Discrete Mathematics **Lent 2009**

MA210

Solutions to Exercises 8

(1) Suppose that G is a graph in which every vertex has degree at least k, where $k \geq 1$, and in which every cycle contains at least 4 vertices.

(a) Show that G contains a path of length at least $2k - 1$.

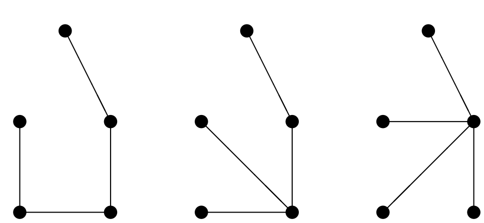
Solution. Take the longest path $v_1, v_2, \ldots, v_\ell$ in G. By maximality, all k neighbours of v_ℓ must be on this path, i.e., in the set $\{v_1, v_2, \ldots, v_{\ell-1}\}$. But it is not possible that two consecutive vertices v_i, v_{i+1} are both neighbours of v_{2k}, for otherwise, we would have a cycle on three vertices v_i, v_{i+1}, v_ℓ. Hence, the set $\{v_1, v_2, \ldots, v_{\ell-1}\}$ must also contain at least $k - 1$ non-neighbours of v_ℓ. Consequently, $\ell - 1 \geq k + (k - 1)$, or, $\ell \geq 2k$ and the path has at least $2k - 1$ edges, that is, it has length $2k - 1$. $\square$

(b) For each $k \geq 1$, give an example of a graph in which every vertex has degree at least k, every cycle contains at least 4 vertices, but which does not contain a path of length $2k$.

Solution. The complete bipartite graph $K_{k,k}$ contains no odd cycles (hence no cycle on 3 vertices), every vertex has degree k, and any path in it can have at most $2k$ vertices because there are no more vertices in $K_{k,k}$. $\square$

(2) How many non-isomorphic trees with five vertices are there? Let $V = \{1, 2, 3, 4, 5\}$. How many different trees with vertex set V are there?

Solution. Removing a leaf from a tree yields a tree. There are only two trees on 4 vertices - a path P_4 and a star $K_{1,3}$. By adding one vertex and examining the possibilities for adding a leaf to these two trees, we obtain the following three non-isomorphic trees on 5 vertices:



So, in how many ways we can assign labels from $\{1, 2, 3, 4, 5\}$ to the trees above so that each time we get a different edge set?

For the first first tree, which a path P_5, each such an assignment is simply an ordering of $\{1, 2, 3, 4, 5\}$ to the line. There are $5! = 120$ orderings, but some of them represent the same graph. Which one are these? If we take any ordering, say $1, 2, 3, 4, 5$, and 'flip it' (to $5, 4, 3, 2, 1$), then these two sequences represent the same path (in this particuar example, its edges are 12, 23, 34, 45). Hence, there are $5!/2 = 60$ trees on V of this type.

For the second tree, any assignment of labels is uniquely determined by the labels on vertices of degree 3, 2 and on the leaf attached to a vertex of degree 2. We have 5 possible labels for the vertex of degree 3. After choosing this label, we have 4 labels for the vertex of degree 2, and, after that selection is made, we have 3 choices for the leaf attached to the vertex of degree 2. By the Multiplication Rule, there are $5 \cdot 4 \cdot 3 = 60$ trees on V of this type.

Finally, for the star $K_{1,4}$, any assignment of labels is uniquely determined by the labels on the vertex of degree 3. We have 5 options for this, so there are 5 trees of this type on V.

Altogether, there are $60 + 60 + 5 = 125$ different trees on V. $\square$

(3) Prove that if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

Solution. We proceed by induction on n. For $n = 1$, the only graph with 1 vertex and 0 edges is K_1, which is a tree.

Suppose that every connected graph with $n - 1$ vertices and $n - 2$ edges is a tree. Let G be a connected graph with n vertices and $n - 1$ edges. First we claim that G has a vertex v of degree 1. Indeed, we see that

$$2(n-1) = 2e(G) = \sum_{v \in V(G)} \deg(v).$$

If every vertex had degree at least 2, then the right-hand side of the above equation would be at least $2n$ and that is not possible. Since G is connected, every vertex has degree at least 1 and, therefore, there must be a vertex v of degree 1. Let u be the only neighbor of v in G.

We claim that $G - v$ (the graph obtained from G by removing v and the edge vu) is connected. Indeed, let x and y be two vertices in $G - v$. Since G is connected, there is a path from x to y in G. Can this path contain v? If it did, then v would not be the endpoint of this path (x and y are the endpoints) and so the degree of v would have to be at least 2. However, v has degree 1 so we would get a contradiction. So, a path connecting x and y in G does not contain v, therefore, it is also a path in $G - v$. Hence, any two vertices in $G - v$ are connected by some path in $G - v$, i.e., $G - v$ is connected.

Now, $G - v$ has $n - 1$ vertices, $n - 2$ edges, and it is connected. By induction assumption, it must be a tree. Hence, $G - v$ does not contain a cycle. By adding v and vu back, we cannot create a cycle. (Such a cycle would have to contain v, forcing it to have degree at least 2. However, v has degree 1.) Also, G is connected by assumption, so G must be a tree. □

(4) Let G be a graph. Prove that G is a tree if and only if for every pair of vertices u and v, there is a unique path between u and v.

Solution. We have two implications to prove.

$\Leftarrow$ Let G be a graph in which for every pair of vertices u and v, there is a unique path between u and v.

In order to show that G is a tree, we **must verify** that <u>G is connected</u> and <u>G has no cycle</u>.

G must be connected because we are given that for every pair of vertices u and v, there is a path between u and v.

Suppose now that there is a cycle in G with vertices $v_1, v_2, \ldots, v_k$ and edges v_1v_2, $v_2v_3, \ldots, v_{k-1}v_k$, v_kv_1. Then there are two paths between v_1 and v_k, namely, v_1, v_k and $v_1, v_2, \ldots, v_k$. This is a contradiction with our assumption that between every two vertices there is a unique path connecting them. Hence, G contains no cycles.

$\Rightarrow$ Suppose that G is a tree, that is, G is connected and acyclic. Since G is connected, there is a path between every two vertices of G. We mus prove that such a path is <u>unique</u>.

Suppose there are two vertices, u and v, for which there are two different paths from u to v. Let the vertices of the first path be $u = x_1, x_2, \ldots, x_{k-1}, x_k = v$ and let the vertices of the second path be $u = y_1, y_2, \ldots, y_{\ell-1}, y_\ell = v$.

We also assume that u and v are chosen in such a way that $k + \ell$ (the sum of lengths of these two paths) is as small as possible among all the pairs of vertices which have at least two paths in between them.

If $x_i \neq y_j$ for all $1 < i < k$ and $1 < j < \ell$, then the two paths together form a cycle in G and that is not possible because G is a tree.

Otherwise, we must have $x_i = y_j = w$ for some $1 < i < k$ and $1 < j < \ell$. What now?

(a) If $u = x_1, x_2, \dots, x_{i-1}, x_i = w$ and $u = y_1, y_2, \dots, y_{j-1}, y_j = w$ are two different paths between u and w, then the sum of lengths of these paths $i + j$ is smaller than $k + \ell$. Thus, we have a contradiction with our initial choice of u and v.

(b) In the same way we obtain a contradiction if $w = x_i, x_{i+1}, \dots, x_{k-1}, x_k = v$ and $w = y_j, y_{j+1}, \dots, y_{\ell-1}, y_\ell = v$ are two different paths between w and v.

(c) However, if neither of the above occurs, then both paths $u = x_1, x_2, \dots, x_{k-1}, x_k = v$ and $u = y_1, y_2, \dots, y_{\ell-1}, y_\ell = v$ are the same! This is a contradiction with our assumption that there are at least two different paths between u and v.

$\square$

(5) Suppose that G is a forest with n vertices and c components. Prove that G has $n - c$ edges.

Solution. Let $G_1, \dots, G_c$ be all the components of G. Each component is then connected and with no cycle, i.e., every G_i is a tree. If we denote by n_i the number of vertices in component G_i, then we have $e(G_i) = n_i - 1$ and $n = n_1 + n_2 + \cdots + n_c$.

As we have no edges between two components, we also have $e(G) = e(G_1) + e(G_2) + \cdots + e(G_c)$, and therefore

$$e(G) = e(G_1) + e(G_2) + \cdots + e(G_c) = (n_1 - 1) + (n_2 - 1) + \cdots + (n_c - 1) = n - c.$$

$\square$

(6) Prove by induction that every tree is a bipartite graph. (Do not use the theorem about the characterization of bipartite graphs from lectures. This problem is easy to prove directly.)

Solution. We will use the property that every tree T contains a vertex v of degree 1, and that $T - v$ is also a tree. (Why is this true?)

Now we proceed by induction on the number n of vertices of a tree T. For $n = 1, 2$, the only trees are K_1 and K_2 and both are bipartite. Suppose that any tree on less than n vertices is bipartite and let T be a tree on n vertices.

Let v be a vertex of degree 1 in T and let u be its only neighbor. We know that $T - v$ is a tree with $n - 1$ vertices, so, by induction assumption, $T - v$ has bipartition X, Y. (This means that every edge in $T - v$ has one endpoint in X and the other one in Y.)

Now, if $u \in X$, then $X, Y \cup \{v\}$ is a bipartition of T. However, if $u \in Y$, then $Y, X \cup \{v\}$ is a bipartition of T. □

(7) (a) How many spanning trees does the graph P_n have?

Solution. P_n is a tree itself; removing any of its edges disconnects it (resulting graph has $n - 2$ edges, so it cannot be a tree). Hence, there is only one spanning tree of P_n. □

(b) How many spanning trees does the graph C_n have?

Solution. Since every tree on n vertices has exactly $n - 1$ edges, we must remove exactly one edge from C_n. For this we have n possibilities, and each time we get a different tree. Hence, C_n has n different spanning trees. □

(c) How many spanning trees does the graph K_4 have?

Solution. There are two trees on 4 vertices: path P_4 and star $K_{1,3}$ (i.e., one vertex adjacent to 3 other vertices).

There are $4!$ ways to order the vertices of K_4 and each such ordering forms the path P_4. However, orderings a, b, c, d and d, c, b, a form the same path (with edges ab, bc, cd), hence K_4 has $4!/2 = 12$ different spanning paths P_4.

Any star $K_{1,3}$ in K_4 is uniquely determined by the center of this star (the vertex that is adjacent to the other 3 vertices). In K_4, we have 4 choices for this center, so K_4 has four different spanning stars $K_{1,3}$.

Altogether, K_4 has $12 + 4 = 16$ different spanning trees. □

(8) Let the graph K_n have vertices $\{1, 2, \ldots, n\}$ and suppose that for each $u, v \in \{1, 2, \ldots, n\}$, the edge uv has weight $c_{uv} = u + v$. Determine the minimum cost spanning tree of this graph. What is the total cost for this minimum cost spanning tree?

Solution. We prove that for every n, Kruskal's algorithm will choose edges $12, 13, \ldots, 1n$. Hence, the total cost of this mimimum cost spanning tree is

$$3 + 4 + \cdots + n + (n + 1) = \tfrac{1}{2}(n + 1)(n + 2) - 3.$$

We order the edges as follows:

$$
\begin{aligned}
&12,\\
&13,\\
&14, 23,\\
&15, 24,\\
&16, 25, 34,\\
&\vdots\\
&1i, \text{all the remaining edges of weight } i+1,\\
&\vdots\\
&1n, \text{all the remaining edges of weight at least } n.
\end{aligned}
$$

We claim that from each row above, Kruskal' Algorithm will chose only the first edge (the only one that contains vertex 1).

This is certainly true for the first 2 rows. So, let suppose that Kruskal's Algorithm selected the edges $12, 13, \ldots, 1(i-1)$, rejected all the others, and it is going to consider the row:

$$1i, \text{all the remaining edges of weight } i+1$$

Since the edge $1i$ together with $12, 13, \ldots, 1(i-1)$ form a star $K_{1,i-1}$, which is a tree, Kruskal's Algorithm will accept it.

The remaining edges with weight $i+1$ are jk, where $2 \le j \le i-1$ and $k = i+1-j < i$. Since $1j$ and $1k$ were already picked by Kruskal's Algorithm, adding jk would create a cycle: $1j$, $1k$, jk. Hence, Kruskal's Algorithm will reject it.

By induction, Kruskal's algorithm will choose edges $12, 13, \ldots, 1n$. After edge $1n$ the algorithm stops because we have a spanning tree. $\square$