Summer 2009 BC examination

MA 2-1-oh

Discrete Mathematics (Half Unit)

This mock exam is meant to give an impression of what can be expected at the real exam. The fact that certain types of questions or topics do not appear in this mock exam, does not mean they cannot appear in the real exam !

Instructions to candidates

Time allowed : 2 hours.

This examination paper contains **5** questions. You should answer **4** questions. If additional questions are answered, only your **BEST 4** answers will count towards the final mark.

All questions carry equal numbers of marks.

Answers should be justified by showing work.

You may **NOT** use an electronic calculator for this examination.

You are supplied with : Answer booklets.

1 (a) One of the teaching rooms at the LSE has 20 seats, which are formed by 5 rows, each 4 tables wide. A certain class using that room has 9 students.

(i) In how many ways can these 9 students be seated in that teaching room? The teacher notes that in fact the students dont choose just any way to sit. There is a group of 5 students who always sit in the front two rows, such that 2 of them (the exact 2 can differ) sit in the front row, and the other 3 sit in the 2nd row. The other 4 students find seats in the final 3 rows.

(ii) In how many ways can the students be seated as described above, where the group of 5 students is predetermined ? At another time a different class of 9 students is taught in this room. They always seat themselves so that there is at least one student in each of the five rows.

(iii) Determine the number of ways that 9 students can be seated in the room so that there is at least one student in each of the five rows.

(b) Suppose the sequence $(a_n)_{n=0}^{\infty}$ has generating function $f(x)$.

(i) What sequence is generated by the function $g(x) = (x^2 - x)f(x)$?

(i) What sequence is generated by the function $h(x) = \frac{f(x)}{1+x}$?

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2 (a) For an integer $m \geq 3$, the graph T_m is defined as follows: The vertex set is the collection of all 2-element subsets of an m-element set (so T_m has $\binom{m}{2}$ vertices). And there is an edge between a pair of different vertices if and only if their intersection has exactly one element.

(i) Make a sketch of T_3, T_4 and T_5.

(ii) Which of T_3, T_4, T_5 contains a Hamilton cycle?

(iii) For which $m \geq 3$ is T_m a bipartite graph?

(iv) Formulate Euler's Theorem on the existence of an Euler tour in a graph. Show that for every $m \geq 3$ the graph T_m satisfies the conditions in Euler's Theorem.

(b) Given the following recurrence relation for the sequence $(a_n)_{n=0}^{\infty}$:

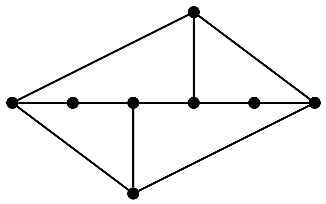
$$\begin{aligned} a_0 &= 0, \\ a_1 &= 1, \\ a_2 &= 1, \\ a_n &= a_{n-1} + a_{n-2} - a_{n-3}, \text{ for } n \geq 3. \end{aligned}$$

(i) Show that the generating function of $(a_n)_{n=0}^{\infty}$ is $f(x) = \dfrac{x}{1 - x - x^2 + x^3}$.

(ii) Find a closed form expression for a_n.

3 (a) What does it mean to say that a graph is *bipartite* ? State König's Theorem (that is, the characterization of bipartite graphs by odd cycles).

(b) Give one reason why the graph below is not bipartite. What is the minimum number of edges needed to be deleted in order to make this graph bipartite ? Justify your answer.



(c) What does it mean to say that a graph is a *tree*? If $d_1, d_2, \ldots, d_n$ are the degrees of an n-vertex tree, prove that $\sum_{i=1}^{n} d_i = 2(n-1)$.

(d) Suppose that positive integers $d_1, d_2, \ldots, d_n$, where $n \geq 2$, satisfy $\sum_{i=1}^{n} d_i = 2(n-1)$. Prove that there exists an n-vertex tree with vertex degrees $d_1, d_2, \ldots, d_n$.

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4 (a) The code C of length 6 has the following codewords :

$$000000; 000111; 111000; 010101; 101010; 111111.$$

(i) What is the maximum number of errors that C can detect ?

(ii) And what is the maximum number of errors that C can correct ?

(iii) Find a word $\bar{\boldsymbol{x}}$ of length 6 that C cannot correct. Justify your answer.

(iv) In order to improve the error-correcting properties of the code C, you add a 7-th bit at the end of each word, which must of course be 0 or 1. Describe all possible ways that a 7-th bit can be added to each word so that the resulting code can correct more errors than C can. Justify your answer.

(v) Construct a linear code C_L of length 6 with the minimum number of elements, but such that $C \subset C_L$. What is the dimension of C_L ?

(b) How many solutions are there for the equation

$$x_1 + x_2 + x_3 + x_4 = 20;$$

(i) if all x_i must be non-negative integers;

(ii) if all x_i must be non-negative integers and x_4 is at most 10?

5 (a) Let an be the number of words of length n that can be formed using the letters A, B, C such that any A or B has to be followed by a C.

(i) Determine a recurrence relation for a_n, including appropriate initial values $a_0, a_1, \ldots$.

(ii) Find a closed form expression for a_n. Let b_n be the number of words of length n that can be formed using the letters A, B, C such that any A or B has to be followed by a C, except if A or B are the last letter in the word.

(iii) Answer the questions (i) and (ii) for the sequence $(b_n)_{n=0}^{\infty}$.

(b) A certain binary code C has length n (for some $n \geq 1$), m codewords (for some $m \geq 1$), and minimum distance δ. The code $C^{(2)}$ of length $2n$ and with again m codewords is formed by replacing for every word in C every occurrence of a 0 by 01 and every occurrence of a 1 by 10. (E.g., if 010 is in C, then the corresponding codeword in $C^{(2)}$ is 011001.)

(i) What is the minimum distance of $C^{(2)}$?

(ii) If C is a linear code, does it mean that $C^{(2)}$ is a linear code as well ? Justify your answer.

(iii) Somebody suggests further improving the error-correcting performance of $C^{(2)}$ by adding a parity check bit as a $(2n+1)$-st bit.

By how much does this additional bit improve the possibilities to detect or correct errors ?

END OF EXAM