Solutions to Exercises 7

(1) The complete bipartite graph $K_{m,n}$ is defined by taking two disjoint sets, $V_1$ of size $m$ and $V_2$ of size $n$, and putting an edge between $u$ and $v$ whenever $u \in V_1$ and $v \in V_2$.

(a) How many edges does $K_{m,n}$ have?

**Solution.** Every vertex of $V_1$ is adjacent to every vertex of $V_2$, hence the number of edges is $mn$. □

(b) What is the degree sequence of $K_{m,n}$?

**Solution.** Every vertex of $V_1$ has degree $n$ because it is adjacent to every vertex of $V_2$. Similarly, every vertex of $V_2$ has degree $m$ because it is adjacent to every vertex of $V_2$. So the degree sequence of $K_{m,n}$ consists of $m$’s and $n$’s listed in non-increasing order.

If $m \geq n$, then the degree sequence is

$$\left(\underbrace{m, \ldots, m}_{n}, \underbrace{n, \ldots, n}_{m}\right).$$

If $m < n$, then the degree sequence is

$$\left(\underbrace{n, \ldots, n}_{m}, \underbrace{m, \ldots, m}_{n}\right).$$

□

(c) Which complete bipartite graphs $K_{m,n}$ are connected?

**Solution.** Take any $m, n \geq 1$. For any vertex $x \in V_1, y \in V_2$, the pair $xy$ is an edge, so $x, y$ is a walk from $x$ to $y$.

For vertices $x, y \in V_1, x \neq y$, take any $w \in V_2$. The pairs $xw, wy$ are edges, so $x, w, y$ is a walk from $x$ to $y$.

For vertices $x, y \in V_2, x \neq y$, take any $w \in V_1$. The pairs $xw, wy$ are edges, so $x, w, y$ is a walk from $x$ to $y$.

Hence, all complete bipartite graphs $K_{m,n}$ are connected. □

(d) Which complete bipartite graphs $K_{m,n}$ have an Euler circuit?

**Solution.** We know that a graph has an Euler circuit if and only if all its degrees are even. As noted above, $K_{m,n}$ has vertices of degree $m$ and $n$, so it has an Euler circuit if and only if both $m$ and $n$ are even. □

(e) Which complete bipartite graphs $K_{m,n}$ have a Hamilton cycle?
Solution. Every cycle in a bipartite graph is even and alternates between vertices from $V_1$ and $V_2$. Since a Hamilton cycle uses all the vertices in $V_1$ and $V_2$, we must have $m = |V_1| = |V_2| = n$.

Suppose that $K_{n,n}$ has partite sets $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{w_1, \ldots, w_n\}$. Since $v_iw_j$ is an edge of $K_{n,n}$ for every $1 \leq i, j \leq n$, we see that $v_1, w_1, v_2, w_2, \ldots, v_n, w_n$ is a Hamiltonian cycle (note that $w_nv_1$ is an edge).

(2) The cube graph $Q_n$ was defined in lectures: the vertices of $Q_n$ are all sequences of length $n$ with entries from $\{0, 1\}$ and two sequences are joined by an edge if they differ in exactly one position.

(a) How many edges does $Q_n$ have?

Solution. Fix any vertex $v$ of $Q_n$. All its neighbors differ from $v$ in exactly one position. There are $n$ positions possible to differ at. Hence, every vertex has degree $n$. Since, the number of edges $e(Q_n)$ satisfies $2e(Q_n) = \sum_{v \in V(Q_n)} \deg(v) = n2^n$, we must have $e(Q_n) = n2^{n-1}$.

(b) What is the degree sequence of $Q_n$?

Solution. Based on part (a), the degree sequence of $Q_n$ is $\underbrace{n, \ldots, n}_{\text{2 times}}$.

(c) Which cube graphs $Q_n$ are connected?

Solution. $Q_1 = K_2$ is certainly connected. Suppose that $Q_{n-1}$ is connected for some $n > 1$, and let’s look at $Q_n$. We split its vertices to two sets: $V_1$ contains all the vertices of $Q_1$ ending with 0 and $V_2$ contains all the vertices of $Q_1$ ending with 1. Clearly, $V_1$ and $V_2$ are disjoint and every vertex of $Q_n$ must be in one of them.

The crucial observation is that the vertices of $V_1$ form the cube $Q_{n-1}$. Why?

Firstly, every vertex in $V_1$ can be written as $v0$, where $v$ is a $0-1$ sequence of length $n-1$. Hence, there is 1-to-1 correspondence between vertices of $V_1$ and the vertices of $Q_{n-1}$: for every $v0 \in V_1$ we have $v \in V(Q_{n-1})$.

Secondly, a pair of vertices $v0, w0 \in V_1$ form an edge if and only if $v0$ and $w0$ differ in exactly one position. But they both have 0 at the end, so $v0$ and $w0$ differ in exactly one position if and only if $v$ and $w$ differ in exactly one position. Hence, $v0, w0 \in V_1$ form an edge in $Q_n$ if and only if $v, w$ form an edge in $Q_{n-1}$.

Similarly, the vertices of $V_2$ form the cube $Q_{n-1}$. In the same way as above, we have that $v1, w1 \in V_2$ form an edge in $Q_n$ if and only if $v, w$ form an edge in $Q_{n-1}$.

So, by induction assumption, we know that there is a walk between any two vertices in $V_1$ and between any two vertices in $V_2$.

Take a vertex $v0 \in V_1$ and $w1 \in V_2$. We know there is a walk between $v0$ and $w0$ (using only the vertices of $V_1$), which together with edge $w0w1$ ($w0w1$ differ in the last coordinate) form a walk from $v0$ to $w1$. 


Hence, we showed that $Q_n$ is connected. □

(d) Which cube graphs $Q_n$ have an Euler tour?

**Solution.** $Q_n$ has an Euler tour if and only if all its degrees are even. Since $Q_n$ is $n$-regular, we obtain that $Q_n$ has an Euler tour if and only if $n$ is even. □

(e) Which cube graphs $Q_n$ have a Hamilton cycle?

**Solution.** For $n = 2$, $Q_2$ is the cycle $C_4$, so it is Hamiltonian.

Assume that $Q_{n-1}$ is Hamiltonian and consider the cube graph $Q_n$. Let $V_1$ and $V_2$ be as defined in part (c).

The vertices of $V_1$ form the cube graph $Q_{n-1}$ and so there is a cycle $C$ covering all the vertices of $V_1$.

Moreover, there is a 1-to-1 correspondence between the vertices of $V_1$ and the vertices of $V_2$: $v_0 \in V_1$ if and only if $v_1 \in V_2$. This means that we can construct a cycle $C'$ covering all the vertices of $V_2$ as follows: if $v_0 w_0$ is an edge in $C$, then we put the edge $v_1 w_1$ to $C'$.

Now we link $C$ and $C'$ to a Hamiltonian cycle in $Q_n$: take and edge $v_0 w_0$ in $C$ and $v_1 w_1$ in $C'$ and replace edges $v_0 w_0$ and $v_1 w_1$ with edges $v_0 v_1$ and $w_0 w_1$.

So, $Q_n$ is Hamiltonian as well. □

(3) Suppose that $G$ is a graph in which every vertex has degree at least $k$, where $k \geq 1$, and in which every cycle contains at least 4 vertices.

(a) Show that $G$ contains a path of length at least $2k - 1$.

(b) For each $k \geq 1$, give an example of a graph in which every vertex has degree at least $k$, every cycle contains at least 4 vertices, but which does not contain a path of length $2k$.

**Solution.** See Exercises 8.

(4) Show that the cube graph $Q_n$ is bipartite.

**Solution.** Let $V_1$ be the set of those vertices of $Q_n$ (i.e., sequences of 0’s and 1’s of length $n$) with an even number of 0’s. Similarly, let $V_1$ be the set of those vertices of $Q_n$ with an odd number of 0’s. Clearly, every vertex must have either an odd or an odd number of 0’s and, hence $V_1$, $V_2$ partition $V(Q_n)$ into two disjoint parts.

Is it possible to have an edge $xy$ with $x, y \in V_1$? This would mean that $x$ and $y$ differ in exactly one position. But this would imply that if one of them has an even number of 0’s then the other one has an odd number of 0’s (one 0 is changed to 1 or one 1 is changed to 0), so these two vertices cannot be both from $V_1$. This is a contradiction. In the same way one proves that it is not possible to have an edge with both vertices from $V_2$.

(5) We call a graph *tree* if it is connected and contains no cycles. Prove that if $G$ is a connected graph with $n$ vertices and $n - 1$ edges, then $G$ is a tree.

**Solution.** See Exercises 8.
(6) Recall that the complement of a graph \( G = (V, E) \) is the graph \( \bar{G} \) with the same vertex \( V \) and for every two vertices \( u, v \in V \), \( uv \) is an edge in \( \bar{G} \) if and only if \( uv \) is not an edge of \( G \).

Suppose that \( G \) is a graph on \( n \) vertices such that \( G \) is isomorphic to its own complement \( \bar{G} \). Prove that \( n \equiv 0 \pmod{4} \) or \( n \equiv 1 \pmod{4} \).

**Solution.** Every pair of vertices in \( V \) is an edge in exactly one of the graphs \( G, \bar{G} \). Hence the number of edges \( e(G) \) of \( G \) and the number of edges \( e(\bar{G}) \) satisfy:

\[
e(G) + e(\bar{G}) = \binom{n}{2}.
\]

Since we assume that \( G \) and \( \bar{G} \) are isomorphic, they must have the same number of edges, i.e., \( e(G) = e(\bar{G}) \). Consequently, we have that

\[
2e(G) = e(G) + e(\bar{G}) = \binom{n}{2} = \frac{n(n-1)}{2}.
\]

Thus, \( \frac{n(n-1)}{4} = e(G) \) must be an integer. Exactly one of the numbers \( n \) and \( n - 1 \) is even, so either 4 divides \( n \) or 4 divides \( n - 1 \). In the first case, we have \( n \equiv 0 \pmod{4} \) and, in the second case, we have \( n \equiv 1 \pmod{4} \). \( \square \)

(7) A mouse intends to eat a \( 3 \times 3 \times 3 \) cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a \( 1 \times 1 \times 1 \) cube, before going on to an adjacent one.

Can the mouse end in the center?

**Solution.** Imagine each \( 1 \times 1 \times 1 \) cube as a vertex. We construct a graph \( G \) by joining two vertices \( x, y \) by an edge if the mouse can move from \( x \) to \( y \) (i.e., when \( x \) and \( y \) have a common side (not corner, not edge!)�).

We claim that \( G \) is bipartite. Indeed, we define its bipartition \( X \cup Y \) as follows: we put the 8 corner cubes and centers of each side (6 of them) to \( X \), all the other \( 1 \times 1 \times 1 \) cubes to \( Y \) (i.e., the center of the \( 3 \times 3 \times 3 \) cube and one central cube from each of 12 edges of the \( 3 \times 3 \times 3 \) cube). Is this really a bipartition? In other words, are there no edges in \( X \) or in \( Y \) ? Clearly, no two corners cubes have a common side, no two center cubes are adjacent as well, and a corner and the center of a side are not adjacent either. Similarly, the center of the \( 3 \times 3 \times 3 \) cube is not adjacent to any of the central cubes from each of 12 edges of the \( 3 \times 3 \times 3 \) cube. These central cubes are non-adjacent as well.

So, the plan of our mouse is to “eat” a path containing all the vertices of \( G \), starting in \( X \) (all corners are there) and ending in \( Y \) (the center of the \( 3 \times 3 \times 3 \) cube is there). Such a path must alternate among vertices in \( X \) and \( Y \) because \( G \) is bipartite. However, \(|X| = 14 > 13 = |Y| \) and we start in \( X \), so this is impossible. \( \square \)