Discrete Mathematics MA 210 — Lent 2009

Notes for lectures 15 and 16

3.5 Extremal problems

Let G be a graph. Define the relation $\sim$ on $V(G)$ by

$$u \sim v \Leftrightarrow u = v \text{ or there exists a walk in } G \text{ with } u \text{ and } v \text{ as ends.}$$

It is easy to show that $\sim$ is an equivalence relation. The equivalence classes of this relation are called the *components* of G. This means that vertices u and v belong to the same component if and only if there exists a walk in G between u and v.

Note that graph is connected if and only if the graph has exactly one component.

Theorem 3.9. *Suppose that G is a graph in which every vertex has degree at least $k \geq 1$.*

1. *Then G contains a path of length at least k.*
2. *Then G contains a cycle of length at least $k+1$.*
3. *If G has no path of length $k+1$, then every component of G is a complete graph on $k+1$ vertices.*

Theorem 3.10 (Dirac's Theorem). *Let G be a graph on $n \geq 3$ vertices, and suppose that every vertex $v \in V(G)$ satisfies $d(v) \geq n/2$. Then G contains a Hamilton cycle.*

3.6 Some special graphs

The *path* P_n is a graph on n vertices with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$.

The *cycle* C_n is a graph on n vertices with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

The *complete* graph K_n on n vertices is a graph with n vertices in which every two vertices are adjacent.

For $n \geq 1$, we define the *cube graph* Q_n as follows: $V(Q_n)$ is the set of all $0-1$ sequences of length n and two sequences are joined by an edge if they differ in exactly one position.

3.7 Bipartite graphs

Definition 3.11. *A graph G is bipartite if we can write $V(G) = A \cup B$ with $A \cap B = \emptyset$, and all edges of G have one end vertex in A and the other end in B.*

Theorem 3.12 (König's Theorem). *A graph G is bipartite if and only if G contains no odd cycles.*