Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 6

(1) Let $V = \{1, 2, \ldots, n\}$. How many different graphs with vertex set V are there?

Solution. Each graph G with vertex set V is uniquely determined by its edge set E. E must be a subset of $\binom{V}{2}$, the set of all pairs in V. We have seen already that every set with m elements has 2^m different subsets. In our case, $m = \left|\binom{V}{2}\right| = \binom{n}{2}$, hence there are $2^{\binom{n}{2}}$ different graphs with vertex set V. $\square$

(2) How many non-isomorphic graphs with four vertices are there? (Hint: the answer is not the same as the answer in Question 1 for $n = 4$.)

Solution. By examining the possibilities, we find 1 graph with 0 edges, 1 graph with 1 edge, 2 non-isomorphic graphs with 2 edges, 3 non-isomorphic graphs with 3 edges, 2 non-isomorphic graphs with 4 edges, 1 graph with 5 edges and 1 graph with 6 edges. Altogether, we have 11 non-isomorphic graphs on 4 vertices

(3) Recall that the degree sequence of a graph is the list of all degrees of its vertices, written in non-increasing order. Prove that two isomorphic graphs must have the same degree sequence. Is it true that every two graphs with the same degree sequence are isomorphic? Justify your answer!

Solution. Let $G = (V, E)$ be a graph isomorphic to a graph $H = \{V', E'\}$. We know that G and H must have the same number n of vertices. Suppose that $V = \{v_1, v_2, \ldots, v_n\}$ and $V' = \{w_1, w_2, \ldots, w_n\}$. We obtain the degree sequence of G by ordering the list of numbers $\deg(v_1), \deg(v_2), \ldots, \deg(v_n)$ and the degree sequence of H by ordering the list of numbers $\deg(w_1), \deg(w_2), \ldots, \deg(w_n)$. Thus, we must show that both lists contain the same numbers (in different orders).

We accomplish this as follows: Let $f : V \to V'$ be the isomorphism of G and H, i.e.,

$$\text{for every } x, y \in V,\ xy \in E \text{ if and only if } f(x)f(y) \in E'.$$

Now, for every v_i, $y \in N_G(v_i)$ if and only if $v_i y \in E$, but $v_i y \in E$ if and only if $f(v_i)f(y) \in E'$, and $f(v_i)f(y) \in E'$ if and only if $f(y) \in N_H(f(v_i))$.

Hence, we showed that for every i, $y \in N_G(v_i)$ if and only if $f(y) \in N_H(f(v_i))$. Therefore, for every i

$$\deg_G(v_i) = |N_G(v_i)| = |N_H(f(v_i))| = \deg_H(f(v_i))$$

and both lists contain the same numbers.

The following two graphs have both degree sequence $(2, 2, 2, 2, 2, 2)$ and they are not isomorphic because one is connected and the other one is not.

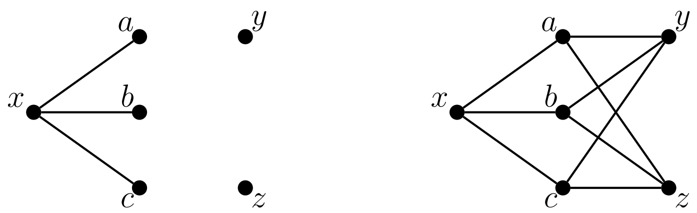
(4) A graph is 3-regular if all its vertices have degree 3. How many non-isomorphic 3-regular graphs with 6 vertices are there? And how many with 7 vertices?

Solution. We know that the sum of the degrees in a graph must be even (because it equals to twice the number of its edges). Hence, there is no 3-regular graph on 7 vertices because its degree sum would be $7 \cdot 3 = 21$, which is not even.

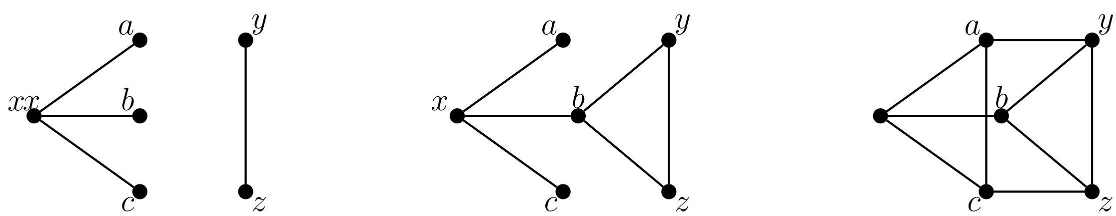
Now we deal with 3-regular graphs on 6 vertices. Let x be any vertex of such 3-regular graph and a, b, c be its three neighbors.

Denote by y and z the remaining two vertices. Notice that both y and z are not adjacent to x. (Why?) We distinguish two possibilities.

If y and z are not adjacent, then both of them must have a, b, c as neighbors to have degree equal to 3.



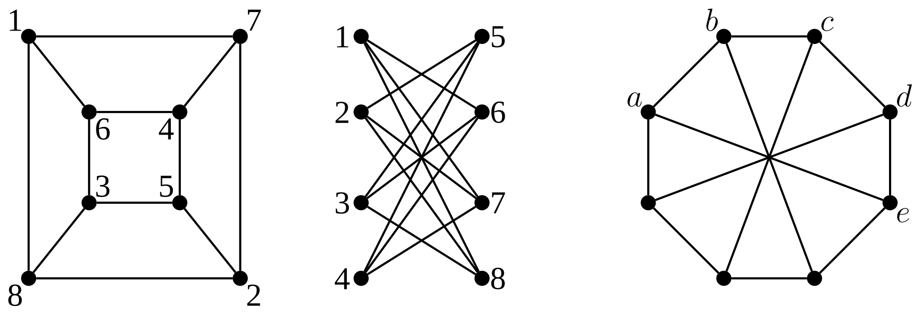
If y and z are adjacent, then both of them must have two neighbors among a, b, c to have degree equal to 3. By Pigeonhole principle, y and z must have a common neighbour. By symmetry, we may assume that b is their common neighbour. At this point, x and b have degree 3, vertices y, z have degree 2 and a and c have degree. Now, between $\{a, c\}$ and $\{y, z\}$ there can be only two edges because each y and z need only one edge to reach the degree 3. By symmetry again, we may assume that ay and cz are the edges. But then there must also be an edge between a and c (so that these two vertices reach degree 3).



It is easy to see that this graph is not isomorphic to the previous one: this graph contains a cycle with 3 vertices and the previous one was bipartite, thus containing no odd vertices. □

(5) Determine which pairs of graphs below are isomorphic. Justify your answer!

Solution. The first two graphs are isomorphic: we assign labels $1, \dots, 8$ to the vertices of both graphs as suggested below and see that both of them have the same vertex set $\{16, 17, 18, 25, 27, 28, 35, 36, 38, 45, 46, 47\}$. Hence $f(i) = i$ is a bijection preserving the edges and non-edges.



The first two graph are also bipartite (with partite sets $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4\}$) and, hence, they cannot contain and cycle of odd length (we have seen this in lectures). On the other hand, the third graph contains an odd cycle on 5 vertices a, b, c, d, e, thus, this graph is not isomorphic to the first two. $\square$

(6) Suppose that we have a graph with at least two vertices. Show that it is not possible that all vertices have different degrees.

Solution. Every vertex of a graph on n vertices has degree between 0 and $n - 1$. If all degrees were different in some graph G, then G would have to contain a vertex of degree i for every $i = 0, 1, \dots, n-1$. However, the vertex of degree 0 (let's call it x) is not adjacent to any other vertex of G and the vertex of degree $n - 1$ is adjacent to every other vertex of G (x amongst them), which is clearly impossible. Hence, there are only $n - 1$ possible values for all the degrees: either $0, 1, \dots, n-2$ or $1, 2, \dots, n-1$. In either case, we have n vertices and $n - 1$ possible values for their degrees. By Pigeonhole principle, two vertices must have the same degree. $\square$

(7) There are four married couples at a party. Various people shake hands, but of course no one shakes hands with his/her own wife or husband. At the end of the party, the host asks everybody else how many hands they shook and he receives seven different answers.

How many hands did the wife of the host shake?

Solution. In a group of 8 people, everybody can shake hand with anywhere between 0 and 7 people. However, since nobody shakes hand with his/her spouse, every person shook hand with at most 6 other people. If we label the host by h, we can attached labels $0, 1, 2, \dots, 6$ to the remaining seven people in such a way that person i shook hands with i other people. We know this because h got a different response from every other person.

Now, person 6 couldn't shake hands with person 0 (because 0 shook hands with nobody) so 6 shook hands with with remaining six people h, 1, 2, 3, 4, and 5. So, 0 is the only one not shaking hands with 6, hence, 0 and 6 must be one married couple.

Person 1 shook hands with 6 so he/she can't shake hands with anybody else. Hence, 5 can shake hands only with the following five people: h, 2, 3, 4, and 6. Also, 5 can be married only to 0 or 1, but 0 is married to 6, thus 1 and 5 is a couple.

A similar reasoning (try to do it yourself!) yields that 2 and 4 is a couple, hence h and 3 must be a couple and so the wife of the host shook hands with three people.

$\square$

(8) Prove the following statements:

(a) If there is a walk between two vertices x and y in some graph G, then there is also a path between x and y in G.

Solution. We proceed by induction on the length of the walk. (Recall that the length of a walk $x = v_1, v_2, \ldots, v_k = y$ from x to y is the number of its edges $k - 1$.)

The base case is $k = 2$, i.e., when $x = v_1, v_2 = y$. This walk does not repeat any vertices, hence it is also a path.

Assume now that every walk from x to y of length at most $k - 1$ contains a path from x to y, and let $x = v_1, v_2, \ldots, v_k, v_{k+1} = y$ be a walk of length k from x to y.

If no vertex is repeated in this walk, then it must be a path from x to y. So, assume that $v_i = v_j$ for some $i < j$. But then, $x = v_1, v_2, \ldots, v_i = v_j, v_{j+1}, \ldots, v_{k+1} = y$ is a walk from x to y of length smaller than k (because we removed at least one edge, namely $v_i v_{i+1}$ from the original walk) and, by the induction assumption, it contains a path from x to y. $\square$

(b) If G has a walk between vertices x and y and a walk between vertices y and z, then G also has a walk between x and z.

Solution. Let $x = v_1, v_2, \ldots, v_k = y$ be a walk from x to y in a graph $G = (V, E)$ (i.e., for all $i = 1, \ldots, k - 1$, $v_i v_{i+1} \in E$) and let $y = w_1, w_2, \ldots, w_\ell = z$ be a walk from y to z, i.e., for all $i = 1, \ldots, \ell - 1$, $w_i w_{i+1} \in E$. We define a sequnce $z_1, \ldots, z_{k+\ell-1}$ by

$$z_i = \begin{cases} v_i & \text{if } 1 \leq i \leq k, \\ w_{i-k+1} & \text{if } k + 1 \leq i \leq k + \ell - 1. \end{cases}$$

Clearly, $z_1 = v_1 = x$ and $z_{k+\ell+1} = w_{(k+\ell-1)-k+1} = w_\ell = z$. Furthermore,

$$z_i z_{i+1} = \begin{cases} v_i v_{i+1} \in E & \text{if } 1 \leq i \leq k - 1, \\ v_k w_{k+1-k+1} = y w_2 = w_1 w_2 \in E & \text{if } i = k \text{ because } v_k = y = w_1, \\ w_{i-k+1} w_{i+1-k+1} = w_{i-k+1} w_{i-k+2} \in E & \text{if } k + 1 \leq i \leq k + \ell - 2. \end{cases}$$

Hence, we found a walk from x to z. $\square$

(c) If G has a path between vertices x and y and a path between vertices y and z, then G also has a path between x and z.

Solution. A path from x to y is also a walk from x to y. A path from y to z is also a walk from y to z. Hence, by part (b), there exists a walk from x to z. Consequently, by part (a), there exists also a path from x to z. $\square$