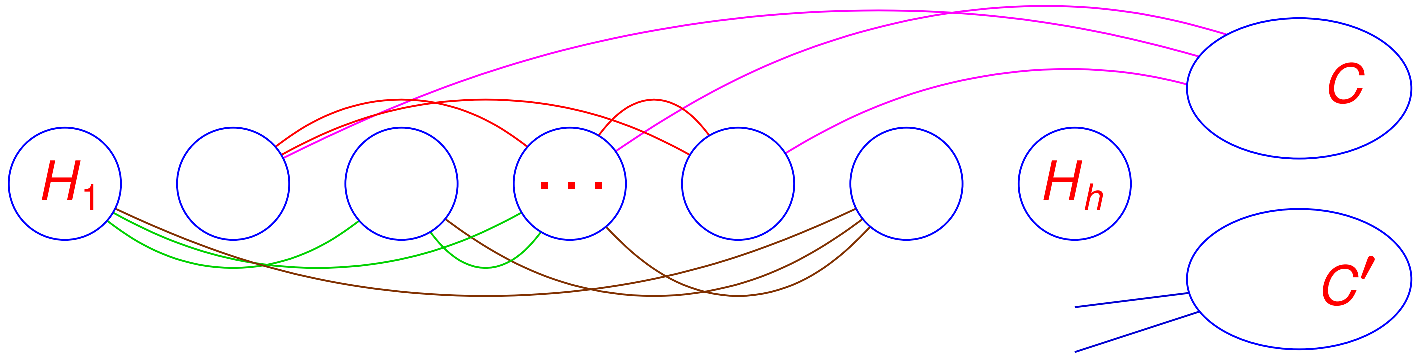


# *The construction method*

- we will construct the $H_m$ one by one such that once $H_1, \ldots, H_h$ are constructed:
  - each $H_m$, $m \leq h$, satisfies the requirements
  - each component $C$ of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$ satisfies:
    - if $C$ is adjacent to $H_{i_1}, \ldots, H_{i_k}$ from $H_1, \ldots, H_{i-1}$, then the $H_{i_1}, \ldots, H_{i_k}$ are pairwise adjacent as well



## *The construction method*

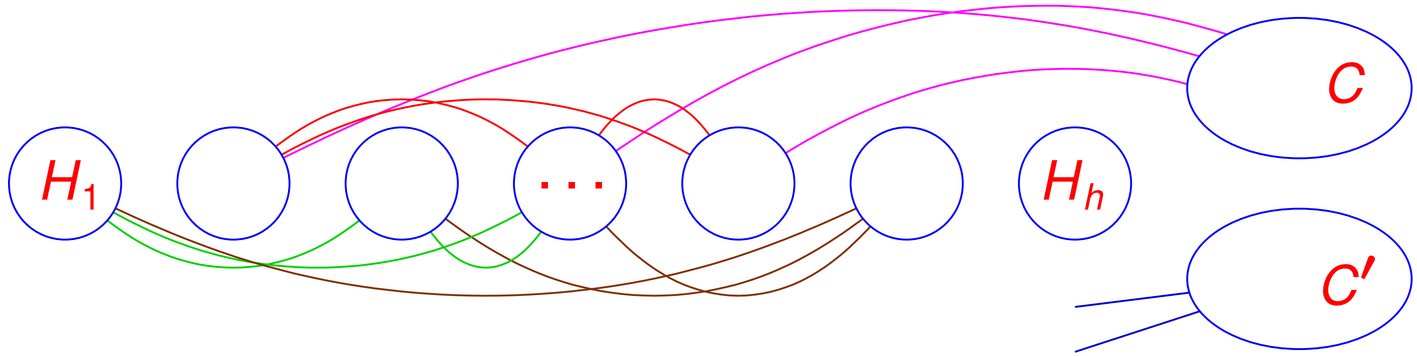



- start with $H_1$ any connected, induced subgraph of $G$

  (good choice for later: $V(H_1) = \{v\}$ for some $v \in V(G)$)

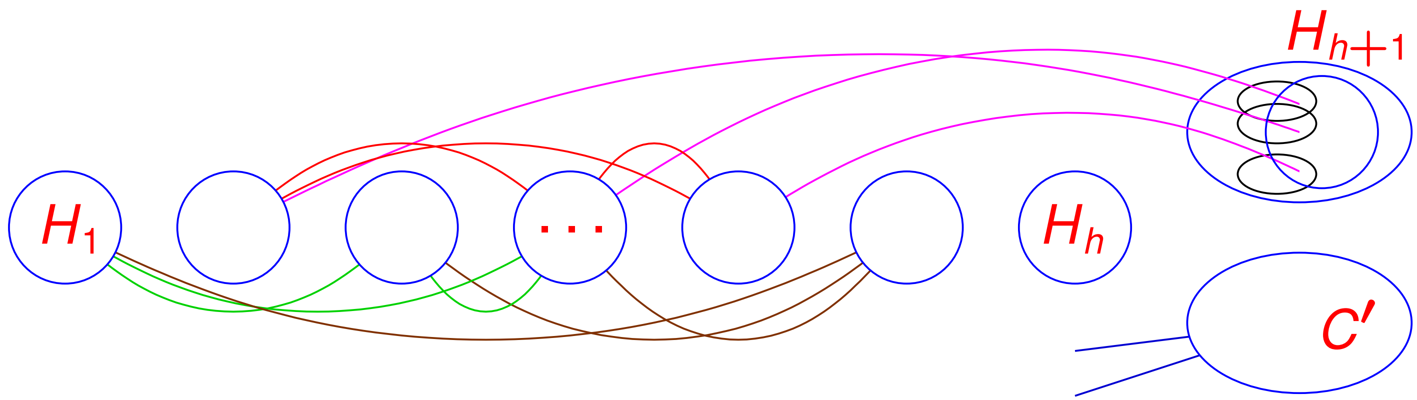
  - all requirements are trivially satisfied

## *The construction method*




- suppose $H_1, \ldots, H_h$ are already constructed and $C$ is some component of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$
  - so if $C$ is adjacent to $H_{i_1}, \ldots, H_{i_k}$, then those neighbours are also pairwise adjacent
- for each $H_{i_r}$, let $A_{i_r}$ be the set of neighbours of $H_{i_r}$ in $C$
- now choose $H_{h+1}$ a connected, induced subgraph of $C$ containing at least one vertex from each $A_{i_r}$

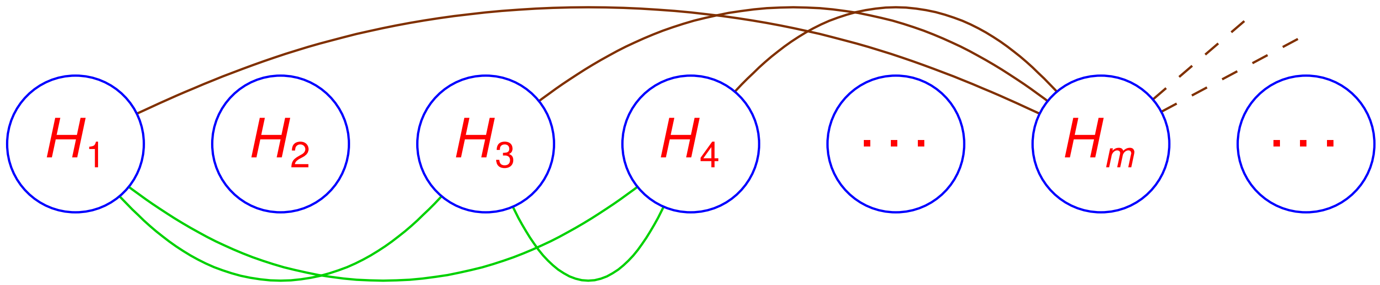
# *The construction method – why does it work*




- $H_{h+1}$ is adjacent to $H_{i_1},\ldots,H_{i_k}$, which are pairwise adjacent
- a component of $G - \big(V(H_1) \cup \cdots \cup V(H_h) \cup V(H_{h+1})\big)$
  - is either a component of $G - \big(V(H_1) \cup \cdots \cup V(H_h)\big)$, and hence still satisfies the requirements
  - or it is a component of $C - V(H_{h+1})$, hence it is adjacent to $H_{h+1}$ and some of $H_{i_1},\ldots,H_{i_k}$, which are all pairwise adjacent

## *Global versus local*

- we can construct (in many ways) a partition of any graph $G$ into induced, connected subgraphs $H_1, \ldots, H_\ell$ such that:
  - for each $H_m$, its adjacent subgraphs $H_{m_1}, \ldots, H_{m_k}$ from the "earlier" subgraphs $H_1, \ldots, H_{m-1}$ are pairwise adjacent as well




- so how can we choose the $H_m$ so that they have a "simple" structure as well?

## *The local structure – choosing $H_m$*

- each $H_m$ was chosen as some induced subgraph $H$ of some connected component $C$, such that:
  - $H$ is connected,
  - $H$ contains at least one vertex from each $A_r$, for some vertex sets $A_1, \ldots, A_k$ in $V(C)$

## *Choosing H – method I*

- given $C$ and the sets $A_1, \ldots, A_k$ in $V(C)$
  - take a vertex $a_r$ from each $A_r$
  - for $r = 2, \ldots, k$, let $P_r$ be a shortest $a_1$-$a_r$-path in $C$
  - take $H$ the subgraph of $C$ induced by $V(P_2) \cup \cdots \cup V(P_k)$

**Corollary**

- for each vertex in $C$, there are at most $2d + 1$ vertices on each $P_r$ at distance at most $d$
- gives best known bounds for generalised colouring numbers of graphs without $K_t$-minor   ( vdH et al., 2017 )

## *Choosing H – method II*

- given $C$ and the sets $A_1,\ldots,A_k$ in $V(C)$
  - take a vertex $a_r$ from each $A_r$
  - take $H$ a minimal connected, induced subgraph of $C$ containing all of $A=\{a_1,\ldots,a_k\}$

**Easy Lemma** (vdH & Wood, 2018)

- every vertex in $H$ has degree at most $|A|$ in $H$
- $H$ has treewidth at most $|A|-1$
- every vertex not in $A$ is a cut-vertex of $H$
  - corollary: there is a 2-colouring of $H$ where each colour class induces components of size at most $\lceil \frac{1}{2}|A| \rceil$

# *Colouring graphs without complete minor*

- recall: $G$ has no $K_t$-minor $\Longrightarrow$
  - each component $C$ is adjacent to at most $t-2$ "earlier $H_i$"
    - so $A$ has size at most $t-2$

this leads to:

- $G$ has no $K_t$-minor $\Longrightarrow$
  $G$ has a chordal decomposition $\mathcal{H} = (H_1, \ldots, H_\ell)$ such that
  - the quotient graph $Q(\mathcal{H})$ has a proper colouring with at most $t-1$ colours
  - in each $H_m$ has maximum degree at most $t-2$
  - each $H_m$ has treewidth at most $t-3$

## *Colouring graphs without complete minor*

**Theorem 1** ( vdH & Wood, 2018 )

- $G$ has no $K_t$-minor $\Longrightarrow$

  $G$ can be coloured with $t-1$ colours such that

  - each colour class induces a graph with maximum degree at most $t-2$
  - each colour class induces a graph with treewidth at most $t-3$

- improves previous degree bound $c'\,t^2 \log t$

  ( Edwards et al., 2015 )

## *Colouring graphs without complete minor*

- no $K_t$-minor $\Longrightarrow$ chordal decomposition $\mathcal{H} = (H_1, \ldots, H_\ell)$
  - the quotient graph $Q(\mathcal{H})$ has a proper colouring with at most $t-1$ colours
  - there is a 2-colouring of $H$ where each colour class induces components of size at most $\lceil \frac{1}{2}(t-1) \rceil$

**Theorem 2** (vdH & Wood, 2018)

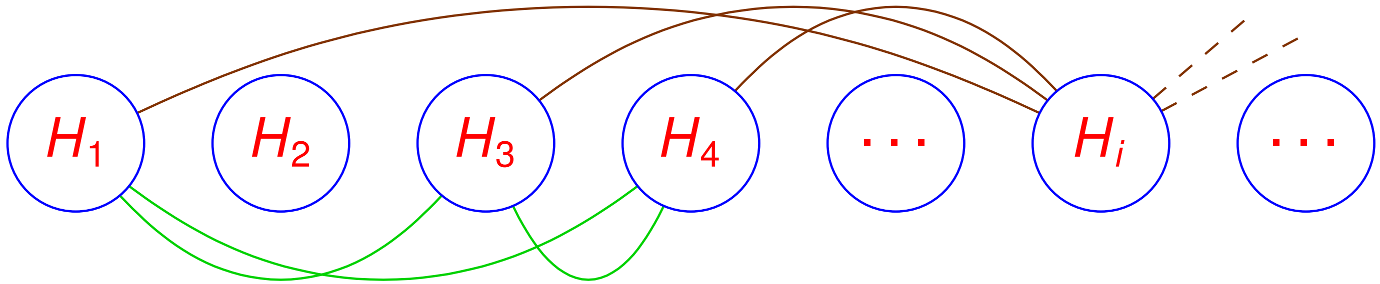
- $G$ has no $K_t$-minor $\Longrightarrow$
  $G$ can be coloured with $2(t-1)$ colours such that
  - each colour graph induces a graph with components of size at most $\lceil \frac{1}{2}(t-2) \rceil$

    (huge improvement of size bound over previous results)

# *A "nice" structure theorem*

**Theorem 3** (vdH & Wood, 2018)

- $G$ has no $K_t$-minor $\Longrightarrow$ $G$ has a partition into connected subgraphs $\mathcal{H} = (H_1, \ldots, H_\ell)$ such that
  - the quotient graph $Q(\mathcal{H})$ has treewidth at most $t-2$
  - each $H_m$ has treewidth at most $t-3$



# *Algorithmic application*

- Dvořák & Siebert (2018+)
  are using the "choose minimal" structure to obtain
  polynomial time, arbitrary good approximation schemes
  for problems such as
  - minimum dominating set
  - minimum $d$-dominating set
  - maximum independent set
  - maximum $d$-independent set
  - induced subgraph cover
  - etc.

  for graphs without some fixed $K_t$-minor

## *An early application*

**Theorem** (Reed & Seymour, 1998)

- $G$ has no $K_t$-minor $\Longrightarrow$
  $G$ has fractional chromatic number at most $2(t-1)$

- the paper introduces chordal decompositions
- and uses chordal decompositions with quite a complicated structure

## *Hidden in the Reed & Seymour proof*

**Lemma**

- any graph $G$ has a chordal decomposition $\mathcal{H} = (H_1, \ldots, H_\ell)$ such that
  - each $H_m$ has an independent set $S_m$ satisfying

$$|S_m| \geq \tfrac{1}{2}(|V(H_m)| + 1)$$

- now use that for a graph $G$ without $K_t$-minor, $Q(\mathcal{H})$ has a proper colouring with at most $t-1$ colours

## *Hidden in the Reed & Seymour proof*

**Theorem**

- $G$ has no $K_t$-minor

  $\Longrightarrow$ there is an induced subgraph $F$ of $G$, such that

  - $F$ has a proper colouring with at most $t-1$ colours
  - $F$ has size $|V(F)| \geq \frac{1}{2}(|V(G)| + t - 1)$

**Corollary** (Duchet & Meyniel, 1982)

- $G$ has no $K_t$-minor

  $\Longrightarrow$ $G$ has an independent set $S$ of size

$$|S| \geq \frac{1}{2}\Big(\frac{|V(G)|}{t-1} + 1\Big)$$

## *Problems – questions*

- other structural applications?
- other algorithmic applications?

## *Problems – questions*

- what can you do for graphs with no $H$-minor, where $H$ is not a complete graph?

in particular, is the following true:

- $G$ has no $(K_t - e)$-minor $\implies$ there is chordal decomposition $\mathcal{H} = (H_1, \ldots, H_\ell)$ such that
  - the quotient graph $Q(\mathcal{H})$ has a proper colouring with at most $(t-1) - 1$ colours
  - each $H_m$ has “low” degree

## *Problems – questions*

- Seymour (2016) already observed that
  - $G$ has no $K_t$-minor $\implies$
    there is an induced subgraph $F$ of $G$
    - which can be properly $(t-1)$-coloured
    - and $F$ has size at least $\frac{1}{2}|V(G)|$
  - follows from ideas in the proof of Duchet & Meyniel (1982)

- can you increase the constant $\frac{1}{2}$?

- Duchet & Meyniel's result has been improved,
  e.g. by Fox (2010) and Balogh & Kostochka (2011)
  - can those proofs be used to increase the constant $\frac{1}{2}$?

I'll better stop now,

before Bill gets nervous about missing his train ...

so

**Thanks for your attention !**