London Taught Course Centre

2012 examination

Graph Theory

Answers

1 (a) We first do the case $k = 2$. For $n = 2\,\ell$ even, we can draw $C^2_{2\,\ell}$ in the plane as follows: first draw the cycle $v_1 v_2 \ldots v_n v_1$. Then draw the additional edges that go between odd-indexed vertices $v_1v_3, v_3v_5, \ldots, v_{2\,\ell-1}v_1$ inside the cycle and the 'even' additional edges outside the cycle.

For n odd, the graph C^2_n is not planar. The easy way to show this is by using arguments similar to the previous paragraph. First draw the cycle in a plane. Suppose the edge v_1v_3 is drawn on the inside of the cycle. Then the edge v_2v_4 must be drawn on the outside of the cycle. And then the edge v_3v_5 must be on the inside again. Etc., until we reach a problem with the last edge. If v_1v_3 is on the outside, we reach a similar contradiction.

For $k \geq 3$, the graph C^k_n is never planar. To see that, again we draw the cycle first. Then for the edges v_1v_{k+1} and v_2v_{k+2} we have: one must be on the inside and one must be on the outside of the cycle. But then there is no way to add the edge v_3v_{k+3} without crossing some other edge.

We obtain that C^k_n is planar if, and only if, n is even and $k = 2$.

(b) The graphs C^k_n with $n \geq 2\,k+1 \geq 5$ have maximum degree $\Delta = 4$ and are connected. From Brooks' Theorem we know that the only connected graphs G that have $\chi(G) = \Delta + 1$ are complete graphs and odd cycles. Obviously, C^k_n is not a cycle; and the only choice for n, k that makes C^k_n a complete graph is $n = 5$, $k = 2$. So we have $\chi(C^2_5) = 5$; while in all other cases $\chi(C^k_n) \leq 4$.

Now take $n = 6\,\ell+1$ for some $\ell \geq 1$ and $k = 2$. We will show that $\chi(C^2_{6\,\ell+1}) = 4$. Note that those graphs are not bipartite (since $6\,\ell+1$ is odd), hence have chromatic number at least three. Suppose we try to colour $C^2_{6\,\ell+1}$ using three colours only, and consider the colouring on the cycle $v_1v_2 \ldots v_{6\,\ell+1}v_1$. Since $6\,\ell+1$ is not a multiple of three, we can't colour the cycle in a regular pattern 1,2,3,1,2,3,.... In particular, at some point on the cycle we see the colour pattern c_1, c_2, c_1 on three consecutive vertices on the cycle. But that means the additional edges between vertices at distance two on the cycle introduces an edge between the two vertices with colour c_1. So the colouring is no longer proper.

So in order to colour $C^2_{6\,\ell+1}$ we need at least four colours. Together with the conclusion in the first paragraph of this part, we get $\chi(C^2_{6\,\ell+1}) = 4$ for all $\ell \geq 1$.

(c) A little bit of playing with small values of n and k should lead to the observation that C^3_6 is isomorphic to the complete bipartite graph $K_{3,3}$. And $K_{3,3}$ is the standard example of a graph for which the choice number is not equal to the chromatic number.

2 (a) Let $C \subset A$ and $D \subset B$ be sets such that $|C| \geq \frac{1}{2}|A|$ and $|D| \geq \frac{1}{2}|B|$. Take $C' \subset C$ and $D' \subset D$ such that $|C'| \geq (2\varepsilon)|C|$ and $|D'| \geq (2\varepsilon)|D|$. We see that $|C'| \geq (2\varepsilon)|A|/2 = \varepsilon|A|$ and $|D'| \geq (2\varepsilon)|B|/2 = \varepsilon|B|$. From ε-regularity, we obtain that $|d(A,B) - d(C',D')| < \varepsilon$.

Furthermore, $|C| \geq \frac{1}{2}|A| > \varepsilon|A|$ and $|D| \geq \frac{1}{2}|B| > \varepsilon|B|$, hence $|d(A,B) - d(C,D)| < \varepsilon$. By the triangle inequality,

$$|d(C,D) - d(C',D')| \leq |d(A,B) - d(C',D')| + |d(A,B) - d(C,D)| < 2\varepsilon.$$

(b) Notice that $e_{\bar{G}}(A,B) = |A|\,|B| - e_G(A,B)$, hence $d_{\bar{G}}(A,B) = 1 - d_G(A,B)$. So,

$$|d_{\bar{G}}(A,B) - d_{\bar{G}}(A',B')| = |1 - d_G(A,B) - (1 - d_G(A',B'))| = |d_G(A,B) - d_G(A',B')|.$$

From this, it follows that A, B is ε-regular in G if, and only if, A, B is ε-regular in $\bar{G}$.

(c) We follow the hint. Let A' be the set of all vertices in A with less than $(1/2 - \varepsilon)|B|$ neighbours in B. Then, $e(A',B) < |A'|(1/2 - \varepsilon)|B|$ and $d(A',B) < 1/2 - \varepsilon$.

On the other hand, if $|A'| \geq \varepsilon|A|$, then $d(A',B) > d(A,B) - \varepsilon \geq 1/2 - \varepsilon$. This is a contradiction.

(d) Since $\varepsilon < 1/10$ and $|A|, |B| \geq 100$, we have that $|N(a) \setminus \{b\}| \geq (1/2 - \varepsilon)|B| - 1 \geq \varepsilon|B|$ and $|N(b) \setminus \{a\}| \geq (1/2 - \varepsilon)|A| - 1 \geq \varepsilon|A|$. So, by ε-regularity, the pair $N(b) \setminus \{a\}, N(a) \setminus \{b\}$ has density at least $d(A,B) - \varepsilon \geq 1/2 - \varepsilon > 0$. Hence, there is an edge xy with $x \in N(a) \setminus \{b\}$ and $y \in N(b) \setminus \{a\}$.

Thus, we have the path a, x, y, b.

In a similar way, we have that $|N(x) \setminus \{a,y\}| \geq (1/2 - \varepsilon)|A| - 2 \geq \varepsilon|A|$ and $|N(y) \setminus \{b,x\}| \geq (1/2 - \varepsilon)|B| - 2 \geq \varepsilon|B|$. So, by ε-regularity, the pair $N(x) \setminus \{a,y\}, N(y) \setminus \{b,x\}$ has density at least $d(A,B) - \varepsilon \geq 1/2 - \varepsilon > 0$. Hence, there is an edge wz with $w \in N(y) \setminus \{b,x\}$ and $z \in N(x) \setminus \{a,y\}$.

Thus, we have the path a, x, z, w, y, b.