LTCC Course on Graph Theory 2013/14

Solutions to Exercises for Notes 2

I think these are all rather easy!

1. *How does Euler's formula for graphs embedded in the plane need to be modified to handle graphs with c components, where c is not necessarily equal to 1?*

Answer.
We claim that, for a graph G with c components, v vertices and e edges, any embedding of G in the plane has f faces, where $v - e + f = 1 + c$.

We proceed by induction on c, noting first that the result is true for the graph with $c = 0$.

Given a graph G with at least one component H, consider any embedding of G in the plane. Then the whole of H lies in one face F of the embedding of $G - H$. The embedding of $G - H$ has $1 + c(G - H) - v(G - H) + e(G - H)$ faces, by the induction hypothesis. The embedding of H inside the face F has – not counting the exterior face – $1 - v(H) + e(H)$ faces. The total number of faces of the embedding is thus $2 + c(G - H) - v(G - H) - v(H) + e(G - H) + e(H) = 1 + c(G) - v(G) + e(G)$, as claimed. The result now follows by induction.

2. *Consider the oriented surface S_k, $k \geq 1$, constructed as above. Draw a graph in S_k by putting a vertex at each corner of the boundary $4k$-gon, and an edge along each segment of the boundary, identifying any of these edges and vertices as necessary.*
Count the number of vertices, edges, and faces in this embedding, and verify the Euler-Poincaré-formula in this case.

Answer.
The key thing here is to see that the identification of boundary segments of the $4k$-gon identifies all the corners as a single vertex. (You should check that, within each section of four segments, all five corners are identified.) So the number of vertices is 1, as is the number of faces. The number of edges is $2k$, as the boundary segments are identified in pairs. Thus $v - e + f = 2 - 2k$, in line with the Euler-Poincaré formula.

3. *Describe the graphs not containing K_3 as a minor.*
Describe the graphs not containing the 4-cycle C_4 as a minor.

Answer.
We claim that G has K_3 as a minor if and only if it contains a cycle.

If G does contain a cycle, then we can exhibit a K_3 minor by partitioning the vertex set of the cycle into three parts, each of which is a consecutive section of vertices on the cycle.

If G has a K_3 minor, then there are three disjoint connected sets of vertices V_1, V_2, V_3 with an edge e_{ij} between each pair (V_i, V_j). Now we can find (possibly trivial) paths inside each V_i

conecting the endpoints of the two edges e_{ij} in that set. Joining these paths with the e_{ij} gives a cycle.

So the graphs with no K_3 minor are exactly the forests.

Similarly, a graph with no C_k minor, where C_k is cycle on k vertices, is exactly one with no cycle of length k or greater.

In particular, if G has no C_4 minor, then all cycles of G are triangles. Another way to say this is that every *block* (maximal 2-connected subgraph) of G is either an edge or a triangle.

4. *Let P be the* Petersen graph. *(If you don't know what this is, find out!)*
Show that P is non-planar:
(a) using Euler's formula;
(b) by showing that P contains $K_{3,3}$ as a topological minor;
(c) by showing that P contains K_5 as a minor.
What is the minimum size of a set F of edges of P whose deletion leaves a planar graph?

Answer.
(a) P has 10 vertices, and its shortest cycle is of length 5. Using this last fact, we see that, in a putative embedding in the plane with f_i i-sided faces, we have $2e = \sum_i if_i \geq 5f$. Thus Euler's formula gives $10 = 5v - 5e + 5f \leq 5v - 3e$, so $e \leq \frac{1}{3}(5v - 10) = 40/3$. Therefore $e \leq 13$. But the Petersen graph has 15 edges.

(b), (c) See the Figures below.

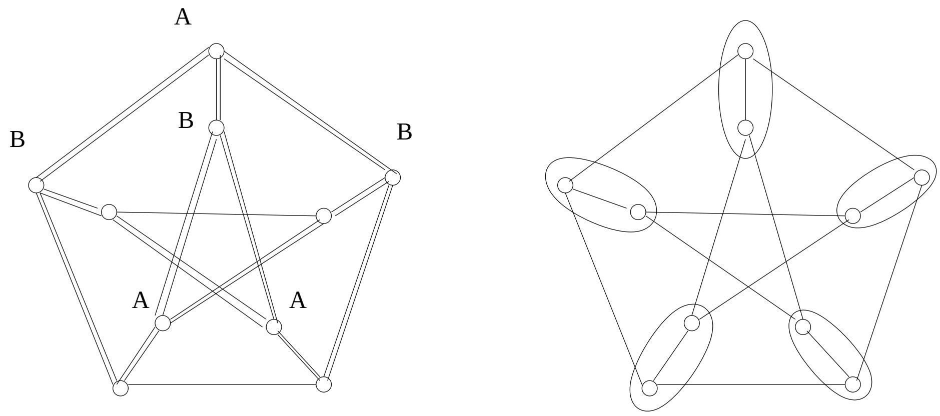

Figure 1: A $K_{3,3}$ topological minor, and a K_5 minor, in the Petersen graph

Method (a) shows that we have to delete at least 2 edges to make P planar. The figure for (b) shows that we can't delete *any* two edges. But there is a set of two edges that can be removed. See below.

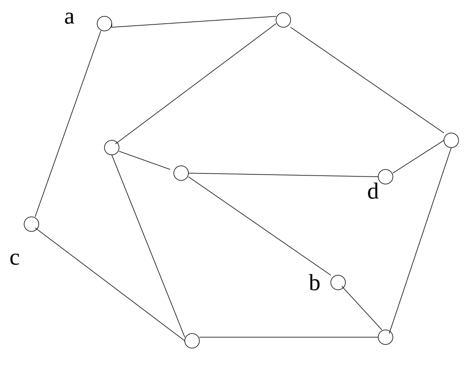

Figure 2: Deleting ab and cd gives a planar graph