

Solutions to Exercises 1

- (1) (a) There are 5 seats in a row and 12 people to choose from. How many different seatings are possible if every seat must be occupied by a person?

Solution. We have 12 choices for the first seat. After we made this choice, there are $12 - 1 = 11$ possibilities for the second seat in the row. In general, having made choices for the first $i - 1$ seats, $i = 2, 3, 4, 5$, we have $12 - (i - 1)$ possibilities for the i -th seat. By the Multiplication Rule, there are $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = (12)_5$ possible seatings. \square

- (b) Now there are 12 seats in a row and 5 people to be seated. If every person must sit in a seat, how many different seatings are possible?

Solution. We have 12 choices of a seat for the first person to sit. After we made this choice, there are $12 - 1 = 11$ possible seats for the second person to sit on. In general, having made choices for the first $i - 1$ persons, $i = 2, 3, 4, 5$, we have $12 - (i - 1)$ possibilities for the seat of the i -th person. By the Multiplication Rule, there are $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = (12)_5$ possible seatings. \square

- (2) How many numbers between 2000 and 5000 can be formed from the digits 1, 2, 3, 4, 5, 6 if repetition of digits is

- (a) allowed;

Solution. The first digit must be 2, 3, or 4; there are no restrictions on the remaining three digits, so for each of them we have 6 possibilities. Hence, there are $3 \cdot 6^3$ such numbers. \square

- (b) not allowed?

Solution. Again, the first digit must be 2, 3, or 4. Having chosen digits at the first i positions, $i = 1, 2, 3$, for the $(i + 1)$ -st digit, we have $6 - i$ possibilities. Hence, there are $3 \cdot 6 \cdot 5 \cdot 4$ such numbers. \square

- (3) (a) How many 10-digit numbers can be formed by using each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once?

Solution. The first cannot be 0. Having chosen digits at the first i positions, $i = 1, 2, 3$, for the $(i + 1)$ -st digit, we have $10 - i$ possibilities. Hence, there are $9 \cdot 9 \cdot 8 \cdot \dots \cdot 1 = 9 \cdot 9!$ such numbers. \square

- (b) And how many of these numbers are larger than 4,000,000,000?

Solution. The first digit cannot be 0, 1, 2 or 3. The rest is as above, hence there are $6 \cdot 9!$ such numbers.

□

- (4) A car registration is to consist of 3 letters followed by a number between 1 and 999. How many car registration numbers are possible?

Solution. There are 26 letters and we can choose them with repetition. So, there are 26^3 choices of 3 letters. There are also 999 choices for one of the numbers. Hence, we have $999 \cdot 26^3$ possible registration numbers.

□

- (5) You have an apple, a banana, a peach, a pear, and an apricot.

- (a) In how many ways can you distribute these pieces of fruit among 8 children if no child can receive more than one piece?

Solution. $(8)_5$ as here are 8 choices for a child getting the apple, 7 choices for a child getting the banana, 6 choices for a child getting the peach, 5 choices for a child getting the pear, and 4 choices for a child getting the apricot.

□

- (b) How many ways are there if any child can receive any number of pieces?

Solution. 8^5 as there are 8 possible choices for a child getting each of the five fruits.

□

- (6) Suppose there are m girls and n boys in a class. What is the number of ways of arranging them in a line so that all the girls are together?

Solution. We treat m girls as one boy B. We can order $n + 1$ boys in $(n + 1)!$ ways. Then we replace B by one of $m!$ possible orderings of girls and obtain an ordering in which all the girls are together. Thus, there are $(n + 1)!m!$ possible orderings.

□

- (7) The names of the 12 months of the year are listed in a certain order. Given that June and July are not next to each other, how many such lists are possible?

Solution. There are $12!$ ways to order 12 months. If we treat June and July as one month, there are $11!$ orderings in which June, July appear in this order, and there are $11!$ orderings in which July, June appear in this order. Hence,

$$12! - 2 \cdot 11! = 10 \cdot 11!$$

orderings do not have June and July together.

□

- (8) Six people are to be seated around a circular table.

- (a) How many ways are there to achieve this?

Solution. Although $1-2-3-4-5-6$, $2-3-4-5-6-1$, $3-4-5-6-1-2$, $4-5-6-1-2-3$, $5-6-1-2-3-4$, $6-1-2-3-4-5$ are different orderings of six people to a line, they are the same ordering if people are seated around a circular table. To avoid this, we can imagine that person 1 always sits at the head of the table, i.e., we only count the

orders that starts with 1. And so there are $5!$ such orders since we need to order persons 2, 3, 4, 5, 6 in every possible way. \square

- (b) And in how many ways can it be done if there are two people who refuse to sit next to each other?

Solution. We first find the number of ways how to sit two fixed persons together. We treat them as one person and, similarly to (a), we obtain that there are $4!$ ways to sit 5 people around a circular table. Now, the two fixed persons can be ordered in two ways, so the total number of ways to sit them together is $2 \cdot 4!$.

To get the number of seating where they are not together, we subtract $2 \cdot 4!$ from the total number of seatings: $5! - 2 \cdot 4! = 3 \cdot 4!$. \square

- (9) An eight-person committee is to be formed from a group of 10 persons from London and 15 persons from outside the city.

- (a) In how many ways can the committee be chosen if the committee must contain 4 persons from each of the 2 groups?

Solution. There are $\binom{10}{4}$ ways to select 4 members out of 10 ‘outsiders’ and $\binom{15}{4}$ ways to choose 4 Londoners. By the Multiplication Principle, there are $\binom{10}{4}\binom{15}{4}$ ways to choose the committee. \square

- (b) In how many ways can the committee be chosen if the committee must contain more persons from outside London than from the city?

Solution. We must choose j ‘outsiders’ and $8 - j$ Londoners, where $5 \leq j \leq 8$. Using our approach from (a), the number of ways to do this is

$$\binom{10}{5}\binom{15}{3} + \binom{10}{6}\binom{15}{2} + \binom{10}{7}\binom{15}{1} + \binom{10}{8}\binom{15}{0}$$

\square

- (c) In how many ways can the committee be chosen if the committee must contain at least 2 Londoners?

Solution. In this case, we must choose $8 - j$ ‘outsiders’ and j Londoners, where $2 \leq j \leq 8$. Hence, the total number of ways is

$$\sum_{j=2}^8 \binom{10}{8-j} \binom{15}{j}$$

\square

- (10) A certain game is played by 13 persons using a (standard) deck of 52 cards: there are four suits (Spades, Diamonds, Hearts, and Clubs) and 13 distinguishable cards (Ace, Two, ..., Ten, Jack, Queen, King) in each of the suits. A k -hand is a set of k different cards from this deck.

- (a) How many 4-hands are there?

Solution. We are choosing 4 cards out of $4 \cdot 13 = 52$, their order does not matter, hence there are $\binom{52}{4}$. \square

A player looks at her 4 cards and sees that she has only cards from one suit.

- (b) How many 4-hands that contain only cards from one suit are there?

Solution. We first select one suit out of 4. Then we choose 4 cards out of 13 possible in the selected suit. Their order does not matter, hence this can be done in $\binom{13}{4}$ ways. By the Multiplication rule, there are $4\binom{13}{4}$ 4-hands with all cards from one suit. \square

Another player looks at his 4 cards and sees that he has one cards from each suit.

- (c) How many 4-hands that contain one card from each suit exist?

Solution. We choose 1 card out of 13 possible in each suit to form the 4-hand, so this is possible in $\binom{13}{1}^4$ ways. \square

The player from (b) peeps at her left neighbour's cards and sees that this neighbour has only cards of one suit, but different from the one she has. Then she looks at her right neighbour's cards and, to her surprise, again sees only cards of one suit, and different from the suit she or the left neighbour has.

- (d) How many ways are there to divide 52 cards into thirteen 4-hands such that player 1 has cards of exactly one suit; player 2 has cards of exactly one suit, but not the same suit as player 1; player 3 has cards of exactly one suit, but not the same suit as player 1 or 2; and there are no restrictions for the hands of the other players?

Solution. For players 1,2,3, we use approach from part (b): we have $4\binom{13}{4}$ ways to assign a 4-hand in one suit to player 1, $3\binom{13}{4}$ ways to assign a 4-hand in one suit to player 2 (one suit is used for player 1), $2\binom{13}{4}$ ways to assign a 4-hand in one suit to player 3 (two suits are used for players 1 and 2).

After this, we have $52 - 12 = 40$ cards to dealt to the remaining 10 players with no restrictions. So, for player 4, we have $\binom{40}{4}$ possible 4-hands; for player 5, we have $\binom{36}{4}$ possible 4-hands; for player 6, we have $\binom{32}{4}$ possible 4-hands, etc.

The total number of ways is (by the Multiplication Rule)

$$4\binom{13}{4} \cdot 3\binom{13}{4} \cdot 2\binom{13}{4} \cdot \binom{40}{4} \binom{36}{4} \binom{32}{4} \cdots \binom{8}{4} \binom{4}{4}.$$

\square