Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 2

(1) The rules for the University of ABC five-a-side soccer competition specify that the members of each team must have birthdays in the same month. How many mathematics students are needed in order to guarantee that they can rise a team?

Solution. We need $4 \cdot 12 + 1 = 49$ students. By Pigeonhole Principle, there is one month with at least $4 + 1 = 5$ students born in it. $\square$

(2) (a) How many solutions are there of the equation $x_1 + x_2 + x_3 = 12$ with x_1, x_2, x_3 non-negative integers?

Solution. This is the same as the number of ways to choose 12 objects from 3 distinct objects with repetition and order does not matter, i.e., $\binom{12+3-1}{3-1} = \binom{14}{2} = 91$. $\square$

(b) How many solutions are there of the equation $x_1 + x_2 + x_3 = 12$ with x_1, x_2, x_3 positive integers?

Solution. We set $x_i = y_i + 1$ for $i = 1, 2, 3$ and obtain $y_1 + y_2 + y_3 = 12 - 3 = 9$. So, there is one-to-one correspondence between the positive integer solutions of $x_1 + x_2 + x_3 = 12$ and the non-negative integer solutions of $y_1 + y_2 + y_3 = 9$. Hence, both equation have the same number of solutions. Since $y_1 + y_2 + y_3 = 9$ has $\binom{9+3-1}{3-1} = \binom{11}{2} = 55$ non-negative integer solutions (we use the same method as in (a)), $x_1 + x_2 + x_3 = 12$ has also 55 solutions in positive integers. $\square$

(3) A baseball team is made up of a pitcher and eight other players. The manager must choose the team from a group of 20 players, six of whom are pitchers. Note that non-pitchers cannot play at the pitcher's position.

(a) In how many ways can manager pick the team if the pitchers do not want to play in any other position than as a pitcher?

Solution. The manager can choose one pitcher in $\binom{6}{1}$ ways and the remaining eight players in $\binom{20-6}{8} = \binom{14}{8}$ ways. By the Multiplication Rule, the team can be picked in $\binom{6}{1}\binom{14}{8}$ ways. $\square$

(b) In how many ways can the team be formed if the pitchers can also play as one of the other players?

Solution. Again, the manager can choose one pitcher in $\binom{6}{1}$ ways. After selecting the pitcher, he has 19 players available to play the field, so he can choose eight of them in $\binom{20-1}{8} = \binom{19}{8}$ ways. By the Multiplication Rule, the team can be picked in $\binom{6}{1}\binom{19}{8}$ ways. $\square$

(4) Prove, by using a counting argument, that for $0 \leq k \leq r \leq n$,

$$\binom{n}{r} \cdot \binom{r}{k} = \binom{n}{k} \cdot \binom{n-k}{r-k}.$$

Solution. We will count in two ways the number N of pairs (A, B) of subsets A, B of $\{1, 2, \ldots, n\}$ satisfying all the following properties:

(i) $|A| = r$,

(ii) $|B| = k$, and

(iii) $B \subset A$.

We can choose A in $\binom{n}{r}$ ways. Once A is selected, we must pick B as a subset of A (to satisfy (iii)), hence we need to choose k elements (to satisfy (ii)) out of r possible. This can be done in $\binom{r}{k}$ ways. By the Multiplication Rule, we have

$$N = \binom{n}{r}\binom{r}{k}.$$

Another way to count is to choose B first; this can be done in $\binom{n}{k}$ ways. To satisfy (i) and (iii), we see that k elements of A are already selected (all elements of B) and so we need to choose only $r - k$ elements from $n - k$ available. Hence

$$N = \binom{n}{k}\binom{n-k}{r-k}.$$

Consequently,

$$\binom{n}{r}\binom{r}{k} = N = \binom{n}{k}\binom{n-k}{r-k}.$$

$\square$

(5) A domino has two numbers from $\{0, 1, 2, 3, 4, 5, 6\}$, one at each side of its (indistinguishable) ends. So, for example, there is a 3-3 domino, with a 3 at each end, and a 4-0 domino, which is the same as 0-4 domino. A set of dominoes contains one copy of each different domino. How many dominoes are there?

Solution. There are 7 dominoes with the same number on both ends $(0-0, \ldots, 6-6)$ and $\binom{7}{2} = 21$ dominoes with different numbers on both ends (the order does not matter because we cannot distinguish between the ends). Thus, there are $7+21 = 28$ dominoes. $\square$

(6) In how many ways can one choose 2 dominoes from the complete set so that at least one domino contains a 1 and at least one contains a 6?

Solution. If one of the dominoes is $1 - 6$, then the other one can be any of the remaining 27 dominoes. Otherwise, one domino must have 1 at one of its ends (but it is not $1 - 6$) and the the other one must have 6 at one of its ends (but it cannot be $1 - 6$). We have 6 options for the first one and 6 for the second one, so, by the Multiplication Rule, we have 36 possibilities. The total number of options is $27 + 36 = 63$. $\square$

(7) What is the coefficient of $x^5y^2z^3$ in $(x+y+z)^{10}$?

Solution. By the Multinomial Theorem, this is $\binom{10}{5,2,3} = \frac{10!}{5!2!3!}$. $\square$

(8) How many different arrangements are there of the letters in the word MICROECONOMICS?

Solution. The total number of ways to order 14 letters is 14! but we cannot distinguish between any two M's or I's or O's or C's so the total number of ways is: $\frac{14!}{2!2!3!3!}$. $\square$

(9) How many integers from 1 to 1000 are divisible by none of 5,7, 11?

Solution. Let A_j will be the set of numbers between 1 and 1000 divisible by j. We want to find the value of $1000 - |A_5 \cup A_7 \cup A_{11}|$. By the Inclusion-Exclusion Principle, we have

$$\begin{aligned} |A_5 \cup A_7 \cup A_{11}| = & |A_5| + |A_7| + |A_{11}| \\ & - |A_5 \cap A_7| - |A_5 \cap A_{11}| - |A_{11} \cap A_7| \\ & + |A_5 \cap A_7 \cap A_{11}| \end{aligned}$$

We have that $|A_5| = 200$, $|A_7| = 142$, $|A_{11}| = 90$. $|A_5 \cap A_7|$ counts the number of numbers divisible by 5 and 7. Since 5 and 7 are primes, we have that $A_5 \cap A_7 = A_{35}$, and $|A_5 \cap A_7| = |A_{35}| = 28$. Similarly, $|A_5 \cap A_{11}| = |A_{55}| = 18$ and $|A_{11} \cap A_7| = |A_{77}| = 12$. Finally, $|A_5 \cap A_7 \cap A_{11}| = |A_{385}| = 2$. Hence,

$$|A_5 \cup A_7 \cup A_{11}| = 200 + 142 + 90 - 28 - 18 - 12 + 2 = 376$$

and there are $1000 - |A_5 \cup A_7 \cup A_{11}| = 1000 - 376 = 624$ numbers between 1 and 1000 not divisible by neither of 5, 7 and 11. $\square$

(10) How many orderings are there of numbers $1, 2 \dots, 8$ in which none of the patterns 12, 34, 56, or 78 appears?

Solution. Let A_{ij} will be the set of all orderings in which the pattern ij appears. The total number of orderings of numbers $1, 2 \dots, 8$ is 8!. What we are looking for is the quantity $8! - |A_{12} \cup A_{34} \cup A_{56} \cup A_{78}|$ (by the Addition Principle). By the Inclusion-Exclusion Principle, we have

$$\begin{aligned} |A_{12} \cup A_{34} \cup A_{56} \cup A_{78}| = & |A_{12}| + |A_{34}| + |A_{56}| + |A_{78}| \\ & - |A_{12} \cap A_{34}| - |A_{12} \cap A_{56}| - |A_{12} \cap A_{78}| \\ & - |A_{34} \cap A_{56}| - |A_{34} \cap A_{78}| - |A_{56} \cap A_{78}| \\ & + |A_{12} \cap A_{34} \cap A_{56}| + |A_{12} \cap A_{34} \cap A_{78}| + |A_{34} \cap A_{56} \cap A_{78}| \\ & - |A_{12} \cap A_{34} \cap A_{56} \cap A_{78}| \end{aligned}$$

How do we find the size of A_{12}? We treat 12 as one number, so we need to find in how many ways we can order 7 numbers $12, 3, 4, 5, 6, 7, 8$. This is $7!$, hence $|A_{12}| = 7!$. In the same way, we obtain that $|A_{34}| = |A_{56}| = |A_{78}| = 7!$.

To find $|A_{12} \cap A_{34}|$, we need to find the number of orderings that fix 12 and 34. Again, we treat 12 as one number, 34 as another one, and so $|A_{12} \cap A_{34}|$ is equal to the number of orderings of 6 numbers, i.e., $|A_{12} \cap A_{34}| = 6!$. In the same way, we get that $|A_{12} \cap A_{56}| = |A_{12} \cap A_{78}| = |A_{34} \cap A_{56}| = |A_{34} \cap A_{78}| = |A_{56} \cap A_{78}| = 6!$.

Analogous reasoning gives $|A_{12} \cap A_{34} \cap A_{56}| = |A_{12} \cap A_{34} \cap A_{78}| = |A_{34} \cap A_{56} \cap A_{78}| = 5!$ and $|A_{12} \cap A_{34} \cap A_{56} \cap A_{78}| = 4!$. Hence,

$$|A_{12} \cup A_{34} \cup A_{56} \cup A_{78}| = 4(7!) - 6(6!) + 4(5!) - 4!$$

and

$$8! - |A_{12} \cup A_{34} \cup A_{56} \cup A_{78}| = 8! - 4(7!) + 6(6!) - 4(5!) + 4!$$

□

(11) A (standard) deck consists of 52 cards: there are four suits (Spades, Diamonds, Hearts, and Clubs) and 13 distinguishable cards (Ace, Two, ..., Ten, Jack, Queen, King) in each of the suits. A k-hand is a set of k different cards from this deck.

(a) How many 13-hands are there?

Solution. $\binom{52}{13}$. □

(b) Use the Inclusion-Exclusion principle to find the number of 13-hands that contain at least one card from each suit.

Solution. Let A_i, $i \in \{\text{Spades, Diamonds, Hearts, Clubs}\}$, be the set of all 13-hands that avoid suit i. Then $A_{\text{Spades}} \cup A_{\text{Diamonds}} \cup A_{\text{Hearts}} \cup A_{\text{Clubs}}$ is the set of all 13-hands that avoid some suit. By (a), the total number of 13-hands is $\binom{52}{13}$. So, by the Addition Principle, we are looking for the value of

$$\binom{52}{13} - |A_{\text{Spades}} \cup A_{\text{Diamonds}} \cup A_{\text{Hearts}} \cup A_{\text{Clubs}}|.$$

The size of A_i is the number of 13-hands drawn from $52 - 13 = 39$ cards avoiding suit i, i.e., $|A_i| = \binom{39}{13}$.

The size of $A_i \cap A_j$ is the number of 13-hands drawn from $52 - 2 \cdot 13 = 26$ cards avoiding suits i, j, i.e., $|A_i \cap A_j| = \binom{26}{13}$.

The size of $A_i \cap A_j \cap A_k$ is the number of 13-hands drawn from $52 - 3 \cdot 13 = 13$ cards avoiding suits i, j, k, i.e., $|A_i \cap A_j \cap A_k| = \binom{13}{13} = 1$.

Since every 14-hand must use some suit, we have $|A_{\text{Spades}} \cap A_{\text{Diamonds}} \cap A_{\text{Hearts}} \cap A_{\text{Clubs}}| = 0$.

By the Inclusion-Exclusion Principle, we have

$$\begin{aligned}
|A_{\text{Spades}} \cup A_{\text{Diamonds}} \cup A_{\text{Hearts}} \cup A_{\text{Clubs}}| =& |A_{\text{Spades}}| + |A_{\text{Diamonds}}| + |A_{\text{Hearts}}| + |A_{\text{Clubs}}| \\
&- |A_{\text{Spades}} \cap A_{\text{Diamonds}}| - |A_{\text{Hearts}} \cap A_{\text{Clubs}}| \\
&- |A_{\text{Spades}} \cap A_{\text{Hearts}}| - |A_{\text{Diamonds}} \cap A_{\text{Clubs}}| \\
&- |A_{\text{Spades}} \cap A_{\text{Clubs}}| - |\cap A_{\text{Diamonds}} \cap A_{\text{Hearts}}| \\
&+ |A_{\text{Spades}} \cap A_{\text{Diamonds}} \cap A_{\text{Hearts}}| \\
&+ |A_{\text{Spades}} \cap A_{\text{Diamonds}} \cap A_{\text{Clubs}}| \\
&+ |A_{\text{Spades}} \cap A_{\text{Hearts}} \cap A_{\text{Clubs}}| \\
&+ |A_{\text{Diamonds}} \cap A_{\text{Hearts}} \cap A_{\text{Clubs}}| \\
&- |A_{\text{Spades}} \cap A_{\text{Diamonds}} \cap A_{\text{Hearts}} \cap A_{\text{Clubs}}| \\
=& 4\binom{39}{13} - 6\binom{26}{13} + 4\binom{13}{13} - 0.
\end{aligned}$$

Consequently,

$$\binom{52}{13} - |A_{\text{Spades}} \cup A_{\text{Diamonds}} \cup A_{\text{Hearts}} \cup A_{\text{Clubs}}| = \binom{52}{13} - 4\binom{39}{13} + 6\binom{26}{13} - 4.$$

□

(c) What is the chance that you obtain a 13-hand in which there are at most three suits?

Solution. This chance is given by

$$\frac{|A_{\text{Spades}} \cup A_{\text{Diamonds}} \cup A_{\text{Hearts}} \cup A_{\text{Clubs}}|}{\text{the total number of 13-hands}} = \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}$$

□

(12) The game of korfball is played with teams of 6 players. Each team contains 3 female and 3 male players. A coach must select a korfball team from 8 male and 7 female candidates.

(a) How many choices are there to choose the team?

Solution. The coach has $\binom{7}{3}$ choices for the female players and $\binom{8}{3}$ choices for the male ones. By the Multiplication Rule, there are $\binom{7}{3}\binom{8}{3}$ ways to choose the team. □

In fact, the rules for forming a korfball team of 6 players are a bit more complicated. A team actually consists of 3 pairs, each containing one female and one male player. One pair will play in the offense, one pair plays mid-field, and one pair plays defense. Our coach still has 8 male and 7 female candidates.

(b) In how many ways can the coach choose 3 pairs (one offensive, one defensive and one for the mid-field), each pair consisting of one female and one male player?

Solution. For the offense, we can chose one woman and one man in $\binom{7}{1}\binom{8}{1} = 56$ ways. (Again, we used the Multiplication Rule.)
Have chosen offense, we can chose one woman and one man for the mid-field in $\binom{6}{1}\binom{7}{1} = 42$ ways. After this, we can chose one woman and one man for defense in $\binom{5}{1}\binom{6}{1} = 30$ ways.
By the Multiplication Rule, the coach can choose the team in

$$56 \cdot 42 \cdot 30 = 70560$$

ways. □

A coach of a different club has 7 female and 7 male players to form a korfball team. These players are actually 7 married couples. The second coach knows from experience that a married couple should not form one of the pairs in a team.

(c) Determine, using the Inclusion-Exclusion principle or otherwise, in how many ways the second coach can choose 3 pairs (one offensive, one defensive and one for the mid-field), each pair consisting of one female and one male player, so that none of the pairs is a married couple.
Solution. Similarly to (b), there are $\binom{7}{1}^2\binom{6}{1}^2\binom{5}{1}^2 = 49 \cdot 36 \cdot 25 = 44100$ ways to form the team.
Let A_i be the set of all teams in which couple i plays as a pair in some position. So, $|A_1 \cup \cdots \cup A_7|$ is the number of teams in which some couple plays as a pair. Hence, we want to find the value of

$$44100 - |A_1 \cup \cdots \cup A_7|.$$

We use the Inclusion-Exclusion Principle. Since there are 3 pairs in the team, the intersection of four or more of A_i's must be empty. We have $|A_i \cap A_j \cap A_k| = 3! = 6$ because for each of the couples we just need to select its position (offense, mid-field, defense).
Similarly, we have $|A_i \cap A_j| = 3 \cdot 2\binom{5}{1}^2 = 150$ because there are 3 possibilities for the position of the couple i, then only 2 possibilities for the position of the couple j, and then we have $\binom{5}{1}$ female candidates and $\binom{5}{1}$ male candidates for the last pair.
Finally, we have $|A_i| = 3 \cdot \binom{6}{1}^2\binom{5}{1}^2 = 2700$ because there are 3 possibilities for the position of the couple i, then we have $\binom{6}{1}$ female candidates and $\binom{6}{1}$ male candidates for the pair in one of the remaining positions and then we have $\binom{5}{1}$ female candidates and $\binom{5}{1}$ male candidates for the last pair.
By the Inclusion-Exclusion Principle,

$$|A_1 \cup \cdots \cup A_7| = \binom{7}{1} \cdot 2700 - \binom{7}{2} \cdot 150 + \binom{7}{3} \cdot 6 = 15960.$$

Consequently, the coach can choose the team in $44100 - 15960 = 28140$ ways. □