LTCC Course on Graph Theory 2013/14

Notes 1

Graph Colouring

More or less everything we will be discussing in the lectures and these notes can be found in great detail in Chapter 12 of Diestel (in particular Section 5.4). Also Chapter V of Bollobás (and, again, in particular Section V.4) has everything we need.

The best source for anything related to graph colouring is the book *Graph Coloring Problems*, by T.R. Jensen and B. Toft, John-Wiley (1995).

1.1 Vertex Colouring

Given a graph $G = (V, E)$ and a finite *colour set* C, a (*proper*) *vertex colouring of* G is a function $\varphi : V \longrightarrow C$ so that for all edges $uv \in E$ we have $\varphi(u) \neq \varphi(v)$. The *chromatic number* $\chi(G)$ of G is the smallest number of colours a colour set must have so that a vertex colouring exists.

We usually use positive integers for the colours, $C = \{1, \ldots, k\}$ for some k. And then we can say that $\chi(G)$ is the minimum k so that there is a vertex colouring $\varphi : V \longrightarrow \{1, \ldots, k\}$.

If G admits a vertex colouring with k colours, we call G *k-colourable*. So saying that G is k-colourable is equivalent to stating that $\chi(H) \leq k$.

- Many of the bounds on the chromatic number are in terms of vertex degrees. For a graph G, denote by $\delta(G)$ the minimum vertex degree and by $\Delta(G)$ the maximum vertex degree.

  The greedy algorithm gives an obvious proof that $\chi(G) \leq \Delta(G) + 1$. With some effort, you can say a little more :

  Theorem 1 (Brooks, 1941)
  A connected graph G that is not a complete graph or an odd cycle satisfies $\chi(G) \leq \Delta(G)$.

  Graphs such as the complete bipartite graphs $K_{n,n}$, where $\chi(K_{n,n}) = 2$ while $\Delta(K_{n,n}) = n$, show that the chromatic number can be much smaller than $\Delta(G)$ in general.

  The *degeneracy* $\deg(G)$ *of* G, is the maximum of $\delta(H)$ over all subgraphs H of G. This number is also known as the *maximin degree*; and this number plus one goes under the name of *colouring number* or *Szekeres-Wilf number*. From the definition, it follows that if a graph G has degeneracy m, then every subgraph of G has a vertex of degree at most m. From this observation, it is easy to construct a greedy algorithm that proves the following bound :

  Property 2
  For all graphs G we have $\chi(G) \leq \deg(G) + 1$.

Again, complete bipartite graphs show that in general the chromatic number and the degeneracy can be very far apart.

- Much research on graph colouring has focussed on finding the best upper bound on the chromatic number of planar graphs. A *planar graph* is a graph that can be drawn in the plane so that the vertices are different points, the edges are simple curves connecting their endvertices, and the interior of each edge contains no vertex and no point from any other edge. We will see a lot more about planar graphs (and more general related concepts) in the next lecture.

  The Four Colour Conjecture can be stated as "every planar graph has chromatic number at most 4". It was originally formulated as a conjecture on the number of colours needed to colour a map of contiguous regions in the plane, by Francis Guthrie in 1852. (Francis asked the question to his brother Fredrick, who was at that time a student of De Morgan at University College London. De Morgan started mentioning the problem to mathematicians he communicated with.)

  From Euler's Formula (see the next lecture), it is easy to obtain that a planar graph $G = (V, E)$ on at least three vertices satisfies $|E| \leq 3|V| - 6$. Since $\sum_{v \in V} = 2|E|$, from this we can derive that a planar graph always has a vertex of degree at most five. So we have $\deg(G) \leq 5$, and hence Theorem 2 guarantees that every planar graph G is 6-colourable.

  A next, non-trivial, improvement was obtained by Heawood, using ideas from Kempe's incorrect proof of the Four Colour Conjecture.

  Theorem 3 (Heawood, 1890; Kempe, 1879)
  All planar graphs are 5-colourable.

  The search for a proof of the Four Colour Conjecture that all planar graphs are 4-colourable, eventually lead to a proof.

  Theorem 4 (Appel & Haken, 1977)
  All planar graphs are 4-colourable.

  Appel and Haken's proof has always been fairly controversial, since it relied on computer assistance in checking a very large number of cases. A more recent proof of the Four Colour Theorem by Robertson, Sanders, Seymour and Thomas, 1997, reduces some of the need for computer checking, but doesn't do away with it completely.

1.2 Vertex List Colouring

Given a finite set of colours C, a *list assignment* is an assignment $L : V \longrightarrow \mathcal{P}(C)$ of subsets of the colours to the vertices. So each vertex v gets attached to it a list $L(v)$ of colours. Given a list assignment L, we call G *L-colourable* if there exists a function $\varphi : V \longrightarrow C$ so that $\varphi(v) \in L(v)$ for all vertices v and such that for all edges $uv \in E$ we have $\varphi(u) \neq \varphi(v)$.

We say that G is *k-list-colourable* or *k-choosable* if G is L-colourable for every list assignment L with $|L(v)| = k$ for all vertices v. And the *list chromatic number* or *choice number* $\mathrm{ch}(G)$ is the smallest k so that G is k-choosable.

By giving all vertices the same list $L(v) = \{1, \ldots, k\}$ of colours, it follows directly that $\mathrm{ch}(G) \geq \chi(G)$. And somehow one would expect that the case when all colours have the same list is the "hardest" to colour, that cases where the lists are not identical are "easier". Surprisingly enough, that is not the case :

Property 5
For all $k \geq 2$, there exist bipartite graphs G (i.e., with $\chi(G) = 2$) for which $\mathrm{ch}(G) \geq k$.

An easy positive result is that the greedy algorithms for colouring using vertex degrees are also applicable for list colouring. So we get :

Property 6
For all graphs G we have $\mathrm{ch}(G) \leq \deg(G) + 1 \leq \Delta(G) + 1$.

- As soon as the concept of list colouring was introduced (usually attributed to Vizing, 1976), determining the choice number of planar graphs became a hot topic. Since there are planar graphs that are not 3-colourable, there are also planar graphs that are not 3-choosable. And Property 6 guarantees that for every planar graph G we have $\mathrm{ch}(G) \leq 6$. So for quite a while it was an open question what the exact number should be. This question was settled in the 1990s with surprisingly simple proofs.

Theorem 7 (Voigt, 1993)
There exist planar graphs that are not 4-choosable.

Theorem 8 (Thomassen, 1994)
All planar graphs are 5-choosable.

Thomassen's proof is stunningly simple and subtle at the same time. Before we describe its main idea, we need one more concept. A drawing of a planar graph in the plane divides the plane into a number of connected "regions". Such a region is called a *face* of the drawing (or *embedding*). All faces have a finite size, except the one on the outside, which is usually called the *outer face*.

To prove Theorem 8, we only need to consider planar graphs that are connected and in which can be drawn in the plane such that each face is bounded by exactly three edges – we call such a face a *triangle*. Indeed, if a planar graph has a face with more than three edges on its boundary, than we can add an edge so that the graph remains planar (see exercise 1). Any colouring of this larger graph is certainly a colouring of the original graph.

A *near-triangulation* is an embedding of a connected planar graph in which each face, except for the outside face, is a triangle, and the outside face is a simple cycle (no repeated vertices). Thomassen then proves the following for near-triangulations :

Theorem 8′ (Thomassen, 1994)
Let G be a planar graph that can be drawn in the plane as a near-triangulation, and suppose that $v_1 v_2 \cdots v_k v_1$ is the sequence of vertices encountered when walking along the boundary of the outside face. Suppose that the vertices of G have been assigned the following lists of colours :
– *v_1 and v_2 have a list of one colour each, with $L(v_1) \neq L(v_2)$;*
– *the other vertices on the outer face have a list of three colours;*
– *all vertices not on the outer face have a list of five colours.*
Then there exists a colouring of G where each vertex receives a colour from its list.

Note that the lists of colours given to each vertex according to Theorem 8′ is always at most five. This means that Theorem 8 is a direct consequence of Theorem 8′. We will discuss the proof of Theorem 8′ in the lectures.

1.3 Edge Colouring

In this section we assume that graphs can have *multiple edges* (but still no loops). When we want to exclude multiple edges, we use the term *simple graph*.

Given a graph $G = (V, E)$ and a finite *colour set* C, a (*proper*) *edge colouring of* G is a function $\varphi : E \longrightarrow C$ so that for any two adjacent edges $e, f \in E$ (i.e., e and f have at least one common endvertex) we have $\varphi(e) \neq \varphi(f)$.

We use terminology similar to vertex colouring. So a graph G can be *k-edge-colourable*, and the minimum k for which G is k-edge-colourable is the *edge chromatic number* or *chromatic index* $\chi'(G)$.

- For a graph $G = (V, E)$, the *line graph* $L(G) = (V_L, E_L)$ is the graph that has the edges of G as vertices : $V_L = E$; and two edges are adjacent in the line graph if they have a common endvertex in G.

  It is easy to see that edge colouring a graph G is the same as vertex colouring the line graph $L(G)$. So in that sense, edge colouring is just a special case of vertex colouring. But by arguing in this way, we ignore the special structural properties of the edge set of a graph that can be used when analysing edge colouring, but are not present in vertex colouring in general.

- For a graph with multiple edges, we count the degree of a vertex as the number of edges incident with that vertex. So $\Delta(G)$ is the maximum number of edges incident with any vertex of G.

  It is obvious that $\chi'(G) \geq \Delta(G)$. Since any edge can be adjacent to at most $2\,(\Delta(G) - 1)$ other edges, this gives the following trivial bounds for the chromatic index.

Property 9
For all graphs G we have $\Delta(G) \leq \chi'(G) \leq 2\,\Delta(G) - 1$.

But in fact, the relationship between chromatic index and maximum degree is much closer than the relationship between chromatic number and maximum degree, as the following classical results illustrate.

Theorem 10 (König, 1916)
For a bipartite graph G we have $\chi'(G) = \Delta(G)$.

Theorem 11 (Shannon, 1949)
For a graph G we have $\chi'(G) \le \frac{3}{2}\,\Delta(G)$.

Note that Theorem 11 is best possible. Form a graph G on three vertices x, y, z by adding m multiple edges between each pair from x, y, z. Then $\Delta(G) = 2\,m$, but all $3\,m$ edges are adjacent, hence $\chi'(G) = 3\,m$.

It is no accident that the graphs that show that Shannon's bound are best possible have multiple edges. Let $\mu(G)$ denote the maximum edge multiplicity of a graph G. Hence G is simple if and only if $\mu(G) \le 1$.

Theorem 12 (Vizing, 1964)
For a graph G we have $\chi'(G) \le \Delta(G) + \mu(G)$.

Corollary 13 (Vizing, 1964)
For a simple graph G we have $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$.

Determining which of the two numbers $\Delta(G), \Delta(G) + 1$ is the right value for the chromatic index of a given simple graph G is not an easy task. This problem is known to be NP-complete. Even if all vertices have degree three, deciding if a the simple graph has chromatic index three or four is NP-complete (both results Holyer, 1981)

1.4 Edge List Colouring

Also for this section we assume that graphs can have multiple edges.

List colourings of edges are defined analogously to vertex list colourings. The essential comparable concepts are *k-edge-list-colourable* or *k-edge-choosable*, and *edge list chromatic number*, *list chromatic index* or *edge choice number*. This latter parameter is denoted $\mathrm{ch}'(G)$.

- Again we trivially have $\mathrm{ch}'(G) \ge \chi'(G)$. For vertex colouring, we have seen that the chromatic number and the choice number can be arbitrarily far apart. For edge colouring, it is conjectured that there actually is no difference! The following List Colouring Conjecture is attributed to Vizing (1975). (But see the Jensen & Toft book for a discussion about its history.)

Conjecture 14
For every graph G, $\mathrm{ch}'(G) = \chi'(G)$.

Using a greedy algorithm, it is again easy to obtain that $\mathrm{ch}'(G) \le 2\,\Delta(G) - 1$. For a long time, not much progress was made beyond that easy observation.

Theorem 15 (Borodin, Kostochka & Woodall, 1997)
For a graph G we have $\mathrm{ch}'(G) \le \frac{3}{2}\,\Delta(G)$ *(so* $\mathrm{ch}'(G) \le \frac{3}{2}\,\chi'(G)$ *).*

Theorem 15 should be compared with Theorem 11. The proof of Theorem 15 uses the techniques developed by Galvin (see the next section). The next result says more, but only if the maximum degree is very large.

Theorem 16 (Kahn, 2000)

For every $\epsilon > 0$ there exists a constant D_ϵ, so that if G is a graph with $\chi'(G) \geq \Delta(G) \geq D_\epsilon$, then $\mathrm{ch}'(G) \leq (1+\epsilon)\,\chi'(G)$.

Kahn's proof is a masterpiece of probabilistic techniques (for which see Lecture 4).

- One of the major breakthroughs in the research on edge list colouring was the following result.

Theorem 17 (Galvin, 1995)

For every bipartite graph G, $\mathrm{ch}'(G) = \chi'(G)$.

Galvin's proof relies on a concept called "kernels in directed graphs". We will view an *oriented graph G^* of an underlying undirected graph G* as an assignment of one of the two possible directions to each edge of G. By the *outdegree* $d^+_{G^*}(v)$ of a vertex v we denote the number of arcs that have v as a tail.

A *kernel* of a directed graph G^* is an independent set $K \subseteq V$, so that for every vertex $v \in V \setminus K$, there is an arc in G^* from v to a vertex in K.

Lemma 18

Let G be a graph and L a vertex list assignment of G. Suppose there exists an oriented graph G^* with G as underlying graph, such that $|L(v)| \geq d^+_{G^*}(v) + 1$ for each vertex v and such that every induced subgraph of G^* has a kernel. Then G is L-colourable.

Recall that the line graph $L(G) = (V_L, E_L)$ is the graph that has the edges of G as vertices : $V_L = E$; and two edges are adjacent in the line graph if they have a common endvertex in G.

For a graph G, for each vertex v choose a linear ordering $\leq_v$ of the edges incident with v. Then we can translate this to an orientation of the line graph $L(G)$ as follows : If two edges e, f have a common endvertex v, and $e \leq_v f$ in the chosen linear ordering around v, then orient the edge ef in $L(G)$ from e to f. If e and f are parallel edges, then it is possible that we have both an arc from e to f and an arc from f to e (if e and f have different ordering around each of their two common endvertices). This causes no problems in what follows.

We call any orientation of $L(G)$ obtained from a system $\leq_v$ of linear orderings as above a *normal orientation*. Notice that an induced subgraph of a line graph with a normal orientation is again a line graph with a normal orientation.

So what does a kernel in a line graph $L(G)^*$ with normal orientation look like ? First we observe that an independent set in a line graph $L(G)$ corresponds to a matching in G. (A *matching* is a set of edges so that no two have a common endvertex.) Next assume the normal orientation of the line graph originated from linear orderings $\leq_v$ of the edges incident with each vertex v. So a kernel in $L(G)^*$ is a matching M in G so that for each edge $e \in E \setminus M$ there is an arc from e to some $f \in M$ in $L(G)^*$. In other words, for each edge $e = uv \in E$ with $e \notin M$, we have that there is an edge $uw = f_1 \in M$ with $e \leq_u f_1$, or an edge $wv = f_2 \in M$ with $e \leq_v f_2$.

The following lemma, together with Lemma 18, is most of the proof of Galvin's Theorem.

Lemma 19

Let G be a bipartite graph and let $L(G)^*$ be a normal orientation of the line graph of G. Then $L(G)^*$ has a kernel.

Proof Assume the normal orientation of the line graph originated from linear orderings $\leq_v$ of the edges incident with each vertex v. And denote the two parts of G by X and Y.

We prove the lemma by induction on the number of edges of G. If there is only one edge, then we can just use that edge as the kernel. So assume there is more than one edge. For each $x \in X$, let e_x be the edge incident with x that is maximal for the linear ordering $\leq_x$. Take $N = \{ e_x \mid x \in X \}$.

If N is a matching, then it is a kernel in $L(G)^*$, since for each other edge $e = xy$ with $x \in X$, we have that $e \leq_x e_x$, hence there is an arc in $L(G)^*$ from e to $e_x \in N$.

So suppose N is not a matching, hence there exists $x, x' \in X$, $x \neq x'$, and $y \in Y$ so that $e_x = xy$ and $e_{x'} = x'y$. Without loss of generality we can assume $e_x \leq_y e_{x'}$. Now remove e_x from G to form G^-, and leave the orderings of the edges around each vertex the same. By induction, $L(G^-)^*$ has a kernel. This kernel corresponds to a matching M in G^-.

If $e_{x'} \in M$, then, since $e_x \leq_y e_{x'}$, there is an arc in $L(G)^*$ from e_x to $e_{x'}$, so M is also a kernel in $L(G)^*$.

If $e_{x'} \notin M$, then there is an edge $f \in M$ so that there is an arc $L(G)^*$ from $e_{x'}$ to f. But since $e_{x'}$ was the maximal element around x', this arc must arise since $e_{x'}$ and f both have y as a common endvertex, and $e_{x'} \leq_y f$. As we also have $e_x \leq_y e_{x'}$, this means $e_x \leq_y f$, and hence also this time we can conclude that M is a kernel in $L(G)^*$. ■

Proof of Theorem 17 Take $k = \chi'(G)$, and let $\varphi : E \longrightarrow \{1, \ldots, k\}$ be an edge colouring of G. Denote the two parts of G by X and Y.

We form the following linear orderings of the edges around a vertex v. If $x \in X$, and e_1, e_2 have x as an endvertex, then we set $e_1 \leq_x e_2$ if $\varphi(e_1) \leq \varphi(e_2)$. While if $y \in Y$, and e_1, e_2 have y as an endvertex, then we set $e_1 \leq_y e_2$ if $\varphi(e_1) \geq \varphi(e_2)$.

Form the orientation $L(G)^*$ of $L(G)$ using the linear orderings above. What can we say about $d^+_{L(G)^*}(e)$ of an edge $e = xy$ of G? This is the number of edges incident with x that have a colour larger than e plus the number of edges incident with y that have a colour smaller than e. Since all edges incident with the same vertex have different colours, and there are k colours in total, this means that for all edges e we have $d^+_{L(G)^*}(e) \leq k - 1$ (we subtract one for the colour e has itself).

So if we give each edge e a list $L(e)$ of $k = \chi'(G)$ colours, then by Lemma 18 we know that the edges are L-colourable, proving the theorem. ■

Exercises

1 For $n \geq 3$, a cycle C_n is a graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

(a) Let C_n, $n \geq 3$, be a cycle. Let us assign lists $L(v)$ of two colours to each vertex v of C_n. Show that there is an L-colouring, except if n is odd and all lists are identical.

(b) Let C_{2k}, $k \geq 2$, be an even cycle. Prove that $\chi(C_{2k}) = \mathrm{ch}(C_{2k}) = \chi'(C_{2k}) = \mathrm{ch}'(C_{2k}) = 2$, using only the definitions.

(c) Let C_{2k-1}, $k \geq 2$, be an odd cycle. Prove that $\chi(C_{2k-1}) = \mathrm{ch}(C_{2k-1}) = \chi'(C_{2k-1}) = \mathrm{ch}'(C_{2k-1}) = 3$.

2 Determine the chromatic index $\chi'(K_n)$ of the complete graphs K_n.

3 Let G be a planar graph and suppose it is drawn in the plane so that at least one face has more than three edges on its boundary. Show that you can add an edge to G so that the larger graph is again a planar, simple, graph. (The issue here is that you need to prevent adding an edge between two vertices that are already connected by an edge. In other words: make sure that after adding an edge the larger graph is still simple.)

4 Explain how Shannon's Theorem 11 follows easily from Vizing's Theorem 12.

5 Give an infinite family of directed graphs without a kernel.

6 A *total colouring* of a graph $G = (V, E)$ is an assignment of colours to vertices *and* edges $V \cup E$ so that adjacent or incident elements get different colours. The smallest k such that G has a total colouring with k colours is called the *total chromatic number*, denoted $\chi''(G)$.

(a) Find lower and upper bounds of $\chi''(G)$ in terms of $\Delta(G)$ and/or $\deg(G)$.

(b) Determine the total chromatic number of cycles C_n and complete graphs K_n.

(c) Show that if the List Colouring Conjecture 14 is true, then for simple graph G we have $\chi''(G) \leq \Delta(G) + 3$.