Solutions to Exercises 4

(1) Define sequence \((b_n)_{n \geq 1}\) by \(b_n = \binom{n}{0} + \binom{n-1}{1} + \ldots\), where we use \(\binom{n}{k} = 0\) for \(k > n\).

Verify that \(b_1 = 1\), \(b_2 = 2\), and that, for every \(n \geq 3\), we have \(b_n = b_{n-1} + b_{n-2}\).

Solution. Using \(\binom{n}{k} = 0\) for \(k > n\), we have
\[
b_1 = \binom{1}{0} + \binom{0}{1} + \cdots = \binom{1}{0} = 1
\]
and
\[
b_2 = \binom{2}{0} + \binom{1}{1} + \binom{0}{2} + \cdots = \binom{2}{0} + \binom{1}{1} = 1 + 1 = 2.
\]
Every \(b_n\) has only finitely many non-zero summands and we can write it as
\[
b_n = \sum_{k=0}^{n} \binom{n-k}{k}.
\]

Then,
\[
b_{n-1} + b_{n-2} = \sum_{k=0}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k}
\]
\[
= \binom{n-1}{0} + \sum_{k=1}^{n-1} \binom{n-1-k}{k} + \sum_{k=0}^{n-2} \binom{n-2-k}{k}
\]
\[
= 1 + \sum_{k=1}^{n-1} \binom{n-1-k}{k} + \sum_{k=1}^{n-1} \binom{n-2-(k-1)}{k-1}
\]
\[
= 1 + \sum_{k=1}^{n-1} \left( \binom{n-1-k}{k} + \binom{n-1-k}{k-1} \right).
\]

We will use the identity \(\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}\) with \(m = n - k\) and, also, that \(\binom{n-0}{0} = \binom{n}{0} = 1\) and \(\binom{n-n}{n} = \binom{0}{n} = 0\):
\[
b_{n-1} + b_{n-2} = 1 + \sum_{k=1}^{n-1} \left( \binom{n-1-k}{k} + \binom{n-1-k}{k-1} \right) + 0
\]
\[
= \binom{n-0}{0} + \sum_{k=1}^{n-1} \binom{n-k}{k} + \binom{n-n}{n}
\]
\[ = \sum_{k=0}^{n} \binom{n-k}{k} = b_n. \]

(2) Let \( a_n \) denote the number of \( n \)-digit sequences in which each digit is either 0 or 1, and no two consecutive 0’s are allowed.

(a) Show that \( a_1 = 2 \) and \( a_2 = 3 \). What would you say \( a_0 \) is?

**Solution.** Both 0 and 1 are valid 1-digit sequences, hence \( a_1 = 2 \). Similarly, 11, 10, 01 are all the valid 2-digit sequences (00 is prohibited), hence \( a_2 = 3 \). Empty sequence (the equivalent of \( \emptyset \) for sets) contains no consecutive 0’s, so \( a_0 = 1 \).

(b) Show that for \( n \geq 3 \) we have \( a_n = a_{n-1} + a_{n-2} \).

**Solution.** Consider any valid \( n \)-digit sequence \( x_1x_2\ldots x_n \).

- If \( x_1 = 1 \), then \( x_2x_3\ldots x_n \) is a valid \( n-1 \)-digit sequence, and we have \( a_{n-1} \) of these.
- If \( x_1 = 0 \), then \( x_2 \) must be 1 to avoid consecutive zeros. All the remaining \( n-2 \) digits do not contain consecutive zeros, so they form a valid \( (n-2) \)-digit sequence, and we have \( a_{n-2} \) of them.

Thus, the total number \( a_n \) of valid \( n \)-digit sequences satisfies \( a_n = a_{n-1} + a_{n-2} \) for \( n \geq 2 \).

(c) Give a closed form expression for \( a_n \).

**Solution.** Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be the generating function of \( (a_n)_{n \geq 1} \). We shall use that \( a_n = a_{n-1} + a_{n-2} \) for \( n \geq 2 \). Indeed, we have

\[
 f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n \\
 = a_0 + a_1 x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\
 = a_0 + a_1 x + \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
 = a_0 + a_1 x + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
 = a_0 + a_1 x + x \sum_{n=1}^{\infty} a_n x^n + x^2 \sum_{n=0}^{\infty} a_n x^n \\
 = a_0 + a_1 x + x(f(x) - a_0) + x^2 f(x) \\
 = 1 + 2x - x + (x + x^2) f(x). 
\]
Hence,
\[ f(x) = \frac{1 + x}{1 - x - x^2}. \]

Equation \( 1 - x - x^2 \) has two solutions \( \alpha = (-1 + \sqrt{5})/2 \) and \( \beta = (-1 - \sqrt{5})/2 \).

We rewrite \( f(x) \) as
\[ f(x) = \frac{-1 - x}{(\alpha - x)(\beta - x)} = \frac{A}{\alpha - x} + \frac{B}{\beta - x} = \frac{A\beta + B\alpha - (A + B)x}{(\alpha - x)(\beta - x)}. \]

Consequently, we have \( A + B = 1 \) and \( A\beta + B\alpha = -1 \). These two equations have solution
\[ A = \frac{-\beta}{\alpha - \beta} = \frac{-\beta}{\sqrt{5}} \quad \text{and} \quad B = \frac{\alpha}{\alpha - \beta} = \frac{\alpha}{\sqrt{5}}, \]

therefore,
\[
\begin{align*}
f(x) &= \frac{A}{\alpha - x} + \frac{B}{\beta - x} = \frac{A}{\alpha} \cdot \frac{1}{1 - \frac{x}{\alpha}} + \frac{B}{\beta} \cdot \frac{1}{1 - \frac{x}{\beta}} \\
&= \frac{A}{\alpha} \sum_{n=0}^{\infty} \left( \frac{x}{\alpha} \right)^n + \frac{B}{\beta} \sum_{n=0}^{\infty} \left( \frac{x}{\beta} \right)^n \\
&= \sum_{n=0}^{\infty} \left( \frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}} \right) x^n.
\end{align*}
\]

From this we deduce that
\[ a_n = \frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}. \]

Since \( \alpha\beta = -1 \), this can be rewritten as
\[ a_n = (-1)^{n+1}(B\alpha^{n+1} + A\beta^{n+1}) = \frac{(-1)^{n+1}}{\sqrt{5}}(\alpha^{n+2} - \beta^{n+2}). \]

(3) Let \( f_n \) denote the \( n \)-th Fibonacci number, i.e., \( f_0 = f_1 = 1 \) and \( f_n = f_{n-1} + f_{n-2} \) for \( n \geq 2 \). Prove that for every \( n \geq 2 \) we have \( f_n^2 - f_{n-1} \cdot f_{n+1} = (-1)^n \).

**Solution.** We proceed by induction on \( n \). For \( n = 2 \), we look at \( f_2^2 - f_1 f_3 \). Using the recurrence \( f_n = f_{n-1} + f_{n-2} \), we have \( f_2 = f_1 + f_0 = 2 \) and \( f_3 = f_2 + f_1 = 3 \). So,
\[ f_2^2 - f_1 f_3 = 2^2 - 1 \cdot 3 = 1 = (-1)^2. \]

Suppose that \( f_n^2 - f_{n-1} \cdot f_{n+1} = (-1)^n \). Then
\[
\begin{align*}
f_{n+1}^2 - f_{n+1} \cdot f_{n+2} &= f_{n+1}^2 - f_n \cdot (f_{n+1} + f_n) \text{ using } f_{n+1} + f_n = f_{n+2} \\
&= f_{n+1}(f_{n+1} - f_n) - f_n^2
\end{align*}
\]
\[ f_{n+1} = f_n + f_{n-1} \]
\[ f_{n+1}^2 = (f_n + f_{n-1})^2 = f_n^2 + 2f_n f_{n-1} + f_{n-1}^2 \]
\[ f_{n+1}^2 - f_n^2 = 2f_n f_{n-1} \]
\[ = (-1)^n \]
\[ = (-1)^{n+1}. \]

Thus our assertion holds by induction. \( \square \)

(4) Use generating functions to solve the following recurrence relation:

\[ a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2, \]
\[ a_0 = 0; \]
\[ a_1 = 3. \]

**Solution.** Using Theorem 3.5 (see the notes for Lectures 5 and 6), we need to solve

\[ x^2 = 5x - 6, \quad \text{or } (x - 2)(x - 3) = 0. \]

Then, the general solution is given by

\[ a_n = A \cdot 2^n + B \cdot 3^n. \]

Using the initial conditions \( a_0 = 0 \) and \( a_1 = 3 \), we obtain

\[ 0 = A \cdot 2^0 + B \cdot 3^0 = A + B \quad \text{and } 3 = A \cdot 2^1 + B \cdot 3^1 = 2A + 3B. \]

Thus, \( A = -3 \) and \( B = 3 \). Consequently,

\[ a_n = 3(3^n - 2^n). \]

(5) Suppose that \( f(x) \) generates the sequence \( a_0, a_1, a_2, \ldots \). Give the expressions, in terms of \( f \), for the generating functions of the following sequences:

(a) \( 0, a_0, 0, a_1, 0, a_2, 0, \ldots \);

**Solution.** For \( f(x) = \sum_{n=0}^{\infty} a_n x^n \), we have \( f(x^2) = \sum_{n=0}^{\infty} a_n x^{2n} \) and \( xf(x^2) = \sum_{n=0}^{\infty} a_n x^{2n+1} = 0 + a_0 x + 0 x^2 + a_1 x^3 + 0 x^4 + a_2 x^5 + 0 x^6 + \ldots \). Hence, \( xf(x^2) \) is the generating function of \( 0, a_0, 0, a_1, 0, a_2, 0, \ldots \). \( \square \)
(a) Show that the generating function of the sequence \(a_n = 1\), \(a_n = b_n\).

**Solution.** For \(f(x) = \sum_{n=0}^{\infty} a_n x^n\), we have \(1 + xf(x) = 1 + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} (-1)^n a_n x^n = 1 + a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \ldots\). Hence, \(1 + xf(x)\) is the generating function of \(1, a_0, a_1, a_2, a_3, a_4, \ldots\).

(c) \(a_0, -a_1, a_2, -a_3, a_4, \ldots\)

**Solution.** For \(f(x) = \sum_{n=0}^{\infty} a_n x^n\), we have \(f(-x) = \sum_{n=0}^{\infty} a_n (-x)^n = \sum_{n=0}^{\infty} (-1)^n a_n x^n = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - \ldots\). Hence, \(f(-x)\) is the generating function of \(a_0, -a_1, a_2, -a_3, a_4, \ldots\).

(b) Find generating functions for the sequences \(b_n = n^2, n \geq 0\), and \(c_n = n^3, n \geq 0\).

**Solution.** Let \(f(x)\) is the generating function of a sequence \((a_n)_{n \geq 1}\), i.e., \(f(x) = \sum_{n=0}^{\infty} a_n x^n\). Then \(f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}\). So,

\[
g(x) = xf'(x) = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n
\]

is the generating function of the sequence \((na_n)_{n \geq 1}\).

For \(a_n = n, n \geq 1\), we proved in part (a) that its generating function is \(f(x) = \frac{x}{(1-x)^2}\). We also notice that \(b_n = n^2 = na_n\) for every \(n \geq 1\). Therefore, by our earlier observation, \(g(x) = xf'(x) = x(\frac{x}{(1-x)^2})' = \frac{x(1+x)}{(1-x)^3}\) is the generating function of \((b_n)_{n \geq 1}\).

For the last part, we have that \(c_n = n^3 = nb_n\) for every \(n \geq 1\). Therefore, by our earlier observation, \(h(x) = xg'(x) = x(\frac{x(1+x)}{(1-x)^3})' = \frac{x(x^2+4x+1)}{(1-x)^4}\) is the generating function of \((c_n)_{n \geq 1}\).
(7) Find the sequences generated by the following functions:

(a) \( f(x) = \frac{x^3}{1+x}; \)

**Solution.** We use that \( \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \) with \( y = -x \), hence \( \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n \). Consequently,

\[
f(x) = \frac{x^3}{1+x} = x^3 \sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^n x^{n+3} = \sum_{n=3}^{\infty} (-1)^n x^n .
\]

Thus, \( f(x) \) is the generating function of \( 0, 0, 0, 1, -1, 1, -1, \ldots \). \( \square \)

(b) \( g(x) = \frac{x}{1-7x+12x^2}; \)

**Solution.** We write

\[
g(x) = \frac{x}{(1-3x)(1-4x)} = \frac{A}{1-3x} + \frac{B}{1-4x} = \frac{(A+B) + (-4A-3B)x}{(1-3x)(1-4x)}.
\]

Hence, \( A + B = 0 \) and \( -4A - 3B = 1 \), from which we obtain \( A = -1 \) and \( B = 1 \).

Using \( \frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \) with \( y = 3x \) and \( y = 4x \), we have

\[
g(x) = \frac{1}{1-3x} - \frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n - \sum_{n=0}^{\infty} (3x)^n = \sum_{n=0}^{\infty} (4^n - 3^n)x^n.
\]

Hence, \( g(x) \) is the generating function of \( (4^n - 3^n)_{n \geq 0} \). \( \square \)

(c) \( h(x) = \frac{x^7}{2-x^7}; \)

**Solution.** We see that

\[
h(x) = \frac{x^7}{2-x^7} = -1 + \frac{2}{2-x^7} = -1 + \frac{1}{1-\left(\frac{x}{2}\right)^7} = -1 + \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = -1 + \sum_{n=0}^{\infty} \frac{2^{-n} x^n}{n!} .
\]

From this we deduce that \( h(x) \) generates \( (x_n)_{n \geq 1} \), where

\[
x_n = \begin{cases} 2^{-n/7} & \text{if } n \geq 1 \text{ and } 7 \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}
\]

\( \square \)

(d) \( k(x) = e^{2x}. \)

**Solution.** We know that \( e^y = \sum_{n=0}^{\infty} \frac{1}{n!} y^n. \) By taking \( y = 2x \), we have \( k(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!} (2x)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n \), so \( k(x) \) is the generating function of \( \left(\frac{2^n}{n!}\right)_{n \geq 1} \). \( \square \)