London Taught Course Centre

2012 examination

Graph Theory

Instructions to candidates

This part of the exam has one question, consisting of several parts. You are expected to **answer all parts**.

Justify all your answers.

1 For integers k, n with $k \geq 2$ and $n \geq 2\,k$, define the graph C_n^k as follows. Start with the cycle C_n with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Now add edges between any pair of vertices at distance k in C_n. In other words, add the edges $v_1v_{k+1}, v_2v_{k+2}, \ldots, v_{n-k}v_n, v_{n-k+1}v_1, v_{n-k+2}v_2, \ldots, v_nv_k$.

(a) Determine all $n \geq 2\,k \geq 4$ for which C_n^k is planar.

For $n \geq 2\,k+1$ we have that every vertex in C_n^k has degree exactly four. This means that the chromatic number and the choice number are at most five.

(b) Show that there is only a finite number of graphs C_n^k with $n \geq 2\,k+1 \geq 5$ so that $\chi(C_n^k) = 5$.

Show that for an infinite number of graphs C_n^k with $n \geq 2\,k+1 \geq 5$ we have $\chi(C_n^k) = 4$.

(c) Give at least one graph C_n^k for which $\mathrm{ch}(C_n^k) \neq \chi(C_n^k)$.

2 For a graph $G = (V, E)$, a pair A, B of disjoint subsets of V is called an *ε-regular pair* if, for every pair $A' \subseteq A$, $B' \subseteq B$, with $|A'| \geq \varepsilon\,|A|$, $|B'| \geq \varepsilon\,|B|$, we have

$$\left| \frac{e(A,B)}{|A|\,|B|} - \frac{e(A',B')}{|A'|\,|B'|} \right| < \varepsilon.$$

Let A, B be an ε-regular pair in a graph $G = (V, E)$, for some $\varepsilon < 1/2$.

(a) Let $C \subset A$ and $D \subset B$ be sets such that $|C| \geq \frac{1}{2}\,|A|$ and $|D| \geq \frac{1}{2}\,|B|$.

Use the definition of ε-regularity to prove that the pair C, D is (2ε)-regular in G.

(b) Let $\bar{G}$ be the complement of the graph G, that is, the graph with the same vertex set V whose edges are the pairs of vertices uv such that $uv \notin E$.

Prove that A, B is an ε-regular pair in $\bar{G}$.

(c) Suppose that the *density* $d(A,B) := \dfrac{e(A,B)}{|A|\,|B|}$ of the pair A, B satisfies $d(A,B) \geq 1/2$.

Let A' be the set of all vertices in A with less than $(1/2 - \varepsilon)\,|B|$ neighbours in B.

Prove that $|A'| < \varepsilon\,|A|$.

(*Hint: assume that $|A'| \geq \varepsilon\,|A|$, set $B' = B$, and estimate $d(A', B')$ from below and from above.*)

(d) Suppose that $\varepsilon < 1/10$, $|A|, |B| \geq 100$, $d(A,B) \geq 1/2$, all vertices in A have at least $(1/2-\varepsilon)\,|B|$ neighbours in B, and all vertices in B have at least $(1/2-\varepsilon)|A|$ neighbours in A. Let $a \in A$ and $b \in B$.

Prove there is a path with 3 edges starting in a and ending in b.

(*Hint: show that the pair $N(b) \setminus \{a\}, N(a) \setminus \{b\}$ has density at least $1/2 - \varepsilon$ and deduce that there must be and edge $xy \neq ab$ such that $x \in N(a)$ and $y \in N(b)$.*)

Prove there is a path with 5 edges starting in a and ending in b.