Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 7

(1) The *complete bipartite graph* $K_{m,n}$ is defined by taking two disjoint sets, V_1 of size m and V_2 of size n, and putting an edge between u and v whenever $u \in V_1$ and $v \in V_2$.

(a) How many edges does $K_{m,n}$ have?

Solution. Every vertex of V_1 is adjacent to every vertex of V_2, hence the number of edges is mn. □

(b) What is the degree sequence of $K_{m,n}$?

Solution. Every vertex of V_1 has degree n because it is adjacent to every vertex of V_2. Similarly, every vertex of V_2 has degree m because it is adjacent to every vertex of V_2. So the degree sequence of $K_{m,n}$ consists of m n's and n m's listed in non-increasing order.

If $m \geq n$, then the degree sequence is

$$(\underbrace{m, \ldots, m}_{n}, \underbrace{n, \ldots, n}_{m}).$$

If $m < n$, then the degree sequence is

$$(\underbrace{n, \ldots, n}_{m}, \underbrace{m, \ldots, m}_{n}).$$

□

(c) Which complete bipartite graphs $K_{m,n}$ are connected?

Solution. Take any $m, n \geq 1$. For any vertex $x \in V_1$, $y \in V_2$, the pair xy is an edge, so x, y is a walk from x to y.

For vertices $x, y \in V_1$, $x \neq y$, take any $w \in V_2$. The pairs xw, wy are edges, so x, w, y is a walk from x to y.

For vertices $x, y \in V_2$, $x \neq y$, take any $w \in V_1$. The pairs xw, wy are edges, so x, w, y is a walk from x to y.

Hence, all complete bipartite graphs $K_{m,n}$ are connected. □

(d) Which complete bipartite graphs $K_{m,n}$ have an Euler circuit?

Solution. We know that a graph has an Euler circuit if and only if all its degrees are even. As noted above, $K_{m,n}$ has vertices of degree m and n, so it has an Euler circuit if and only if both m and n are even. □

(e) Which complete bipartite graphs $K_{m,n}$ have a Hamilton cycle?

Solution. Every cycle in a bipartite graph is even and alternates between vertices from V_1 and V_2. Since a Hamilton cycle uses all the vertices in V_1 and V_2, we must have $m = |V_1| = |V_2| = n$.

Suppose that $K_{n,n}$ has partite sets $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{w_1, \ldots, w_n\}$. Since $v_i w_j$ is an edge of $K_{n,n}$ for every $1 \leq i, j \leq n$, we see that $v_1, w_1, v_2, w_2, \ldots, v_n, w_n$ is a Hamiltonian cycle (note that $w_n v_1$ is an edge). $\square$

(2) The *cube graph* Q_n was defined in lectures: the vertices of Q_n are all sequences of length n with entries from $\{0, 1\}$ and two sequences are joined by an edge if they differ in exactly one position.

(a) How many edges does Q_n have?

Solution. Fix any vertex v of Q_n. All its neighbors differ from v in exactly one position. There are n positions possible to differ at. Hence, every vertex has degree n. Since, the number of edges $e(Q_n)$ satisfies $2e(Q_n) = \sum_{v \in V(Q_n)} \deg(v) = n2^n$, we must have $e(Q_n) = n2^{n-1}$. $\square$

(b) What is the degree sequence of Q_n?

Solution. Based on part (a), the degree sequence of Q_n is $\underbrace{n, \ldots, n}_{2^n \text{ times}}$.

(c) Which cube graphs Q_n are connected?

Solution. $Q_1 = K_2$ is certainly connected. Suppose that Q_{n-1} is connected for some $n > 1$, and let's look at Q_n. we split its vertices to two sets: V_1 contains all the vertices of Q_1 ending with 0 and V_2 contains all the vertices of Q_1 ending with 1. Clearly, V_1 and V_2 are disjoint and every vertex of Q_n must be in one of them.

The crucial observation is that the vertices of V_1 form the cube Q_{n-1}. Why?

Firstly, every vertex in V_1 can be written as $v0$, where v is a $0-1$ sequence of length $n-1$. Hence, there is 1-to-1 correspondence between vertices of V_1 and the vertices of Q_{n-1}: for every $v0 \in V_1$ we have $v \in V(Q_{n-1})$.

Secondly, a pair of vertices $v0, w0 \in V_1$ form an edge if and only if $v0$ and $w0$ differ in exactly one position. But they both have 0 at the end, so $v0$ and $w0$ differ in exactly one position if and only if v and w differ in exactly one position. Hence, $v0, w0 \in V_1$ form an edge in Q_n if and only if v, w form an edge in Q_{n-1}.

Similarly, the vertices of V_2 form the cube Q_{n-1}. In the same way as above, we have that $v1, w1 \in V_2$ form an edge in Q_n if and only if v, w form an edge in Q_{n-1}.

So, by induction assumption, we know that there is a walk between any two vertices in V_1 and between any two vertices in V_2.

Take a vertex $v0 \in V_1$ and $w1 \in V_2$. We know there is a walk between $v0$ and $w0$ (using only the vertices of V_1), which together with edge $w0w1$ ($w0w1$ differ in the last coordinate) form a walk from $v0$ to $w1$.

Hence, we showed that Q_n is connected. $\square$

(d) Which cube graphs Q_n have an Euler tour?

Solution. Q_n has an Euler tour if and only if all its degrees are even. Since Q_n is n-regular, we obtain that Q_n has an Euler tour if and only if n is even. $\square$

(e) Which cube graphs Q_n have a Hamilton cycle?

Solution. For $n = 2$, Q_2 is the cycle C_4, so it is Hamiltonian.

Assume that Q_{n-1} is Hamiltonian and consider the cube graph Q_n. Let V_1 and V_2 be as defined in part (c).

The vertices of V_1 form the cube graph Q_{n-1} and so there is a cycle C covering all the vertices of V_1.

Moreover, there is a 1-to-1 correspondence between the vertices of V_1 and the vertices of V_2: $v0 \in V_1$ if and only if $v1 \in V_2$. This mean that we can construct a cycle C' covering all the vertices of V_2 as follows: if $v0\,w0$ is an edge in C, then we put the edge $v1\,w1$ to C'.

Now we link C and C' to a Hamiltonian cycle in Q_n: take and edge $v0\,w0$ in C and $v1\,w1$ in C' and replace edges $v0\,w0$ and $v1\,w1$ with edges $v0\,v1$ and $w0\,w1$.

So, Q_n is Hamiltonian as well. $\square$

(3) Suppose that G is a graph in which every vertex has degree at least k, where $k \geq 1$, and in which every cycle contains at least 4 vertices.

(a) Show that G contains a path of length at least $2k - 1$.

(b) For each $k \geq 1$, give an example of a graph in which every vertex has degree at least k, every cycle contains at least 4 vertices, but which does not contain a path of length $2k$.

Solution. See Exercises 8.

(4) Show that the cube graph Q_n is bipartite.

Solution. Let V_1 be the set of those vertices of Q_n (i.e., sequences of 0's and 1's of length n) with an even number of 0's. Similarly, let V_1 be the set of those vertices of Q_n with an odd number of 0's. Clearly, every vertex must have either an odd or an odd number of 0's and, hence V_1, V_2 partition $V(Q_n)$ into two disjoint parts.

Is it possible to have an edge xy with $x, y \in V_1$? This would mean that x and y differ in exactly one position. But this would imply that if one of them has an even number of 0's then the other one has an odd number of 0's (one 0 is changed to 1 or one 1 is changed to 0), so these two vertices cannot be both from V_1. This is a contradiction. In the same way one proves that it is not possible to have an edge with both vertices from V_2.

(5) We call a graph *tree* if it is connected and contains no cycles. Prove that if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Solution. See Exercises 8.

(6) Recall that the complement of a graph $G = (V, E)$ is the graph $\bar{G}$ with the same vertex V and for every two vertices $u, v \in V$, uv is an edge in $\bar{G}$ if and only if uv is <u>not</u> and edge of G.

Suppose that G is a graph on n vertices such that G is isomorphic to its own complement $\bar{G}$. Prove that $n \equiv 0 (\mod 4)$ or $n \equiv 1 (\mod 4)$.

Solution. Every pair of vertices in V is an edge in exactly one of the graphs G, $\bar{G}$. Hence the number of edges $e(G)$ of G and the number of edges $e(\bar{G})$ satisfy:

$$e(G) + e(\bar{G}) = \binom{n}{2}.$$

Since we assume that G and $\bar{G}$ are isomorphic, they must have the same number of edges,i.e., $e(G) = e(\bar{G})$. Consequently, we have that

$$2e(G) = e(G) + e(\bar{G}) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Thus, $\frac{n(n-1)}{4} = e(G)$ must be an integer. Exactly one of the numbers n and $n - 1$ is even, so either 4 divides n or 4 divides $n - 1$. In the first case, we have $n \equiv 0 (\mod 4)$ and, in the second case, we have $n \equiv 1 (\mod 4)$. □

(7) A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube, before going on to an adjacent one.

Can the mouse end in the center ?

Solution. Imagine each $1 \times 1 \times 1$ cube as a vertex. We construct a graph G by joining two vertices x, y by an edge if the mouse can move from x to y (i.e., when x and y have a common side (not corner, not edge!)).

We claim that G is bipartite. Indeed, we define its bipartition $X \cup Y$ as follows: we put the 8 corner cubes and centers of each side (6 of them) to X, all the other $1 \times 1 \times 1$ cubes to Y (i.e., the center of the $3 \times 3 \times 3$ cube and one central cube from each of 12 edges of the $3 \times 3 \times 3$ cube). Is this really a bipartition? In other words, are there no edges in X or in Y ? Clearly, no two corners cubes have a common side, no two center cubes are adjacent as well, and a corner and the center of a side are not adjacent either. Similarly, the center of the $3 \times 3 \times 3$ cube is not adjacent to any of the central cubes from each of 12 edges of the $3 \times 3 \times 3$ cube. These central cubes are non-adjacent as well.

So, the plan of our mouse is to "eat" a path containing all the vertices of G, starting in X (all corners are there) and ending in Y (the center of the $3 \times 3 \times 3$ cube is there). Such a path must alternate among vertices in X and Y because G is bipartite. However, $|X| = 14 > 13 = |Y|$ and we start in X, so this is impossible. □