HYPERGRAPHS, QUASI-RANDOMNESS, AND CONDITIONS FOR REGULARITY

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Dedicated to Professors Vera T. Sós and András Hajnal on the occasion of their 70th birthdays

Abstract. Haviland and Thomason and Chung and Graham were the first to investigate systematically some properties of quasi-random hypergraphs. In particular, in a series of articles, Chung and Graham considered several quite disparate properties of random-like hypergraphs of density $1/2$ and proved that they are in fact equivalent. The central concept in their work turned out to be the so called *deviation* of a hypergraph. They proved that having small deviation is equivalent to a variety of other properties that describe quasi-randomness. In this paper, we consider the concept of *discrepancy* for k-uniform hypergraphs with an arbitrary constant density d $(0 < d < 1)$ and prove that the condition of having asymptotically vanishing discrepancy is equivalent to several other quasi-random properties of $\mathcal{H}$, similar to the ones introduced by Chung and Graham. In particular, we prove that the correct 'spectrum' of the s-vertex subhypergraphs is equivalent to quasi-randomness for any $s \geq 2k$. Our work may be viewed as a continuation of the work of Chung and Graham, although our proof techniques are different in certain important parts.

1. Introduction and the main result

The rich interplay between the investigation of *deterministic* combinatorial structures and *random* combinatorial structures has been an important feature of modern combinatorics. One aspect of this interaction focuses on the study of deterministic structures that 'mimic' the behavior of random ones, from certain specific points of view.

In this paper, we are interested in 'quasi-random' hypergraphs, in the sense of Chung and Graham [5, 6]. Haviland and Thomason [9, 10], Chung [4], and Chung and Graham [5, 6] have already established the fundamental results in this area. Babai, Nisan, and Szegedy [3] have implicitly found a connection between communication complexity and what is known as 'hypergraph discrepancy', a key concept, as we shall see, in the study of quasi-random hypergraphs. This connection was explored further by Chung and Tetali [7]. Here, we carry out our investigation very much along the lines of Chung and Graham [5, 6], except that we focus on

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hypergraphs of arbitrary constant density, making use of different techniques in certain delicate parts.

In the remainder of this introduction, we carefully discuss a result of Chung and Graham [5] and state our main result, Theorem 1.3 below.

1.1. **The result of Chung and Graham.** We need to start with some definitions. For a set V and an integer $k \geq 2$, let $[V]^k$ denote the system of all k-element subsets of V. A subset $\mathcal{G} \subset [V]^k$ is called a *k-uniform hypergraph*. If $k = 2$, we have a *graph*. We sometimes use the notation $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$. If there is no danger of confusion, we shall identify the hypergraphs with their edge sets. In particular, we write $|\mathcal{H}|$ for the number of edges in $\mathcal{H}$. Throughout this paper, the integer k is assumed to be a fixed constant.

For any l-uniform hypergraph $\mathcal{G}$ and $k \geq l$, let $\mathcal{K}_k(\mathcal{G})$ be the set of all k-element sets that span a clique $K_k^{(l)}$ on k vertices. We also denote by $K_k(2)$ the complete k-partite k-uniform hypergraph whose every partite set contains precisely two vertices. We refer to $K_k(2)$ as the *generalized octahedron*, or, simply, the *octahedron*.

We also consider a function $\mu_{\mathcal{H}}\colon [V]^k \to \{-1, 1\}$ such that, for all $e \in [V]^k$, we have

$$\mu_{\mathcal{H}}(e) = \begin{cases} -1, & \text{if } e \in \mathcal{H} \\ 1, & \text{if } e \notin \mathcal{H}. \end{cases}$$

Let $[k] = \{1, 2, \ldots, k\}$ and let V^{2k} denote the set of all $2k$-tuples $(v_1, v_2, \ldots, v_{2k})$, where $v_i \in V$ $(1 \leq i \leq 2k)$. Furthermore, let $\Pi_{\mathcal{H}}^{(k)}\colon V^{2k} \to \{-1, 1\}$ be given by

$$\Pi_{\mathcal{H}}^{(k)}(u_1, \ldots, u_k, v_1, \ldots, v_k) = \prod_{\varepsilon = (\varepsilon_i)_{i=1}^k} \mu_{\mathcal{H}}(\varepsilon_1, \ldots, \varepsilon_k),$$

where the product is over all vectors $\varepsilon = (\varepsilon_i)_{i=1}^k$ with $\varepsilon_i \in \{u_i, v_i\}$ for all i and we understand $\mu_{\mathcal{H}}$ to be 1 on arguments with repeated entries.

The deviation $\operatorname{dev}(\mathcal{H})$ of $\mathcal{H}$ is defined by

$$\operatorname{dev}(\mathcal{H}) = \frac{1}{m^{2k}} \sum_{u_i, v_i \in V,\ i \in [k]} \Pi_{\mathcal{H}}^{(k)}(u_1, \ldots, u_k, v_1, \ldots, v_k).$$

Note that the quantity $m^{2k} \operatorname{dev}(\mathcal{H})$ is essentially the difference between the number of $2k$-tuples that induce an even number of edges and the number of $2k$-tuples that induce an odd number of edges.

For two hypergraphs $\mathcal{G}$ and $\mathcal{H}$, we denote by $\binom{\mathcal{H}}{\mathcal{G}}$ the set of all induced subhypergraphs of $\mathcal{H}$ that are isomorphic to $\mathcal{G}$. We also write $\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}$ for the number of *weak* (i.e., not necessarily induced) subhypergraphs of $\mathcal{H}$ that are isomorphic to $\mathcal{G}$. Furthermore, we need the notion of the *link* of a vertex.

Definition 1.1. *Let $\mathcal{H}$ be a k-uniform hypergraph and $x \in V(\mathcal{H})$. We shall call the $(k-1)$-uniform hypergraph*

$$\mathcal{H}(x) = \{e \setminus \{x\}\colon e \in \mathcal{H},\ x \in e\}$$

the link of the vertex x in $\mathcal{H}$. For a subset $W \subset V(\mathcal{H})$, we define $\mathcal{H}(W)$ by

$$\mathcal{H}(W) = \bigcap_{x \in W} \mathcal{H}(x).$$

For simplicity, if $W = \{x_1, \ldots, x_k\}$, we write $\mathcal{H}(x_1, \ldots, x_k)$.

Observe that if $\mathcal{H}$ is k-partite, then $\mathcal{H}(x)$ is $(k-1)$-partite for every $x \in V$. Furthermore, if $k = 2$, then $\mathcal{H}(x)$ may be identified with the set of all vertices connected to x, i.e., $\mathcal{H}(x)$ is the neighborhood of x; furthermore, $\mathcal{H}(x, x')$ is the set of all vertices connected to both x and x', i.e., $\mathcal{H}(x, x')$ is the 'joint neighborhood' of x and x'.

In [5], Chung and Graham proved that if the density of an m-vertex k-uniform hypergraph $\mathcal{H}$ is $1/2$, i.e., $|\mathcal{H}| = (1/2 + o(1))\binom{m}{k}$, where $o(1) \to 0$ as $m \to \infty$, then the following statements are equivalent:

$\boldsymbol{Q_1(s)}$**:** for all k-uniform hypergraphs $\mathcal{G}$ on $s \geq 2k$ vertices and automorphism group $\mathrm{Aut}(\mathcal{G})$,

$$\left|\binom{\mathcal{H}}{\mathcal{G}}\right| = (1 + o(1))\binom{m}{s} 2^{-\binom{s}{k}} \frac{s!}{|\,\mathrm{Aut}(\mathcal{G})|},$$

$\boldsymbol{Q_2}$**:** for all k-uniform hypergraphs $\mathcal{G}$ on $2k$ vertices and automorphism group $\mathrm{Aut}(\mathcal{G})$,

$$\left|\binom{\mathcal{H}}{\mathcal{G}}\right| = (1 + o(1))\binom{m}{2k} 2^{-\binom{2k}{k}} \frac{(2k)!}{|\,\mathrm{Aut}(\mathcal{G})|},$$

$\boldsymbol{Q_3}$**:** $\mathrm{dev}(\mathcal{H}) = o(1)$,

$\boldsymbol{Q_4}$**:** for almost all choices of x, $y \in V$, the $(k-1)$-uniform hypergraph $\overline{\mathcal{H}(x) \triangle \mathcal{H}(y)}$, that is, the *complement* $[V]^{k-1} \setminus (\mathcal{H}(x) \triangle \mathcal{H}(y))$ of the symmetric difference of $\mathcal{H}(x)$ and $\mathcal{H}(y)$, satisfies Q_2 with k replaced by $k-1$,

$\boldsymbol{Q_5}$**:** for $k - 1 \leq r < 2k$ and almost all x, $y \in V$,

$$\left|\binom{\mathcal{H}(x, y)}{K_r^{(k-1)}}\right| = (1 + o(1))\binom{m}{r} 2^{-\binom{r}{k-1}}.$$

The equivalence of these properties is understood in the following sense. For two properties involving $o(1)$ terms $\boldsymbol{P} = \boldsymbol{P}(o(1))$ and $\boldsymbol{P'} = \boldsymbol{P'}(o(1))$, the implication "$\boldsymbol{P} \Rightarrow \boldsymbol{P'}$" means that for every $\varepsilon > 0$ there is a $\delta > 0$ so that any k-uniform hypergraph $\mathcal{H}$ on m vertices satisfying $\boldsymbol{P}(\delta)$ must also satisfy $\boldsymbol{P'}(\varepsilon)$, provided $m > M_0(\varepsilon)$.

Chung and Graham [5] stated that "it would be profitable to explore quasi-randomness extended to simulating random k-uniform hypergraphs $G_p(n)$ for $p \neq 1/2$, or, more generally, for $p = p(n)$, especially along the lines carried out so fruitfully by Thomason [12, 13]." Our present aim is to take the first steps in this direction. In this paper, we concentrate on the case in which p is an arbitrary constant. In certain crucial parts, our methods are different from the ones of Chung and Graham. In fact, it seems to us that the fact that the density of $\mathcal{H}$ is $1/2$ is essential in certain proofs in [5] (especially those involving the concept of deviation).

1.2. **Discrepancy and the subgraph counting formula.** The following concept was proposed by Frankl and Rödl and was later used by Chung [4] and Chung and Graham in [5, 6]. For an m-vertex k-uniform hypergraph $\mathcal{H}$ with vertex set V, we define the *density* $d(\mathcal{H})$ and the *discrepancy* $\mathrm{disc}_{1/2}(\mathcal{H})$ of $\mathcal{H}$ as follows:

$$d(\mathcal{H}) = |\mathcal{H}|\binom{m}{k}^{-1}$$

and

$$\mathrm{disc}_{1/2}(\mathcal{H}) = \frac{1}{m^k} \max_{\mathcal{G} \subset [V]^{k-1}} \left| |\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| - |\bar{\mathcal{H}} \cap \mathcal{K}_k(\mathcal{G})| \right|, \tag{1}$$

where the maximum is taken over all $(k-1)$-uniform hypergraphs $\mathcal{G}$ with vertex set V, and $\bar{\mathcal{H}}$ is the complement $[V]^k \setminus \mathcal{H}$ of $\mathcal{H}$.

To accommodate arbitrary densities, we extend the latter concept as follows.

Definition 1.2. *Let $\mathcal{H}$ be a k-uniform hypergraph with vertex set V with $|V| = m$. We define the discrepancy* $\operatorname{disc}(\mathcal{H})$ *of $\mathcal{H}$ as follows:*

$$\operatorname{disc}(\mathcal{H}) = \frac{1}{m^k} \max_{\mathcal{G} \subset [V]^{k-1}} \big| |\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| - d(\mathcal{H}) |\mathcal{K}_k(\mathcal{G})| \big|, \tag{2}$$

where the maximum is taken over all $(k-1)$-uniform hypergraphs $\mathcal{G}$ with vertex set V.

Observe that if $d(\mathcal{H}) = 1/2$, then $\operatorname{disc}(\mathcal{H}) = (1/2) \operatorname{disc}_{1/2}(\mathcal{H})$, so both notions are equivalent. Following some initial considerations by Frankl and Rödl, Chung and Graham investigated the relation between discrepancy and deviation. In fact, Chung [4] succeeded in proving the following inequalities closely connecting these quantities:

(i) $\operatorname{dev}(\mathcal{H}) < 4^k (\operatorname{disc}_{1/2}(\mathcal{H}))^{1/2^k}$,
(ii) $\operatorname{disc}_{1/2}(\mathcal{H}) < (\operatorname{dev}(\mathcal{H}))^{1/2^k}$.

For simplicity, we state the inequalities for the density 1/2 case. For the general case, see Section 5 of [4].

Before we proceed, we need to introduce a new concept. If the vertex set of a hypergraph is totally ordered, we say that we have an *ordered* hypergraph. Given two ordered hypergraphs $\mathcal{G}_\prec$ and $\mathcal{H}_{\prec'}$, where $\prec$ and $\prec'$ denote the orderings on the vertex sets of $\mathcal{G} = \mathcal{G}_\prec$ and $\mathcal{H} = \mathcal{H}_{\prec'}$, we say that a function $f \colon V(\mathcal{G}) \to V(\mathcal{H})$ is an *embedding of ordered hypergraphs* if (i) it is an injection, (ii) it respects the orderings, i.e., $f(x) \prec' f(y)$ whenever $x \prec y$, and (iii) $f(g) \in \mathcal{H}$ if and only if $g \in \mathcal{G}$, where $f(g)$ is the set formed by the images of all the vertices in g. Furthermore, if $\mathcal{G} = \mathcal{G}_\prec$ and $\mathcal{H} = \mathcal{H}_{\prec'}$, we write

$$\binom{\mathcal{H}}{\mathcal{G}}_\prec = \binom{\mathcal{H}_{\prec'}}{\mathcal{G}_\prec}$$

for the number of such embeddings. We use the same symbol '$\prec$' for the orders involved in case this causes no confusion.

As our main result, we shall prove the following extension of Chung and Graham's result (in the sense that the density of $\mathcal{H}$ is allowed to be different from 1/2).

Theorem 1.3. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph of density d on m vertices. Then the following statements are equivalent:*

- $\boldsymbol{P_1}$**:** $\operatorname{disc}(\mathcal{H}) = o(1)$*,*
- $\boldsymbol{P_2}$**:** $\operatorname{disc}(\mathcal{H}(x)) = o(1)$ *and* $d(\mathcal{H}(x)) = (1 + o(1))d$ *for all but* $o(m)$ *vertices* $x \in V$ *and* $\operatorname{disc}(\mathcal{H}(x, y)) = o(1)$ *and* $d(\mathcal{H}(x, y)) = (1 + o(1))d^2$ *for all but* $o(m^2)$ *pairs* x, $y \in V$*,*
- $\boldsymbol{P_3}$**:** $\operatorname{disc}(\mathcal{H}(x, y)) = o(1)$ *and* $d(\mathcal{H}(x, y)) = (1 + o(1))d^2$ *for all but* $o(m^2)$ *pairs* x, $y \in V$*,*
- $\boldsymbol{P_4}$**:** *the number of non-induced copies of $K_k(2)$ in $\mathcal{H}$ is asymptotically minimized among all k-uniform hypergraphs of density d; indeed,*

$$\left| \binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}} \right| = (1 + o(1)) \frac{m^{2k}}{2^k k!} d^{2^k}, \tag{3}$$

$\boldsymbol{P_5}$**:** *for every $s \geq 2k$ and all k-uniform hypergraphs $\mathcal{G}$ on s vertices with $e(\mathcal{G})$ edges and automorphism group* $\mathrm{Aut}(\mathcal{G})$*,*

$$\left|\binom{\mathcal{H}}{\mathcal{G}}\right| = (1+o(1))\binom{m}{s} d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}\frac{s!}{|\mathrm{Aut}(\mathcal{G})|},$$

$\boldsymbol{P_5'}$**:** *for every ordering $\mathcal{H}_{\prec'}$ of $\mathcal{H}$ and for every fixed integer $s \geq 2k$, every ordered k-uniform hypergraph $\mathcal{G}_\prec$ on s vertices with $e(\mathcal{G})$ edges is such that*

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_\prec\right| = (1+o(1))\binom{m}{s} d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})},$$

$\boldsymbol{P_6}$**:** *for all k-uniform hypergraphs $\mathcal{G}$ on $2k$ vertices with $e(\mathcal{G})$ edges and automorphism group* $\mathrm{Aut}(\mathcal{G})$*,*

$$\left|\binom{\mathcal{H}}{\mathcal{G}}\right| = (1+o(1))\binom{m}{2k} d^{e(\mathcal{G})}(1-d)^{\binom{2k}{k}-e(\mathcal{G})}\frac{(2k)!}{|\mathrm{Aut}(\mathcal{G})|}.$$

$\boldsymbol{P_6'}$**:** *for every ordering $\mathcal{H}_{\prec'}$ of $\mathcal{H}$, every ordered k-uniform hypergraph $\mathcal{G}_\prec$ on $2k$ vertices with $e(\mathcal{G})$ edges is such that*

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_\prec\right| = (1+o(1))\binom{m}{2k} d^{e(\mathcal{G})}(1-d)^{\binom{2k}{k}-e(\mathcal{G})}.$$

The equivalence between properties is understood in the sense of Chung and Graham's approach.

Note that, similarly to the case where $k = 2$ (see, e.g., [1, 2]), the equivalence among the above properties may be used to develop a fast algorithm for checking whether a given hypergraph is quasi-random. While it is hard to check whether $\mathrm{disc}(\mathcal{H}) \leq \delta$ directly from the definition of $\mathrm{disc}(\mathcal{H})$, one may check property $\boldsymbol{P_4}$ in $O(m^{2k})$ time. This may be further improved using techniques from [1, 11].

Some of the implications in Theorem 1.3 are fairly easy or are by now quite standard. There are, however, two implications that appear to be quite difficult.

The proof of Chung and Graham that $\mathrm{dev}_{1/2}(\mathcal{H}) = o(1)$ implies P_5 (the 'subgraph counting formula') is based on an approach that has its roots in a seminal paper of Wilson [14]. This beautiful proof seems to make non-trivial use of the fact that $d(\mathcal{H}) = 1/2$. Our proof of the implication that small discrepancy implies the subgraph counting formula ($P_1 \Rightarrow P_5'$) is based on a different technique, which works well in the arbitrary constant density case (see Section 6).

The second implication with a rather technical proof is $P_2 \Rightarrow P_1$. This proof is based on the observation that in k-uniform k-partite hypergraphs the regularity of links and pair-links implies the regularity of the whole hypergraph. For details, we refer the reader to Sections 3.1 and 4.

Remark. Let us make some remarks on the asymptotic notation that we shall use. Unless otherwise stated, we understand by $o(1)$ a function approaching zero as the number of vertices of a given hypergraph goes to infinity. We also use $x \sim y$ if $x = (1+o(1))y$ and $x \gtrsim y$ if $x \geq (1+o(1))y$. Finally, we write $O_1(x)$ for a term y such that $|y| \leq x$.

2. Definitions

Besides introducing some definitions and notation, our aim in this section is to argue that, for most of the purposes of this paper, we may restrict ourselves to the case of k-partite k-uniform hypergraphs. To this end, we first set up a few facts concerning k-partite hypergraphs.

2.1. **Definitions for partite hypergraphs.** For simplicity, we first introduce the term *cylinder* to mean partite hypergraphs.

Definition 2.1. *Let $k \geq l \geq 2$ be two integers. We shall refer to any k-partite l-uniform hypergraph $\mathcal{H} = (V_1 \cup \dots \cup V_k, E)$ as a k-partite l-cylinder or (k,l)-cylinder. If $l = k-1$, we shall often write $\mathcal{H}_i$ for the subhypergraph of $\mathcal{H}$ induced on $\bigcup_{j\neq i} V_j$. Clearly, $\mathcal{H} = \bigcup_{i=1}^k \mathcal{H}_i$. We shall also denote by $K_k^{(l)}(V_1,\dots,V_k)$ the complete (k,l)-cylinder with vertex partition $V_1 \cup \dots \cup V_k$.*

Definition 2.2. *For a (k,l)-cylinder $\mathcal{H}$, we shall denote by $\mathcal{K}_j(\mathcal{H})$, $l \leq j \leq k$, the (k,j)-cylinder whose edges are precisely those j-element subsets of $V(\mathcal{H})$ that span cliques of order j in $\mathcal{H}$.*

Clearly, the quantity $|\mathcal{K}_j(\mathcal{H})|$ counts the total number of cliques of order j in $\mathcal{H}$. In the case in which $l = 1$, the (k,j)-cylinder $\mathcal{K}_j(\mathcal{H})$ is the complete k-partite j-uniform hypergraph on $\bigcup \mathcal{H} = \bigcup_{h\in\mathcal{H}} h$.

When we deal with cylinders, we have to measure density according to their natural vertex partitions.

Definition 2.3. *Let $\mathcal{H}$ be a (k,k)-cylinder with k-partition $V = V_1 \cup \dots \cup V_k$. We define the k-partite density or simply the density $d(\mathcal{H})$ of $\mathcal{H}$ by*

$$d(\mathcal{H}) = \frac{|\mathcal{H}|}{|V_1|\dots|V_k|}.$$

To be precise, we should have a distinguished piece of notation for the notion of k-partite density. However, the context will always make it clear which notion we mean when we talk about the density of a (k,k)-cylinder.

We should also be careful when we talk about the discrepancy of a cylinder.

Definition 2.4. *Let $\mathcal{H}$ be a (k,k)-cylinder with vertex set $V = V_1 \cup \dots \cup V_k$. We define the discrepancy $\operatorname{disc}(\mathcal{H})$ of $\mathcal{H}$ as follows:*

$$\operatorname{disc}(\mathcal{H}) = \frac{1}{|V_1|\dots|V_k|} \max_{\mathcal{G}\subset[V]^{k-1}} \big||\mathcal{H}\cap\mathcal{K}_k(\mathcal{G})| - d(\mathcal{H})|\mathcal{K}_k(\mathcal{G})|\big|, \tag{4}$$

where the maximum is taken over all $(k,k-1)$-cylinders $\mathcal{G}$ with vertex set $V = V_1 \cup \dots \cup V_k$.

We now introduce a simple but important concept concerning the "regularity" of a (k,k)-cylinder.

Definition 2.5. *Let $\mathcal{H}$ be a (k,k)-cylinder with k-partition $V = V_1 \cup \dots \cup V_k$ and let $\delta < \alpha$ be two positive real numbers. We say that $\mathcal{H}$ is (δ,α)-regular if the following condition is satisfied: if $\mathcal{G}$ is any $(k,k-1)$-cylinder such that*

$$|\mathcal{K}_k(\mathcal{G})| \geq \delta|V_1|\dots|V_k|, \tag{5}$$

then

$$(\alpha-\delta)|\mathcal{K}_k(\mathcal{G})| \leq |\mathcal{H}\cap\mathcal{K}_k(\mathcal{G})| \leq (\alpha+\delta)|\mathcal{K}_k(\mathcal{G})|. \tag{6}$$

One should observe that the (δ,α)-regularity of a hypergraph $\mathcal{H}$ does not imply that $\mathcal{H}$ has density α; we may only conclude that the density of a (δ,α)-regular hypergraph is between $\alpha-\delta$ and $\alpha+\delta$. Moreover, the following simple facts hold.

Fact 2.6. *Let $\mathcal{H}$ be a (δ,α)-regular (k,k)-cylinder. Then $\operatorname{disc}(\mathcal{H}) \leq 2\delta$.*

Fact 2.7. *Suppose $\mathcal{H}$ is a (k,k)-cylinder with k-partition $V = V_1 \cup \dots \cup V_k$. Put $\alpha = d(\mathcal{H})$ and assume that $\operatorname{disc}(\mathcal{H}) \le \delta$. Then $\mathcal{H}$ is $(\delta^{1/2}, \alpha)$-regular.*

2.2. **The k-partite reduction.** Suppose $\mathcal{H}$ is a k-uniform hypergraph and let $\mathcal{H}'$ be one of its k-partite spanning subhypergraphs. In this section, we try to relate the deviation and the discrepancy of $\mathcal{H}'$ to those of $\mathcal{H}$.

Definition 2.8. *Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph with m vertices and let $\mathcal{P} = (V_i)_1^k$ be a partition of the vertex set V. We denote by $\mathcal{H}_{\mathcal{P}}$ the (k,k)-cylinder consisting of the edges $h \in \mathcal{H}$ satisfying $|h \cap V_i| = 1$ for all $i \in [k]$.*

The following proposition holds.

Proposition 2.9. *For any partition $\mathcal{P} = (V_i)_1^k$ of V, we have*
  (i) $\operatorname{disc}(\mathcal{H}) \ge |d(\mathcal{H}_{\mathcal{P}}) - d(\mathcal{H})||V_1|\dots|V_k|/m^k$, *and*
  (ii) $\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) \le 2\operatorname{disc}(\mathcal{H}) m^k/|V_1|\dots|V_k|$.

Proof. Let $\mathcal{P} = (V_i)_1^k$ be any partition of V. Observe that $\mathcal{H}_{\mathcal{P}}$ consists precisely of the vertex sets of those copies of $K_k^{(k-1)}$ in $K = K_k^{(k-1)}(V_1, \dots, V_k)$ which are also edges in $\mathcal{H}$; that is, $\mathcal{H}_{\mathcal{P}} = \mathcal{H} \cap \mathcal{K}_k(K)$. Since $|\mathcal{K}_k(K)| = |\mathcal{K}_k(K_k^{(k-1)}(V_1, \dots, V_k))| = |V_1|\dots|V_k|$, this implies the first part of the proposition by taking $\mathcal{G} = K$ in (2).

On the other hand, let $\mathcal{G}_0 \subset [V]^{k-1}$ be a $(k, k-1)$-cylinder for which the maximum is attained in (4), the definition of $\operatorname{disc}(\mathcal{H}_{\mathcal{P}})$. Observe that $\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(\mathcal{G}_0) = \mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0)$. Then

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) &= \frac{1}{|V_1|\dots|V_k|}\big||\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(\mathcal{G}_0)| - d(\mathcal{H}_{\mathcal{P}})|\mathcal{K}_k(\mathcal{G}_0)|\big| \\
&\le \frac{1}{|V_1|\dots|V_k|}\big||\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0)| - d(\mathcal{H})|\mathcal{K}_k(\mathcal{G}_0)|\big| \\
&\qquad + \frac{1}{|V_1|\dots|V_k|}\big|d(\mathcal{H})|\mathcal{K}_k(\mathcal{G}_0)| - d(\mathcal{H}_{\mathcal{P}})|\mathcal{K}_k(\mathcal{G}_0)|\big| \\
&\le \frac{m^k}{|V_1|\dots|V_k|}\operatorname{disc}(\mathcal{H}) + \frac{1}{|V_1|\dots|V_k|}|\mathcal{K}_k(\mathcal{G}_0)|\,|d(\mathcal{H}) - d(\mathcal{H}_{\mathcal{P}})| \\
&\le \frac{m^k}{|V_1|\dots|V_k|}\operatorname{disc}(\mathcal{H}) + |d(\mathcal{H}) - d(\mathcal{H}_{\mathcal{P}})| \\
&\le \frac{2m^k}{|V_1|\dots|V_k|}\operatorname{disc}(\mathcal{H}),
\end{aligned}$$

where in the last inequality we used (i). □

We shall also need the following fact, which follows easily from, say, Chebyshev's inequality.

Fact 2.10. *Let $\mathcal{H} = (V, E)$ be an m-vertex k-uniform hypergraph. Then $(1 - o(1))k^m$ partitions $\mathcal{P} = (V_i)_1^k$ of V satisfy*
  (i) $|V_i| = (1 + o(1))m/k$ *for all* $i \in [k]$,
  (ii) $|\mathcal{H}_{\mathcal{P}}| = (1 + o(1))(k!/k^k)|\mathcal{H}|$, *and*
  (iii) $d(\mathcal{H}_{\mathcal{P}}) = (1 + o(1))d(\mathcal{H})$,

where $o(1) \to 0$ as $|\mathcal{H}|/m^{k-1} \to \infty$.

An immediate consequence of the previous proposition and Fact 2.10 is the following.

Claim 2.11. *If* $\operatorname{disc}(\mathcal{H}) = o(1)$*, then* $\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$ *for* $(1-o(1))k^m$ *partitions* $\mathcal{P} = (V_i)_1^k$ *of* V*.*

With some more effort, one may prove a converse to Claim 2.11.

Claim 2.12. *Suppose there exists a positive real number* $\gamma > 0$ *such that* $\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$ *for* γk^m *partitions* $\mathcal{P} = (V_i)_1^k$ *of* V*. Then* $\operatorname{disc}(\mathcal{H}) = o(1)$*.*

Proof. Let $\boldsymbol{\mathcal{S}}$ be a set of partitions $\mathcal{P}$ for which $\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$ and $|\boldsymbol{\mathcal{S}}| \geq \gamma k^m$. Suppose $\operatorname{disc}(\mathcal{H}) \geq \delta$ for some fixed $\delta > 0$, and let $\mathcal{G}_0$ be a $(k-1)$-uniform hypergraph for which the maximum is attained in (2), the definition of $\operatorname{disc}(\mathcal{H})$. Let $\mathcal{P} = (V_i)_1^k \in \boldsymbol{\mathcal{S}}$ be a partition satisfying the conclusion of Fact 2.10 with respect to $\mathcal{H}$, $\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0)$, and $\mathcal{K}_k(\mathcal{G}_0)$. Such a partition must exist since $\gamma k^m + (1-o(1))k^m > k^m$. Observe that, then,

$$|\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(\mathcal{G}_0)| = |(\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0))_{\mathcal{P}}| = (1+o(1))\frac{k!}{k^k}|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0)|,$$

and

$$|\mathcal{K}_k(\mathcal{G}_0 \cap K_k^{(k-1)}(V_1, \ldots, V_k))| = |(\mathcal{K}_k(\mathcal{G}_0))_{\mathcal{P}}| = (1+o(1))\frac{k!}{k^k}|\mathcal{K}_k(\mathcal{G}_0)|,$$

and, from (*iii*) of Fact 2.10,

$$d(\mathcal{H}_{\mathcal{P}}) = (1+o(1))d(\mathcal{H}).$$

For convenience, put $K = K_k^{(k-1)}(V_1, \ldots, V_k)$. We use an approach similar to the one in Proposition 2.9 to get

$$\begin{aligned}
\operatorname{disc}(\mathcal{H}) &= \frac{1}{m^k}\big||\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}_0)| - d(\mathcal{H})|\mathcal{K}_k(\mathcal{G}_0)|\big| \\
&= \frac{1}{m^k}\left|(1+o(1))\frac{k^k}{k!}|\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(\mathcal{G}_0)| - (1+o(1))d(\mathcal{H}_{\mathcal{P}})\frac{k^k}{k!}|\mathcal{K}_k(\mathcal{G}_0 \cap K)|\right| \\
&\leq \frac{1}{m^k}\frac{k^k}{k!}\big||\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(K \cap \mathcal{G}_0)| - d(\mathcal{H}_{\mathcal{P}})|\mathcal{K}_k(\mathcal{G}_0 \cap K)|\big| \\
&\qquad + \frac{1}{m^k}\frac{k^k}{k!}o(1)\left(|\mathcal{H}_{\mathcal{P}} \cap \mathcal{K}_k(K \cap \mathcal{G}_0)| + d(\mathcal{H}_{\mathcal{P}})|\mathcal{K}_k(\mathcal{G}_0 \cap K)|\right) \\
&\leq (1+o(1))\frac{1}{k!}\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) + \frac{1}{m^k}\frac{k^k}{k!}o(1)2\binom{m}{k} \leq \frac{2}{k!}\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) + \frac{2k^k}{(k!)^2}o(1).
\end{aligned}$$

Since by our assumptions $\operatorname{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$, we immediately obtain that $\operatorname{disc}(\mathcal{H}) < \delta$ for large enough m, which is a contradiction. □

We now state the k-partite version of a part of our main result, Theorem 1.3.

Theorem 2.13. *Suppose* $V = V_1 \cup \cdots \cup V_k$*,* $|V_1| = \cdots = |V_k| = n$*, and let* $\mathcal{H} = (V, E)$ *be a* (k,k)*-cylinder with* $|\mathcal{H}| = dn^k$*. Then the following four conditions are equivalent:*

- $\boldsymbol{C_1}$**:** $\mathcal{H}$ *is* $(o(1), d)$*-regular;*
- $\boldsymbol{C_2}$**:** $\mathcal{H}(x)$ *is* $(o(1), d)$*-regular for all but* $o(n)$ *vertices* $x \in V_k$ *and* $\mathcal{H}(x,y)$ *is* $(o(1), d^2)$*-regular for all but* $o(n^2)$ *pairs* x*,* $y \in V_k$*;*
- $\boldsymbol{C_3}$**:** $\mathcal{H}(x,y)$ *is* $(o(1), d^2)$*-regular for all but* $o(n^2)$ *pairs* x*,* $y \in V_k$*;*
- $\boldsymbol{C_4}$**:** *the number of copies of* $K_k(2)$ *in* $\mathcal{H}$ *is asymptotically minimized among all such* (k,k)*-cylinders of density* d*, and equals* $(1+o(1))n^{2k}d^{2^k}/2^k$*.*

Remark. The simplifying condition $|V_1| = \cdots = |V_k| = n$ has the sole purpose of making the proof more readable and transparent. The immediate generalization of Theorem 2.13 for $V_1, \ldots, V_k$ of arbitrary sizes holds.

The proof of Theorem 2.13 will be given in Sections 4 and 5.

3. The derivation of the general case

In this part, we prove Theorem 1.3. We divide this proof into five sections. In Section 3.1, we show the equivalence of properties P_1, P_2, and P_3. The proof of $P_4 \Rightarrow P_1$ is in Section 3.2. Both sections use Theorem 2.13 as the main tool. In Section 3.3, we prove $P_1 \Rightarrow P_5'$ using the 'subhypergraph counting formula' from Section 6. Then we show that $P_5' \Rightarrow P_5 \Rightarrow P_6$ and $P_5' \Rightarrow P_6' \Rightarrow P_6$ (see Section 3.4). Finally, we prove $P_6 \Rightarrow P_4$ in Section 3.5. The flow of the whole proof is described in the following diagram.

$$\begin{array}{ccccc}
P_2 & \Rightarrow & P_3 & \Rightarrow & P_1 \\
\Uparrow & & & & \Downarrow \\
P_1 & & P_6' & \Leftarrow & P_5' \\
\Uparrow & & \Downarrow & & \Downarrow \\
P_4 & \Leftarrow & P_6 & \Leftarrow & P_5
\end{array}$$

3.1. **Proof of $\boldsymbol{P_1 \Leftrightarrow P_2 \Leftrightarrow P_3}$.** We are now ready to show that, in the first part of the proof of Theorem 1.3, we may assume the hypergraph $\mathcal{H}$ to be k-partite. To be more precise, we show that the equivalence $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3$ in Theorem 1.3 follows from Theorem 2.13. We shall illustrate this on $P_1 \Rightarrow P_2$; the other implications are handled similarly. In fact, we shall be somewhat sketchy; we shall only indicate the double counting argument that gives this result.

Suppose that we have a k-uniform hypergraph $\mathcal{H}$ with density d such that $\mathrm{disc}(\mathcal{H}) = o(1)$. From Fact 2.10 and Claim 2.11 we know that for all but $o(1)k^m$ partitions $\mathcal{P}$ of V we have $\mathrm{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$ and $d(\mathcal{H}_{\mathcal{P}}) = (1 + o(1))d$. For every partition $\mathcal{P}$, denote by $X(\mathcal{P})$ the set of all vertices $x \in V$ such that either $\mathrm{disc}(\mathcal{H}_{\mathcal{P}}(x)) \neq o(1)$ or $\mathrm{disc}(\mathcal{H}_{\mathcal{P}}) \neq o(1)$. From Theorem 2.13 and Facts 2.6 and 2.7, we know that for all but $o(1)k^m$ partitions $\mathcal{P}$, we have $|X(\mathcal{P})| = o(m)$. For the remaining $o(1)k^m$ partitions $\mathcal{P}$, we use $|X(\mathcal{P})| \leq m$. For a vertex $x \in V$, we define $\boldsymbol{\mathcal{P}}(x)$ to be the collection of all partitions $\mathcal{P}$ for which $x \in X(\mathcal{P})$. One can easily see that

$$\sum_{\mathcal{P}} |X(\mathcal{P})| = \sum_{x \in V} |\boldsymbol{\mathcal{P}}(x)|.$$

Let S be the set of vertices $x \in V$ for which $|\boldsymbol{\mathcal{P}}(x)| > (1/2)k^m$. Then

$$o(1)k^m m + o(m)k^m \geq \sum_{\mathcal{P}} |X(\mathcal{P})| = \sum_{x \in V} |\boldsymbol{\mathcal{P}}(x)| > \frac{1}{2}|S|k^m,$$

hence $|S| = o(m)$. This means that, for almost all vertices $x \in V$, we have $\mathrm{disc}(\mathcal{H}(x)_{\mathcal{P}}) = \mathrm{disc}(\mathcal{H}_{\mathcal{P}}(x)) = o(1)$ for at least $(1/2)k^m$ partitions $\mathcal{P}$. By Claim 2.12, it follows that $\mathrm{disc}(\mathcal{H}(x)) = o(1)$ for all but $o(m)$ vertices $x \in V$. We proceed similarly in order to show that $\mathrm{disc}(\mathcal{H}(x, y)) = o(1)$ for all but $o(m^2)$ pairs x, $y \in V$.

3.2. **The minimization of the number of octahedra.** The aim of this section is to show that property P_1 can be derived from property P_4 using the equivalence of the k-partite properties C_1 and C_4. We start with the following lemma.

Lemma 3.1. *Suppose a k-uniform hypergraph $\mathcal{H}$ on m vertices with density $d = d(\mathcal{H})$ is such that*

$$\left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}\right| \sim \frac{m^{2k}}{2^k k!} d^{2^k}. \tag{7}$$

Then, for almost all partitions $\mathcal{P}$ of $V = V(\mathcal{H})$,

$$\left|\binom{\mathcal{H}_{\mathcal{P}}}{K_k(2)}^{\mathrm{w}}\right| \sim \frac{n^{2k}}{2^k} d^{2^k} \tag{8}$$

holds for $n = m/k$.

Proof. Put

$$X(\mathcal{P}) = \left|\binom{\mathcal{H}_{\mathcal{P}}}{K_k(2)}^{\mathrm{w}}\right|.$$

We consider X as a r.v. on the uniform probability space of all partitions $\mathcal{P}$ of the vertex set V of $\mathcal{H}$. Clearly, we may write X as a sum of 0–1 indicator random variables as follows:

$$X = \sum_{\mathcal{K}} X_{\mathcal{K}},$$

where the sum is over all $\mathcal{K} \subset \mathcal{H}$ with $\mathcal{K} \cong K_k(2)$ and $X_{\mathcal{K}}(\mathcal{P}) = 1$ if and only if $\mathcal{K} \subset \mathcal{H}_{\mathcal{P}}$. Note that $\mathbb{P}(X_{\mathcal{K}} = 1) = \mathbb{P}(\mathcal{K} \subset \mathcal{H}_{\mathcal{P}}) = k! k^{m-2k}/k^m$. Therefore, using (7), we have

$$\mathbb{E}(X) = \sum_{\mathcal{K}} \mathbb{E}(X_{\mathcal{K}}) = \sum_{\mathcal{K}} \mathbb{P}(X_{\mathcal{K}} = 1) \sim \frac{m^{2k}}{2^k k!} d^{2^k} \frac{k! k^{m-2k}}{k^m} \sim \frac{n^{2k}}{2^k} d^{2^k}. \tag{9}$$

We now invoke a lemma that will be proved in Section 5.1. Indeed, Claim 5.2 states that

$$\left|\binom{\mathcal{G}}{K_k(2)}^{\mathrm{w}}\right| \gtrsim \frac{n^{2k}}{2^k} d^{2^k} \tag{10}$$

for all (k,k)-cylinders $\mathcal{G}$ of density d with n vertices in each of its vertex classes. Comparing (9) and (10), we deduce that the expectation of X is asymptotically equal to $\min X$, and hence

$$\mathbb{P}\left(X \geq (1+\eta)\frac{n^{2k}}{2^k} d^{2^k}\right) = o(1)$$

for any fixed $\eta > 0$. This completes the proof of Lemma 3.1.

□

We turn to the proof of the implication $P_4 \Rightarrow P_1$.

Proof of $P_4 \Rightarrow P_1$. Let $\mathcal{H}$ be a k-uniform hypergraph on m vertices such that

$$\left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}\right| \sim \frac{m^{2k}}{2^k k!} d^{2^k}, \tag{11}$$

where d is the density $d(\mathcal{H})$ of $\mathcal{H}$. Lemma 3.1 then implies that almost all vertex partitions $\mathcal{P} = (V_i)_1^k$ are such that

$$\left|\binom{\mathcal{H}_{\mathcal{P}}}{K_k(2)}^{\mathrm{w}}\right| \sim \frac{n^{2k}}{2^k} d^{2^k}, \tag{12}$$

where $n = m/k$. The implication $C_4 \Rightarrow C_1$ of Theorem 2.13 gives that, then, the (k,k)-cylinder $\mathcal{H}_{\mathcal{P}}$ is $(o(1), d(\mathcal{H}_{\mathcal{P}}))$-regular for almost all $\mathcal{P}$. We now use Fact 2.6 to conclude that $\mathcal{H}_{\mathcal{P}}$ satisfies $\mathrm{disc}(\mathcal{H}_{\mathcal{P}}) = o(1)$ for a.a. $\mathcal{P}$. We may then apply Claim 2.12 with, say, $\gamma = 1/2$, to deduce that $\mathrm{disc}(\mathcal{H}) = o(1)$; in other words, property P_1 holds for $\mathcal{H}$. □

3.3. **Proof of $P_1 \Rightarrow P_5'$.** In this section, we prove that any k-uniform hypergraph $\mathcal{H}$ with $\mathrm{disc}(\mathcal{H}) = o(1)$ must be such that any k-uniform hypergraph $\mathcal{G}$ on s vertices must occur as an induced subhypergraph of $\mathcal{H}$ as many times as one would expect if $\mathcal{H}$ were a truly random hypergraph with density d. Our proof will be based on a certain technical result, the 'subhypergraph counting formula', which will be proved in Section 6.

Proof of $P_1 \Rightarrow P_5'$. We need to show that for every given integer $s \geq 2k$, real number $\varepsilon > 0$, and density $d \in (0,1)$, there exists a real number $\delta > 0$ such that property $\boldsymbol{P_1}(\delta)$ (i.e., property $\boldsymbol{P_1}$ with $o(1)$ replaced by δ) implies property $\boldsymbol{P_5'}(\varepsilon, s)$ (i.e., property $\boldsymbol{P_5'}$ with given s and $o(1)$ replaced by $O_1(\varepsilon)$).

Let $\delta_0 = \delta_0(d,\varepsilon)$ be the positive real number determined by Corollary 6.13 and $d_0 = \min\{d, 1-d\}$. Choose $\delta > 0$ of the form $1/t^{2k}$, where $t \in \mathbb{N}$, satisfying

$$\delta = \frac{1}{t^{2k}} \leq \left(\frac{\delta_0}{2}\right)^4, \tag{13a}$$

and

$$\delta^{1/2k} = \frac{1}{t} \leq \frac{\varepsilon d_0^{\binom{s}{k}}}{100 s^2} \leq \frac{1}{s}. \tag{13b}$$

Let $m \geq m(d,\varepsilon)$ be an integer divisible by t and set $n = m/t$.

Suppose that $\mathcal{H} = \mathcal{H}_{\prec'}$ and $\mathcal{G} = \mathcal{G}_{\prec}$ are two ordered k-uniform hypergraphs such that $V(\mathcal{H}) = \{v_1 \prec' v_2 \prec' \cdots \prec' v_m\}$, $V(\mathcal{G}) = \{w_1 \prec w_2 \prec \cdots \prec w_s\}$, $d(\mathcal{H}) = d$, and $\mathrm{disc}(\mathcal{H}) \leq \delta$.

For every $i \in [t]$ set $V_i = \{v_{j+n(i-1)} \colon j \in [n]\}$ and note that $V(\mathcal{H}) = \bigcup_{i=1}^t V_i$ is a partition of $V(\mathcal{H})$. An s-tuple $\{u_1, \ldots, u_s\}$ is *crossing*, or *transversal*, if $\big|\{u_1, \ldots, u_s\} \cap V_i\big| \leq 1$ for all $i \in [t]$. Note that the number of non-crossing s-tuples is bounded from above by

$$t\binom{n}{2}\binom{m-2}{s-2} = \frac{1}{2} m \left(\frac{m}{t} - 1\right)\binom{m-2}{s-2}\frac{(s)_2}{(s)_2}$$
$$= \frac{m(m-t)}{t(m)_2}\binom{s}{2}\binom{m-2}{s-2}\frac{(m)_2}{(s)_2} \leq \frac{1}{t}\binom{s}{2}\binom{m}{s}.$$

Since the number of crossing s-tuples is $\binom{t}{s} n^s$, we have the following fact.

Fact 3.2. $\binom{m}{s} - \binom{t}{s} n^s \leq \binom{s}{2}\binom{m}{s}/t$.

For $I \subset [t]$, put $\mathcal{H}_I = \mathcal{H}\left[\bigcup_{i \in I} V_i\right]$ and observe that $\mathcal{H}_I$ is an $(|I|, k)$-cylinder. One can mimic the proof of Proposition 2.9 and obtain the following fact.

Fact 3.3. *For every $I \in [t]^k$, we have $d(\mathcal{H}_I) = d + O_1(\delta^{1/2})$ and*

$$\mathrm{disc}(\mathcal{H}_I) \leq 2\delta \times t^k = 2\delta^{1/2}.$$

Consequently, owing to Fact 2.7, the cylinder $\mathcal{H}_I$ is $(2\delta^{1/4}, d)$-regular.

Thus, the (s,k)-cylinder $\mathcal{H}_I$ satisfies the assumptions of Corollary 6.13 for every $I \in [t]^s$. Therefore, there exist $(1+O_1(\varepsilon))d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}n^s$ transversal copies of $\mathcal{G}$ in each $\mathcal{H}_I$ ($I \in [t]^s$).

Let $I = \{i_1 < i_2 < \cdots < i_s\} \in [t]^s$ and consider one transversal copy of $\mathcal{G}$ in $\mathcal{H}$ on vertices $u_1 \prec u_2 \prec \cdots \prec u_s$, where $u_j \in V_{i_j}$ for every $j \in [s]$. Then the mapping $\varphi\colon V(\mathcal{G}) \to V(\mathcal{H})$ defined by $\varphi(w_i) = u_i$ ($i \in [s]$) is an injection preserving order, and preserving edges and non-edges of $\mathcal{G}$; thus, $\varphi \in \binom{\mathcal{H}}{\mathcal{G}}_\prec$. In view of the previous paragraph, we have

$$(1+O_1(\varepsilon))d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}n^s \times \binom{t}{s} \tag{14}$$

such mappings.

On the other hand, let $\varphi \in \binom{\mathcal{H}}{\mathcal{G}}_\prec$. The s-tuple $\{\varphi(w_1) \prec' \varphi(w_2) \prec' \cdots \prec' \varphi(w_s)\}$ is either crossing or not.

In the first case, this s-tuple induces a transversal copy of $\mathcal{G}$ in $\mathcal{H}$, and, therefore, equation (14) yields the number of mappings φ for which this case occurs. In the second case, $\{\varphi(w_1) \prec' \varphi(w_2) \prec' \cdots \prec' \varphi(w_s)\}$ is not crossing. By Fact 3.2, there are at most $\binom{s}{2}\binom{m}{s}/t$ mappings φ yielding this case.

Combining these two cases together implies that

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_\prec\right| = (1+O_1(\varepsilon))d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}n^s \times \binom{t}{s} + O_1\left(\frac{1}{t}\binom{s}{2}\binom{m}{s}\right).$$

To complete the proof, it suffices to show that

$$\begin{aligned}(1+O_1(\varepsilon))d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}n^s \times \binom{t}{s} + O_1\left(\frac{1}{t}\binom{s}{2}\binom{m}{s}\right)&\\ = (1+O_1(2\varepsilon))\binom{m}{s}d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}.&\end{aligned} \tag{15}$$

Owing to (13b), we have

$$\frac{1}{t}\binom{s}{2}\binom{m}{s} < \frac{\varepsilon}{100}\binom{m}{s}d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}, \tag{16a}$$

and

$$t^s - \binom{s}{2}t^{s-1} > t^s - \frac{\varepsilon}{100}t^s. \tag{16b}$$

Since (16a) holds, (15) follows from the following inequality

$$(1-\varepsilon/2)\binom{m}{s} \le n^s\binom{t}{s} \le (1+\varepsilon/2)\binom{m}{s}.$$

While the right-hand side of this inequality is immediate, the left-hand side is a consequence of (16b). □

3.4. **Proof of $P_5' \Rightarrow P_6' \Rightarrow P_6$ and $P_5' \Rightarrow P_5 \Rightarrow P_6$.** Implications $P_5 \Rightarrow P_6$ and $P_5' \Rightarrow P_6'$ are trivial since P_6 (respectively, P_6') is a special case of P_5 (respectively, P_5'). Moreover, $P_6' \Rightarrow P_6$ is a special case of $P_5' \Rightarrow P_5$, therefore, it suffices to prove that $P_5' \Rightarrow P_5$.

Proof of $P_5' \Rightarrow P_5$. Given two hypergraphs $\mathcal{G}$ and $\mathcal{H}$, let us denote by $\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}}$ the set of injections $\varphi\colon V(\mathcal{G}) \to V(\mathcal{H})$ such that $\varphi(g) \in \mathcal{H}$ if and only if $g \in \mathcal{G}$. Moreover, we write $\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}}$ for the set of such injections such that $\varphi(g) \in \mathcal{H}$ whenever $g \in \mathcal{G}$. Thus, $\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}}$ is the set of embeddings φ of $\mathcal{G}$ into $\mathcal{H}$ such that $\varphi(V(\mathcal{G}))$ induces an isomorphic copy of $\mathcal{G}$ in $\mathcal{H}$, whereas $\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}}$ is the set of embeddings φ such that $\varphi(V(\mathcal{G}))$ induces a superhypergraph of $\mathcal{G}$ in $\mathcal{H}$. If $\mathcal{G}$ has automorphism group $\mathrm{Aut}(\mathcal{G})$, it is easy to verify that we have

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}}\right| = \left|\binom{\mathcal{H}}{\mathcal{G}}\right| \,\big|\,\mathrm{Aut}(\mathcal{G})\big|, \tag{17}$$

and similarly for $\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}}$ and $\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}$.

Suppose now that $\mathcal{H}$ is an ordered hypergraph with ordering $\prec'$. Then,

$$\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}} = \bigcup_{\prec} \binom{\mathcal{H}_{\prec'}}{\mathcal{G}_{\prec}}, \tag{18}$$

where the union ranges over the set of all total orderings $\prec$ of $V(\mathcal{G})$. Furthermore, a moment's thought shows that the union in (18) is a disjoint union. Hence

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}}\right| = \sum_{\prec} \left|\binom{\mathcal{H}_{\prec'}}{\mathcal{G}_{\prec}}\right|. \tag{19}$$

Since P_5' holds, we have

$$\left|\binom{\mathcal{H}_{\prec'}}{\mathcal{G}_{\prec}}\right| = (1+o(1))\binom{m}{s} d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}$$

for every total ordering $\prec$ of $V(\mathcal{G})$. Since there exist $s!$ total orderings of $V(\mathcal{G})$, combining (17) and (19) yields P_5. □

3.5. **Minimization of octahedra from subhypergraph counting.** We now prove that property P_6 (which concerns a certain 'subhypergraph counting formula' for induced subhypergraphs) implies property P_4 (which concerns the number of (weak) subhypergraphs isomorphic to octahedra). The proof will have two parts. In the first part we shall show that, for every hypergraph $\mathcal{H}$ with density d, the number of copies of $\mathcal{K}_k(2)$ in $\mathcal{H}$ is bounded from below by

$$(1+o(1))\frac{m^{2k}}{2^k k!} d^{2^k}$$

(see Lemma 3.5). In the second part, we shall prove that P_6 implies the asymptotic equality (3) given in property P_4 of Theorem 1.3. We start with the following lemma.

Lemma 3.4. *Suppose the k-uniform hypergraph $\mathcal{H}$ with m vertices and with density $d = d(\mathcal{H})$ is such that $\mathcal{H}_{\mathcal{P}}$ satisfies*

$$\left|\binom{\mathcal{H}_{\mathcal{P}}}{K_k(2)}^{\mathrm{w}}\right| \gtrsim \frac{n^{2k}}{2^k} d^{2^k} \tag{20}$$

for almost all partitions $\mathcal{P} = (V_i)_1^k$ of the vertex set $V = V(\mathcal{H})$, where $n = m/k$. Then

$$\left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}\right| \gtrsim \frac{m^{2k}}{2^k k!} d^{2^k}. \tag{21}$$

Proof. This lemma follows easily from a double counting argument. Let us consider the family of pairs $(\mathcal{K}, \mathcal{P})$ such that

(i) $K_k(2) \cong \mathcal{K} \subset \mathcal{H}$,
(ii) $\mathcal{P} = (V_i)_1^k$ is such that (20) holds and $|V_i| \sim n = m/k$ $(1 \le i \le k)$,

and, finally,

(iii) $\mathcal{K} \subset \mathcal{H}_{\mathcal{P}}$.

On the one hand, the number N of such pairs $(\mathcal{K}, \mathcal{P})$ is

$$\sum_{\mathcal{P}} \big|\big\{\mathcal{K}\colon (i) \text{ and } (iii) \text{ hold}\big\}\big|, \tag{22}$$

where the sum is over all $\mathcal{P}$ for which (ii) holds. Thus, because of our assumption on $\mathcal{H}$ and Fact 2.10, we have that

$$N \gtrsim k^m \frac{n^{2k}}{2^k} d^{2^k}. \tag{23}$$

On the other hand, we have that

$$N = \sum_{\mathcal{K}} \big|\big\{\mathcal{P}\colon (ii) \text{ and } (iii) \text{ hold}\big\}\big| \sim \left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}\right| k! k^{m-2k}, \tag{24}$$

where the sum is over all $\mathcal{K}$ that satisfy (i). Above, we again made use of Fact 2.10 to estimate the number of relevant partitions $\mathcal{P}$ for each fixed $\mathcal{K}$. Comparing (23) and (24), we deduce (21). □

The proof of the lower bound on the number of $\mathcal{K}_k(2)$ in $\mathcal{H}$ is straightforward now.

Lemma 3.5. *For any m-vertex k-uniform hypergraph $\mathcal{H}$ with density $d = d(\mathcal{H})$,*

$$\left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}\right| \gtrsim \frac{m^{2k}}{2^k k!} d^{2^k}. \tag{25}$$

Proof. We know that all but $o(1)k^m$ partitions $\mathcal{P}$ satisfies $|V_i| = n \sim m/k$ and $d(\mathcal{H}_{\mathcal{P}}) \sim d$ (see Fact 2.10). By Claim 5.2 we know that

$$\left|\binom{\mathcal{H}_{\mathcal{P}}}{\mathcal{K}_k(2)}\right| \gtrsim \frac{n^{2k}}{2^k} d^{2^k},$$

and, therefore, by Lemma 3.4, (25) holds. □

Proof of $P_6 \Rightarrow P_4$. Let $\mathcal{H}$ be a k-uniform hypergraph on m vertices such that, for any k-uniform hypergraph $\mathcal{G}$ on $2k$ vertices, we have

$$\left|\binom{\mathcal{H}}{\mathcal{G}}\right| \sim \binom{m}{2k} d^{e(\mathcal{G})} (1-d)^{\binom{2k}{k} - e(\mathcal{G})} \frac{(2k)!}{|\operatorname{Aut}(\mathcal{G})|}, \tag{26}$$

where $d = d(\mathcal{H})$ is the density of $\mathcal{H}$ and $\operatorname{Aut}(\mathcal{G})$ is the automorphism group of $\mathcal{G}$. From (17) it follows that, for any such $\mathcal{G}$, we have

$$\left|\binom{\mathcal{H}}{\mathcal{G}}_{\mathrm{inj}}\right| \sim (m)_{2k} d^{e(\mathcal{G})} (1-d)^{\binom{2k}{k} - e(\mathcal{G})}, \tag{27}$$

where, as usual, $(a)_b = a(a-1)\dots(a-b+1)$. We are interested in estimating $\left|\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}}\right|$. Clearly,

$$\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}} = \bigcup_{\mathcal{G}'} \binom{\mathcal{H}}{\mathcal{G}'}_{\mathrm{inj}}, \tag{28}$$

where the union ranges over all k-uniform hypergraphs $\mathcal{G}'$ with the same vertex set as $\mathcal{G}$ and $\mathcal{G}' \supset \mathcal{G}$. Furthermore, a moment's thought shows that the union in (28) is a disjoint union. Hence

$$\begin{aligned}
\left|\binom{\mathcal{H}}{\mathcal{G}}^{\mathrm{w}}_{\mathrm{inj}}\right| &= \sum_{\mathcal{G}'} \left|\binom{\mathcal{H}}{\mathcal{G}'}_{\mathrm{inj}}\right| \\
&\sim \sum_{\mathcal{G}'} (m)_{2k} d^{e(\mathcal{G}')}(1-d)^{\binom{2k}{k}-e(\mathcal{G}')} \\
&= \sum_{t\geq 0} \sum_{e(\mathcal{G}')=e(\mathcal{G})+t} (m)_{2k} d^{e(\mathcal{G})+t}(1-d)^{\binom{2k}{k}-e(\mathcal{G})-t} \\
&= \sum_{t\geq 0} \binom{\binom{2k}{k}-e(\mathcal{G})}{t} (m)_{2k} d^{e(\mathcal{G})+t}(1-d)^{\binom{2k}{k}-e(\mathcal{G})-t} \\
&= (m)_{2k} d^{e(\mathcal{G})} \sum_{t\geq 0} \binom{\binom{2k}{k}-e(\mathcal{G})}{t} d^t (1-d)^{\binom{2k}{k}-e(\mathcal{G})-t} \\
&= (m)_{2k} d^{e(\mathcal{G})}.
\end{aligned}$$

Thus,

$$\left|\binom{\mathcal{H}}{K_k(2)}^{\mathrm{w}}_{\mathrm{inj}}\right| \sim (m)_{2k} d^{2^k}.$$

It now suffices to recall the analogue of (17) for weak subhypergraphs to conclude the proof of P_4, since $|\operatorname{Aut}(K_k(2))| = k!2^k$. □

4. Proof of $C_1 \Leftrightarrow C_2$

In this section, we shall prove the equivalence of conditions C_1 and C_2 in Theorem 2.13. We start with a fairly standard proof of $C_1 \Rightarrow C_2$ (see Section 4.1), and then, in Section 4.2, we prove the converse $C_2 \Rightarrow C_1$.

4.1. Proof of $C_1 \Rightarrow C_2$.

The proof follows from the two claims below.

Claim 4.1. *Suppose $0 < \varepsilon^{1/2} < d$, $V = V_1 \cup \dots \cup V_k$, $|V_1| = \dots = |V_k| = n$, and let $\mathcal{H} = (V, E)$ be a (ε, d)-regular (k,k)-cylinder. Then for all but at most $2\varepsilon^{1/2}n$ vertices $x \in V_k$, the link $\mathcal{H}(x)$ is $(\varepsilon^{1/2}, d)$-regular.*

Proof. Let X^- be the set of all vertices $x \in V_k$ with the following property: there exists a $(k-1, k-2)$-cylinder $\mathcal{F}_x$ with $(k-1)$-partition $V_1 \cup \dots \cup V_{k-1}$ such that

$$|\mathcal{K}_{k-1}(\mathcal{F}_x)| \geq \varepsilon^{1/2} n^{k-1}, \tag{29}$$

but

$$|\mathcal{H}(x) \cap \mathcal{K}_{k-1}(\mathcal{F}_x)| < \left(d - \varepsilon^{1/2}\right) |\mathcal{K}_{k-1}(\mathcal{F}_x)|. \tag{30}$$

We also define X^+ to be the set of all vertices $x \in V_k$ satisfying (29) for which we have

$$|\mathcal{H}(x) \cap \mathcal{K}_{k-1}(\mathcal{F}_x)| \geq \left(d + \varepsilon^{1/2}\right) |\mathcal{K}_{k-1}(\mathcal{F}_x)|.$$

Suppose that $|X^-| > \varepsilon^{1/2} n$ and define a $(k, k-1)$-cylinder $\mathcal{G}$ by

$$\mathcal{G} = K_{k-1}^{(k-1)}(V_1, \dots, V_{k-1}) \cup \bigcup_{x \in X^-} \{e \cup \{x\} \colon e \in \mathcal{F}_x\}$$

Observe that

$$|\mathcal{K}_k(\mathcal{G})| = \sum_{x \in X^-} |\mathcal{K}_{k-1}(\mathcal{F}_x)| \geq |X^-| \varepsilon^{1/2} n^{k-1} \geq \varepsilon n^k,$$

and, therefore, by the regularity of $\mathcal{H}$,

$$|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| \geq (d - \varepsilon) |\mathcal{K}_k(\mathcal{G})| = (d - \varepsilon) \sum_{x \in X^-} |\mathcal{K}_{k-1}(\mathcal{F}_x)|.$$

On the other hand, from (30) we obtain

$$|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| = \sum_{x \in X^-} |\mathcal{H}(x) \cap \mathcal{K}_{k-1}(\mathcal{F}_x)| < \left(d - \varepsilon^{1/2}\right) \sum_{x \in X^-} |\mathcal{K}_{k-1}(\mathcal{F}_x)|,$$

which is a contradiction.

Hence $|X^-| \leq \varepsilon^{1/2} n$. Similarly we obtain $|X^+| \leq \varepsilon^{1/2} n$. □

Claim 4.2. *Suppose $0 < \varepsilon < 1/16$, $2\varepsilon^{1/2} < d$, $V = V_1 \cup \dots \cup V_k$, $|V_1| = \dots = |V_k| = n$, and let $\mathcal{H} = (V, E)$ be a (ε, d)-regular (k, k)-cylinder. Then, $\mathcal{H}(x, y)$ is $(\varepsilon^{1/4}, d^2)$-regular for all but at most $4\varepsilon^{1/4} n^2$ pairs of vertices x, $y \in V_k$.*

Proof. From the previous claim we know that there are at most $2\varepsilon^{1/2} n$ vertices x in V_k with $(\varepsilon^{1/2}, d)$-irregular link $\mathcal{H}(x)$. These vertices form at most $2\varepsilon^{1/2} n^2$ pairs and we shall exclude them from further considerations.

For a vertex $x \in V_k$ denote by Y_x^- the set of all vertices $y \in V_k$ with the following property: there exists a $(k-1, k-2)$-cylinder $\mathcal{F}_y$ with $(k-1)$-partition $V_1 \cup \dots \cup V_{k-1}$ such that

$$|\mathcal{K}_{k-1}(\mathcal{F}_y)| \geq \varepsilon^{1/4} n^{k-1}, \tag{31}$$

but

$$|\mathcal{H}(x, y) \cap \mathcal{K}_{k-1}(\mathcal{F}_y)| < \left(d^2 - \varepsilon^{1/4}\right) |\mathcal{K}_{k-1}(\mathcal{F}_y)|. \tag{32}$$

We also denote by Y_x^+ the set of all vertices $y \in V_k$ for which there is a $(k-1, k-2)$-cylinder $\mathcal{F}_y$ that satisfies (31), but

$$|\mathcal{H}(x, y) \cap \mathcal{K}_{k-1}(\mathcal{F}_y)| > \left(d^2 + \varepsilon^{1/4}\right) |\mathcal{K}_{k-1}(\mathcal{F}_y)|.$$

Suppose there exists a vertex $x \in V_k$ with $(\varepsilon^{1/2}, d)$-regular link $\mathcal{H}(x)$ for which $|Y_x^-| \geq \varepsilon^{1/4} n$. Define a $(k, k-1)$-cylinder $\mathcal{G}$ by

$$\mathcal{G} = \mathcal{H}(x) \cup \bigcup_{y \in Y_x^-} \{e \cup \{y\} \colon e \in \mathcal{F}_y\}.$$

Then

$$|\mathcal{K}_k(\mathcal{G})| = \sum_{y \in Y_x^-} |\mathcal{H}(x) \cap \mathcal{K}_{k-1}(\mathcal{F}_y)|.$$

Note that (31) together with the $(\varepsilon^{1/2}, d)$-regularity of $\mathcal{H}(x)$ implies that

$$|\mathcal{H}(x) \cap \mathcal{K}_{k-1}(\mathcal{F}_y)| \geq \left(d - \varepsilon^{1/2}\right) |\mathcal{K}_{k-1}(\mathcal{F}_y)|$$

for all $y \in Y_x^-$. Hence

$$|\mathcal{K}_k(\mathcal{G})| \geq \sum_{y \in Y_x^-} \left(d - \varepsilon^{1/2}\right) |\mathcal{K}_{k-1}(\mathcal{F}_y)| \geq \left(d - \varepsilon^{1/2}\right) |Y_x^-| \varepsilon^{1/4} n^{k-1} \geq \left(d - \varepsilon^{1/2}\right) \varepsilon^{1/2} n^k \geq \varepsilon n^k. \quad (33)$$

By the (ε, d)-regularity of $\mathcal{H}$ and (33), we have

$$|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| \geq (d - \varepsilon)|\mathcal{K}_k(\mathcal{G})| \geq (d - \varepsilon) \left(d - \varepsilon^{1/2}\right) \sum_{y \in Y_x^-} |\mathcal{K}_{k-1}(\mathcal{F}_y)|. \quad (34)$$

On the other hand, the size of $\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})$ can be bounded from above using (32) as follows:

$$|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| = \sum_{y \in Y_x^-} |\mathcal{H}(x, y) \cap \mathcal{K}_{k-1}(\mathcal{F}_y)| < \left(d^2 - \varepsilon^{1/4}\right) \sum_{y \in Y_x^-} |\mathcal{K}_{k-1}(\mathcal{F}_y)|. \quad (35)$$

Comparing (34) and (35), we get

$$(d - \varepsilon) \left(d - \varepsilon^{1/2}\right) < d^2 - \varepsilon^{1/4},$$

which implies

$$\varepsilon^{1/4} < \left(\varepsilon + \varepsilon^{1/2}\right) d < \varepsilon + \varepsilon^{1/2},$$

which is not true for $\varepsilon < 1/16$.

Hence we have $|Y_x^-| \leq \varepsilon^{1/4} n$. We also obtain $|Y_x^+| \leq \varepsilon^{1/4} n$ in exactly the same way. Consequently, the number of "bad" pairs is bounded by $2\varepsilon^{1/2} n^2 + 2\varepsilon^{1/4} n \times n \leq 4\varepsilon^{1/4} n^2$. □

4.2. **Proof of $C_2 \Rightarrow C_1$.** The objective of this section is to prove the following theorem.

Theorem 4.3. *For every $\delta > 0$ and $d > 0$ there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. If $\mathcal{H}$ is a $(k+1, k+1)$-cylinder with $(k+1)$-partition $V_1 \cup \cdots \cup V_{k+1}$ such that*

(i) $|V_i| = n \geq n_0$ *for all* $i \in [k+1]$,
(ii) $|\mathcal{H}| = dn^{k+1}$,
(iii) $\mathcal{H}(x)$ *is* (ε, d)*-regular for all but at most* εn *vertices* $x \in V_{k+1}$, *and*
(iv) $\mathcal{H}(x, y)$ *is* (ε, d^2)*-regular for all but at most* εn^2 *pairs* x, $y \in V_{k+1}$,

then $\mathcal{H}$ is (δ, d)-regular.

Remark. Here, we work with $(k+1, k+1)$-cylinders to simplify the notation. With this choice, we shall encounter $(k+1)$-, k-, and $(k-1)$-uniform hypergraphs.

Proof. Let $\mathcal{H}$ be a $(k+1, k+1)$-cylinder satisfying assumptions (i)–(iv). We shall assume that

$$\varepsilon = (\delta/4)^{32} < \delta < d \leq 1. \quad (36)$$

Suppose that $\mathcal{H}$ is not (δ, d)-regular, i.e., Definition 2.5 fails. Without loss of generality (by taking complements) we may assume that the second inequality in

(6) is not true, therefore, there exists a $(k+1,k)$-cylinder $\mathcal{G} = \bigcup_{i=1}^{k+1} \mathcal{G}_i$ with $(k+1)$-partition $V_1 \cup \cdots \cup V_{k+1}$ such that

$$|\mathcal{K}_{k+1}(\mathcal{G})| \geq \delta n^{k+1} \tag{37}$$

but

$$\frac{|\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G})|}{|\mathcal{K}_{k+1}(\mathcal{G})|} > d + \delta. \tag{38}$$

We now work on this 'witness' $\mathcal{G}$ for the irregularity of $\mathcal{H}$.

Fact 4.4. *There exist subcylinders $\mathcal{G}'_i \subset \mathcal{G}_i$ ($i \in [k]$) such that $\mathcal{G}^* = \mathcal{G}'_1 \cup \cdots \cup \mathcal{G}'_k \cup \mathcal{G}_{k+1}$ satisfies the following four conditions:*

1. $\mathcal{K}_k(\mathcal{G}^*(x)) = \emptyset$ *or* $|\mathcal{K}_k(\mathcal{G}^*(x))| \geq \varepsilon n^k$ *for all* $x \in V_{k+1}$,
2. $\mathcal{H}(x)$ *is* (ε, d)*-regular for all* $x \in V_{k+1}$ *with* $|\mathcal{K}_k(\mathcal{G}^*(x))| \geq \varepsilon n^k$,
3. $|\mathcal{K}_{k+1}(\mathcal{G}^*)| \geq (\delta/2) n^{k+1}$,
4. $|\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G}^*)| \geq (d + \delta/2)|\mathcal{K}_{k+1}(\mathcal{G}^*)|$.

Proof. For every vertex $x \in V_{k+1}$ with $|\mathcal{K}_k(\mathcal{G}(x))| < \varepsilon n^k$ or with (ε, d)-irregular link $\mathcal{H}(x)$, delete all edges in $\mathcal{G}$ that contain x. Notice that this operation does not remove any edge from $\mathcal{G}_{k+1}$ and produces a subhypergraph $\mathcal{G}^* = \mathcal{G}'_1 \cup \cdots \cup \mathcal{G}'_k \cup \mathcal{G}_{k+1}$ that satisfies conditions (1) and (2).

Moreover, every removal reduces the size of $\mathcal{K}_{k+1}(\mathcal{G})$ (and $\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G})$) by at most εn^k if $|\mathcal{K}_k(\mathcal{G}(x))| < \varepsilon n^k$ or by at most n^k if $\mathcal{H}(x)$ is (ε, d)-irregular. Since there are at most εn vertices x with (ε, d)-irregular link, we obtain that

$$|\mathcal{K}_{k+1}(\mathcal{G}^*)| \geq |\mathcal{K}_{k+1}(\mathcal{G})| - n \times \varepsilon n^k - \varepsilon n \times n^k,$$

and

$$|\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G}^*)| \geq |\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G})| - n \times \varepsilon n^k - \varepsilon n \times n^k,$$

The first inequality together with assumption (37) shows that

$$|\mathcal{K}_{k+1}(\mathcal{G}^*)| \geq |\mathcal{K}_{k+1}(\mathcal{G})| - 2\varepsilon n^{k+1} \geq \delta n^{k+1} - 2(\delta/4)^{32} n^{k+1} \geq (\delta/2) n^{k+1}.$$

Similarly, the second inequality, (37), and (38) yield

$$\begin{aligned} |\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G}^*)| &\geq |\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G})| - 2\varepsilon n^{k+1} \geq (d+\delta)|\mathcal{K}_{k+1}(\mathcal{G})| - 2(\delta/4)^{32} n^{k+1} \\ &\geq (d + \delta/2)|\mathcal{K}_{k+1}(\mathcal{G})| + (\delta/2)\delta n^{k+1} - 2(\delta/4)^{32} n^{k+1} \\ &\geq (d + \delta/2)|\mathcal{K}_{k+1}(\mathcal{G}^*)|, \end{aligned}$$

and the proof is complete. □

We have to work on $\mathcal{G}^*$ further to obtain a witness with more structure. We shall need the following definition.

Definition 4.5. *For each $e \in \mathcal{G}_{k+1}$ define two parameters $g(e)$ and $h(e)$ by*

$$\begin{aligned} g(e) &= \big|\{x \in V_{k+1} \colon \{x\} \cup e \in \mathcal{K}_{k+1}(\mathcal{G}^*)\}\big|, \\ h(e) &= \big|\{x \in V_{k+1} \colon \{x\} \cup e \in \mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G}^*)\}\big|. \end{aligned}$$

Fact 4.6. *Put $\delta' = \delta^2/16$. Then there exists a subcylinder $\mathcal{G}'_{k+1} \subset \mathcal{G}_{k+1}$ such that $\mathcal{G}' = \mathcal{G}'_1 \cup \cdots \cup \mathcal{G}'_k \cup \mathcal{G}'_{k+1}$, where the $\mathcal{G}'_i$ ($i \in [k]$) are taken from Fact 4.4, satisfies the following five conditions:*

1. $\mathcal{K}_k(\mathcal{G}'(x)) = \emptyset$ *or* $|\mathcal{K}_k(\mathcal{G}'(x))| \geq \varepsilon n^k$ *for all* $x \in V_{k+1}$,
2. $\mathcal{H}(x)$ *is* (ε, d)*-regular for all* $x \in V_{k+1}$ *with* $|\mathcal{K}_k(\mathcal{G}'(x))| \geq \varepsilon n^k$,
3. $|\mathcal{K}_{k+1}(\mathcal{G}')| \geq \delta' n^{k+1}$,

(4) *$h(e)/g(e) \geq d + \delta/4$ for all $e \in \mathcal{G}'_{k+1}$,*
(5) *$g(e) \geq \delta' n$ for all $e \in \mathcal{G}'_{k+1}$.*

Proof. We decompose $\mathcal{G}_{k+1}$ into two subcylinders $\mathcal{G}^*_{k+1} \cup \mathcal{G}^{**}_{k+1}$, where

$$\mathcal{G}^{**}_{k+1} = \left\{ e \in \mathcal{G}_{k+1} \colon \frac{h(e)}{g(e)} \geq d + \frac{\delta}{4} \right\}$$

and

$$\mathcal{G}^*_{k+1} = \mathcal{G}_{k+1} \backslash \mathcal{G}^{**}_{k+1}.$$

Let $\mathcal{G}^*$ be as in Fact 4.4. Observe that

$$\sum_{e \in \mathcal{G}_{k+1}} g(e) = |\mathcal{K}_{k+1}(\mathcal{G}^*)| \overset{\text{Fact } 4.4\,(3)}{\geq} (\delta/2) n^{k+1}$$

and

$$\sum_{e \in \mathcal{G}_{k+1}} h(e) = |\mathcal{H} \cap \mathcal{K}_{k+1}(\mathcal{G}^*)| \overset{\text{Fact } 4.4\,(4)}{\geq} (d + \delta/2) |\mathcal{K}_{k+1}(\mathcal{G}^*)|.$$

Then for $\mathcal{G}^{**} = \mathcal{G}'_1 \cup \cdots \cup \mathcal{G}'_k \cup \mathcal{G}^{**}_{k+1}$ we have

$$\begin{aligned}
|\mathcal{K}_{k+1}(\mathcal{G}^{**})| &= \sum_{e \in \mathcal{G}^{**}_{k+1}} g(e) \geq \sum_{e \in \mathcal{G}^{**}_{k+1}} h(e) = \sum_{e \in \mathcal{G}_{k+1}} h(e) - \sum_{e \in \mathcal{G}^*_{k+1}} h(e) \\
&> \left(d + \frac{\delta}{2} \right) |\mathcal{K}_{k+1}(\mathcal{G}^*)| - \left(d + \frac{\delta}{4} \right) \sum_{e \in \mathcal{G}_{k+1}} g(e) \\
&= \left(d + \frac{\delta}{2} \right) |\mathcal{K}_{k+1}(\mathcal{G}^*)| - \left(d + \frac{\delta}{4} \right) |\mathcal{K}_{k+1}(\mathcal{G}^*)| = \frac{\delta}{4} |\mathcal{K}_{k+1}(\mathcal{G}^*)| \geq \frac{\delta^2}{8} n^{k+1}.
\end{aligned}$$

Note that at least $(\delta^2/16) n^k$ edges e of $\mathcal{G}^{**}_{k+1}$ must have $g(e) \geq (\delta^2/16) n$, otherwise we would have

$$|\mathcal{K}_{k+1}(\mathcal{G}^*)| < \frac{\delta^2}{16} n^k \cdot n + n^k \cdot \frac{\delta^2}{16} n = \frac{\delta^2}{8} n^{k+1},$$

which would be a contradiction. Remove all edges e with $g(e) < (\delta^2/16) n$ from $\mathcal{G}^{**}_{k+1}$ and obtain $\mathcal{G}'_{k+1}$. Then, $\mathcal{G}'_{k+1}$ satisfies condition (4) because of the definition of $\mathcal{G}^{**}_{k+1} \supset \mathcal{G}'_{k+1}$. It satisfies condition (5) because all edges e with $g(e) < (\delta^2/16) n$ have been removed, and $\mathcal{G}'$ also satisfies condition (3) because

$$|\mathcal{K}_{k+1}(\mathcal{G}')| \geq |\mathcal{K}_{k+1}(\mathcal{G}^{**})| - n^k \cdot \frac{\delta^2}{16} n \geq \frac{\delta^2}{8} n^{k+1} - \frac{\delta^2}{16} n^{k+1} = \frac{\delta^2}{16} n^{k+1}.$$

Finally, $\mathcal{G}'$ must satisfy (1) and (2) because we did not change any of $\mathcal{G}'_i$ ($i \in [k]$). □

Before we come back to the proof of Theorem 4.3, we state an auxiliary lemma. Let $0 < \alpha \leq 1$ and $0 < \mu < 1$ be given. Let G be a bipartite graph with vertex classes $X_1 \cup X_2$ and let H be a subgraph of G. We call an ordered pair of vertices $(x, y) \in X_1 \times X_1$ *good* if

$$|H(x) \cap G(y)| = \alpha (1 + O_1(\mu)) |G(x, y)| \tag{39}$$

and

$$|H(x, y)| = \alpha^2 (1 + O_1(\mu)) |G(x, y)|. \tag{40}$$

We also call a pair *bad* if it is not good.

The auxiliary lemma is as follows.

Lemma 4.7 (Dementieva, Haxell, Nagle, and Rödl [8])**.** *Let $0 < \alpha \leq 1$ and $0 < \mu < 1$ be given. Suppose the bipartite graphs G and H are such that*

$$\sum_{\text{bad } (x,y)\in X_1\times X_1} |G(x,y)| < \frac{\mu\alpha^2}{(1-\alpha)^2+\alpha^2} \sum_{z\in X_2} |G(z)|^2. \tag{41}$$

Then

$$\sum_{z\in X_2} (|H(z)| - \alpha|G(z)|)^2 \leq 5\mu\alpha^2 \sum_{z\in X_2} |G(z)|^2. \tag{42}$$

The following corollary of the above lemma holds.

Corollary 4.8. *Let $0 < \alpha \leq 1$, $0 < \mu \leq 1$, and $\nu > 0$ be given. Let G and H be bipartite graphs as in Lemma 4.7. Denote by W the set of all vertices $z \in X_2$ such that $|H(z)| \geq (\alpha+\nu)|G(z)|$. Then*

$$\sum_{z\in W} |G(z)| \leq \left(5\mu/\nu^2\right)^{1/2} |X_1||X_2|. \tag{43}$$

Proof. Clearly, $5\mu\alpha^2 \sum_{z\in X_2} |G(z)|^2 \leq 5\mu\alpha^2|X_1|^2|X_2| \leq 5\mu|X_1|^2|X_2|$. On the other hand,

$$\begin{aligned}\sum_{z\in X_2} (|H(z)| - \alpha|G(z)|)^2 &\geq \sum_{z\in W} (|H(z)| - \alpha|G(z)|)^2 \geq \nu^2 \sum_{z\in W} |G(z)|^2 \\ &\geq \frac{\nu^2}{|W|}\left(\sum_{z\in W} |G(z)|\right)^2 \geq \frac{\nu^2}{|X_2|}\left(\sum_{z\in W} |G(z)|\right)^2.\end{aligned}$$

Finally, using inequality (42), we get (43). □

We now turn back to the proof of Theorem 4.3. We define two auxiliary bipartite graphs $G = (X_1 \cup X_2, E(G))$ and $H = (X_1 \cup X_2, E(H))$ in the following way:

$$\begin{aligned} X_1 &= V_{k+1}, \\ X_2 &= V_1 \times \cdots \times X_k, \\ E(G) &= \{(x,e)\colon e \in \mathcal{K}_k(\mathcal{G}'(x))\}, \\ E(H) &= \{(x,e)\colon e \in \mathcal{H}(x) \cap \mathcal{K}_k(\mathcal{G}'(x))\}. \end{aligned} \tag{44}$$

Notice that $|X_1| = n$ and $|X_2| = n^k$. Then

$$\sum_{x\in X_1} |G(x)| = \sum_{e\in X_2} |G(e)| \geq \sum_{e\in \mathcal{G}'_{k+1}} |G(e)| = |\mathcal{K}_{k+1}(\mathcal{G}')| \geq \delta' n^{k+1} = \delta'|X_1||X_2|. \tag{45}$$

Now we prove that the graphs G and H defined by (44) satisfy the assumptions of Corollary 4.8 with $\alpha = d$, $\mu = \varepsilon^{1/2}$ and $\nu = \delta/4$. Indeed, observe that

- if $|\mathcal{K}_k(\mathcal{G}'(x,y))| \geq \varepsilon n^k$ and $\mathcal{H}(x)$ is (ε, d)-regular, then $|\mathcal{H}(x) \cap \mathcal{K}_k(\mathcal{G}'(x,y))| = (d + O_1(\varepsilon))|\mathcal{K}_k(\mathcal{G}'(x,y))|$, i.e.,

  $$|H(x) \cap G(y)| = d(1 + O_1(\varepsilon^{1/2}))|G(x,y)|,$$

  and
- if $|\mathcal{K}_k(\mathcal{G}'(x,y))| \geq \varepsilon n^k$ and $\mathcal{H}(x,y)$ is (ε, d^2)-regular, then $|\mathcal{H}(x,y) \cap \mathcal{K}_k(\mathcal{G}'(x,y))| = (d^2 + O_1(\varepsilon))|\mathcal{K}_k(\mathcal{G}'(x,y))|$, i.e.,

  $$|H(x,y)| = d^2(1 + O_1(\varepsilon^{1/2}))|G(x,y)|.$$

Denote by I_1 the set of all pairs (x,y) such that $|G(x,y)| < \varepsilon|X_2|$, by I_2 the set of all pairs (x,y) such that $\mathcal{H}(x,y)$ is (ε,d^2)-irregular, and by I_3 the set of all pairs (x,y) such that $\mathcal{H}(x)$ is (ε,d)-irregular.

Both observations above imply that every pair $(x,y)\in X_1\times X_1$ such that $\mathcal{H}(x)$ is (ε,d)-regular, $\mathcal{H}(x,y)$ is (ε,d^2)-regular, and $|G(x,y)|\geq\varepsilon|X_2|$ is good. In other words, the set of bad pairs is a subset of $I_1\cup I_2\cup I_3$ and, therefore,

$$\sum_{\text{bad }(x,y)}|G(x,y)| \leq \sum_{(x,y)\in I_1}|G(x,y)| + \sum_{(x,y)\in I_2}|G(x,y)| + \sum_{(x,y)\in I_3}|G(x,y)|.$$

One can easily see that $|I_1|\leq|X_1|^2$, $|I_2|\leq\varepsilon|X_1|^2$, $|I_3|\leq\varepsilon|X_1|\times|X_1|$, and

- $\sum_{(x,y)\in I_1}|G(x,y)| \leq \varepsilon|X_2|\times|I_1| \leq \varepsilon|X_1|^2|X_2|$,
- $\sum_{(x,y)\in I_2}|G(x,y)| \leq |X_2|\times|I_2| \leq \varepsilon|X_1|^2|X_2|$,
- $\sum_{(x,y)\in I_3}|G(x,y)| \leq |X_2|\times|I_3| \leq \varepsilon|X_1|^2|X_2|$.

Consequently, $\sum_{\text{bad }(x,y)}|G(x,y)| \leq 3\varepsilon|X_1|^2|X_2|$.

Suppose that condition (41) of Lemma 4.7 is not satisfied, i.e.,

$$\sum_{\text{bad }(x,y)}|G(x,y)| \geq \frac{d^2\varepsilon^{1/2}}{(1-d)^2+d^2}\sum_{z\in X_2}|G(z)|^2.$$

Since $\sum_{z\in X_2}|G(z)|^2 \geq \left(\sum_{z\in X_2}|G(z)|\right)^2/|X_2|$, we get

$$\frac{d^2\varepsilon^{1/2}}{(1-d)^2+d^2}\left(\sum_{z\in X_2}|G(z)|\right)^2 \leq 3\varepsilon|X_1|^2|X_2|^2.$$

In other words,

$$\sum_{z\in X_2}|G(z)| \leq \left(\frac{3\varepsilon\left((1-d)^2+d^2\right)}{d^2\varepsilon^{1/2}}\right)^{1/2}|X_1||X_2|.$$

On the other hand, we know (see (45)) that $\sum_{z\in X_2}|G(z)| \geq (\delta^2/16)|X_1||X_2|$, and, therefore, comparing both inequalities yields

$$\frac{\delta^2}{16} \leq \left(\frac{3\varepsilon\left((1-d)^2+d^2\right)}{d^2\varepsilon^{1/2}}\right)^{1/2}.$$

This is a contradiction since $3\varepsilon\times\left((1-d)^2+d^2\right)/(d^2\varepsilon^{1/2}) \leq \varepsilon^{1/2}/d^2 = (\delta/4)^{16}/d^2 \leq (\delta/4)^{14} < \delta^4/256$. Thus, G and H also satisfy condition (41) of Lemma 4.7.

Set $W=\mathcal{G}'_{k+1}$. Then, because of property (3), clearly $|W|\geq(\delta^2/16)n^{k+1}/n = (\delta^2/16)n^k = (\delta^2/16)|X_2|$, and for every $e\in W$ we have $|H(e)|\geq(d+\delta/4)|G(e)|$. We apply Corollary 4.8 and obtain that

$$\sum_{z\in W}|G(z)| \leq \left(\frac{80\varepsilon^{1/2}}{\delta^2}\right)^{1/2}|X_1||X_2|.$$

On the other hand, since $|G(e)| \geq (\delta^2/16)|X_1|$ for every $e \in W$, we get

$$\sum_{z\in W} |G(z)| \geq |W| \times \frac{\delta^2}{16}|X_1| \geq \frac{\delta^4}{256}|X_1||X_2|.$$

This is a contradiction because

$$\left(\frac{80\varepsilon^{1/2}}{\delta^2}\right)^{1/2} \leq 4\left(\frac{\delta}{4}\right)^7 < \frac{\delta^4}{256}.$$

□

5. Proof of $\boldsymbol{C_2} \Leftrightarrow \boldsymbol{C_3} \Leftrightarrow \boldsymbol{C_4}$

Sections 5.1, 5.2, and 5.3 are devoted to the proofs of $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_4}$, $\boldsymbol{C_4} \Rightarrow \boldsymbol{C_3}$, and $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_2}$ (note that implication $\boldsymbol{C_2} \Rightarrow \boldsymbol{C_3}$ trivially holds). In these sections, we shall be sketchy in places because the arguments are standard or somewhat repetitive.

5.1. **Proof of $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_4}$.** We start with a standard "cherry counting lemma" for bipartite graphs (a *cherry* is a path of length 2).

Claim 5.1. *Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = n$, $|Y| = m$, and assume that $|E| = dmn$. For x, $x' \in X$, put $d_{x,x'} = |G(x, x')|/m$. Then*

$$\sum \{d_{x,x'} : x,\ x' \in X,\ x \neq x'\} \geq \binom{dn}{2}.$$

Proof. Observe first that

$$\sum_{x\in X} |G(x)| = |E| = \sum_{y\in Y} |G(y)|$$

and

$$\sum\nolimits_{x\neq x'}^{X} |G(x, x')| = \sum_{y\in Y} \binom{|G(y)|}{2},$$

where we write $\sum\nolimits_{x\neq x'}^{X}$ for the sum over all pairs $\{x, x'\}$ of distinct vertices from X. Then

$$\begin{aligned} m\sum\nolimits_{x\neq x'}^{X} d_{x,x'} &= \sum\nolimits_{x\neq x'}^{X} |G(x, x')| \\ &= \sum_{y\in Y} \binom{|G(y)|}{2} \geq m\binom{m^{-1}\sum_{y\in Y}|G(y)|}{2} = m\binom{dn}{2}, \end{aligned}$$

where, naturally, we used the convexity of $\binom{x}{2}$. □

Claim 5.2. *Let $\mathcal{H} = (V_1 \cup \cdots \cup V_k, E)$ be a (k, k)-cylinder with $|E| = dn^k$, where $|V_i| = n$ for all $i \in [k]$. Then the number of copies of $K_k(2)$ in $\mathcal{H}$ is bounded from below by $(1 + o(1))(n^{2k}/2^k)d^{2^k}$.*

Proof. We proceed by induction on k. For $k = 2$, the statement follows from the previous claim and the Cauchy–Schwarz inequality. Indeed, let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = |Y| = n$, and assume that $|E| = dn^2$. For x, $x' \in X$, put $d_{x,x'} = |G(x,x')|/n$. Here and below we use the notation $\sum_{x\neq x'}^{X}$ introduced in the proof of Claim 5.1. Then the number of copies of $K_2(2)$ in G is given by

$$\left|\binom{G}{K_2(2)}\right| = \sum\nolimits_{x\neq x'}^{X} \binom{|G(x,x')|}{2} = \sum\nolimits_{x\neq x'}^{X} \binom{d_{x,x'}n}{2}.$$

By the Cauchy–Schwarz inequality and Claim 5.1, we have

$$\begin{aligned}\left|\binom{G}{K_2(2)}\right| &= \sum\nolimits_{x\neq x'}^{X} \binom{d_{x,x'}n}{2} = (1+o(1))\binom{n}{2}\sum\nolimits_{x\neq x'}^{X} d_{x,x'}^2 \\ &\geq (1+o(1))\left(\sum\nolimits_{x\neq x'}^{X} d_{x,x'}\right)^2 \geq (1+o(1))\binom{dn}{2}^2 \qquad (46)\\ &= (1+o(1))\frac{n^4}{4}d^4.\end{aligned}$$

We now proceed to the induction step. Suppose $k \geq 3$, suppose that the claim is true for $k-1$, and let $\mathcal{H} = (V_1 \cup \cdots \cup V_k, E)$ be a (k,k)-cylinder such that $|E| = dn^k$. Consider an auxiliary bipartite graph with bipartition $X = V_k$ and $Y = V_1 \times \cdots \times V_{k-1}$ and edge set

$$E = \{(x,y) \in X \times Y : y \in \mathcal{H}(x)\}.$$

Then $|X| = n$ and $|Y| = m = n^{k-1}$. For x, $x' \in X$, put $d_{x,x'} = |G(x,x')|/m$. Using the induction assumption, $\mathcal{H}(x,x')$ contains $\gtrsim n^{2(k-1)}d_{x,x'}^{2^{k-1}}/2^{k-1}$ copies of $K_{k-1}(2)$. Furthermore, from the previous claim we know that $\sum_{x\neq x'}^{X} d_{x,x'} \geq \binom{dn}{2}$. Then

$$\begin{aligned}\left|\binom{\mathcal{H}}{K_k(2)}\right| &= \sum\nolimits_{x\neq x'}^{X} \left|\binom{\mathcal{H}(x,x')}{K_{k-1}(2)}\right| \\ &\geq (1+o(1))\sum\nolimits_{x\neq x'}^{X} n^{2(k-1)}\frac{1}{2^{k-1}}d_{x,x'}^{2^{k-1}} \\ &\geq (1+o(1))\frac{n^{2(k-1)}}{2^{k-1}}\binom{n}{2}\left(\binom{dn}{2}\Big/\binom{n}{2}\right)^{2^{k-1}} \\ &= (1+o(1))\frac{n^{2k}}{2^k}d^{2^k},\end{aligned}$$

as required. □

The proof of $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_4}$ is then straightforward.

Proof of $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_4}$. The first part (i.e., the inequality) follows from the previous claim. To obtain the asymptotic equality in the case in which the joint links are almost all (ε, d^2)-regular, we observe the following.

For $k = 2$ we use the fact that (ε, d^2)-regularity of joint links means that $d_{x,x'} \sim d^2$ for almost all pairs of vertices x, $x' \in V_k$. Then we have the asymptotic equality at every step of equation (46), which is exactly what we need to show.

For $k > 2$, since $\mathcal{H}(x,x')$ is (ε, d^2)-regular for almost all pairs of vertices x, $x' \in V_k$, by the induction assumption $\mathcal{H}(x,x')$ contains $(1+o(1))n^{2(k-1)}(d^2)^{2^{k-1}}/2^{k-1}$ copies of $K_{k-1}(2)$. Hence the number of copies of $K_k(2)$ containing x, x' is

$$(1+o(1))n^{2(k-1)}\frac{1}{2^{k-1}}(d^2)^{2^{k-1}},$$

and so, summing over all x, $x' \in X$ with $x \neq x'$, we have that the number of copies of $K_k(2)$ in $\mathcal{H}$ is

$$(1+o(1))\binom{n}{2}n^{2(k-1)}\frac{1}{2^{k-1}}(d^2)^{2^{k-1}} = (1+o(1))\frac{n^{2k}}{2^k}d^{2^k},$$

as required. □

5.2. **Proof of $C_4 \Rightarrow C_3$.** The proof of this implication will be based on Claims 5.1 and 5.2 and on a standard application of the Cauchy–Schwarz inequality.

Proof of $C_4 \Rightarrow C_3$. For $k = 2$, this implication follows from the following. Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = |Y| = n$, and $|E| = dn^2$, and assume that G contains $(1+o(1))n^4d^4/4$ copies of $K_2(2)$, i.e., $\left|\binom{G}{K_2(2)}\right| = (1+o(1))n^4d^4/4$. But then we must have equality everywhere in (46), which means that $d_{x,x'} \sim d^2$ for almost all pairs of vertices x, $x' \in X$. This shows, however, that $G(x,x')$ is (ε, d^2)-regular for almost all pairs of vertices x, $x' \in X$.

Assume now we have $k > 2$. Let $\mathcal{H} = (V_1 \cup \cdots \cup V_k, E)$ be a (k,k)-cylinder with $|V_i| = n$ for all $i \in [k]$ and $|E| = dn^k$. Suppose that $\mathcal{H}$ contains $(1+o(1))n^{2k}d^{2^k}/2^k$ copies of $K_k(2)$.

Consider an auxiliary bipartite graph with bipartition $X = V_k$ and $Y = V_1 \times \cdots \times V_{k-1}$ and edge set

$$E = \{(x,y) \in X \times Y : y \in \mathcal{H}(x)\}.$$

Then $|X| = n$ and $|Y| = m = n^{k-1}$. For x, $x' \in X$, put $d_{x,x'} = |G(x,x')|/m$. From Claim 5.1 we obtain

$$\sum\nolimits_{x\neq x'}^{X} d_{x,x'} \geq \binom{dn}{2},$$

and so

$$\begin{aligned}\sum\nolimits_{x\neq x'}^{X} d_{x,x'}^{2^{k-1}} &\geq \binom{n}{2}\left(\binom{n}{2}^{-1}\sum\nolimits_{x\neq x'}^{X} d_{x,x'}\right)^{2^{k-1}}\\ &\geq \binom{n}{2}\left(\binom{dn}{2}\Big/\binom{n}{2}\right)^{2^{k-1}} \geq (1+o(1))\frac{n^2}{2}d^{2^k}. \qquad (47)\end{aligned}$$

We apply Claim 5.2 to $\mathcal{H}(x,x')$ and obtain that it contains at least

$$(1+o(1))n^{2(k-1)}d_{x,x'}^{2^{k-1}}/2^{k-1}$$

copies of $K_{k-1}(2)$. Consequently

$$\begin{aligned}\left|\binom{\mathcal{H}}{K_k(2)}\right| &= \sum\nolimits_{x\neq x'}^{X} \left|\{\text{copies of } K_k(2) \text{ containing } x,\ x'\}\right|\\ &\geq (1+o(1))\sum\nolimits_{x\neq x'}^{X} \frac{n^{2(k-1)}}{2^{k-1}}d_{x,x'}^{2^{k-1}} \qquad (48)\\ &\overset{(47)}{\geq} (1+o(1))\frac{n^{2(k-1)}}{2^{k-1}}\frac{n^2}{2}d^{2^k}\\ &= (1+o(1))\frac{n^{2k}}{2^k}d^{2^k}.\end{aligned}$$

On the other hand, by C_4 we have that

$$\left|\binom{\mathcal{H}}{K_k(2)}\right| = (1+o(1))\frac{n^{2k}}{2^k}d^{2^k}. \tag{49}$$

From (48) and (49) we conclude that $d_{x,x'} \sim d^2$ for almost all pairs x, $x' \in X = V_k$, and therefore that $\mathcal{H}(x,x')$ contains $(1+o(1))n^{2(k-1)}(d^2)^{2^{k-1}}/2^{k-1}$ copies of $K_{k-1}(2)$. In view of the induction assumption this means that $\mathcal{H}(x,x')$ is (ε, d^2)-regular for almost all pairs x, $x' \in V_k$, i.e., C_3 holds. □

5.3. **Proof of $C_3 \Rightarrow C_2$.** We start with the following claim.

Claim 5.3. *Let $c > 0$ be a fixed constant. Let $G = (X \cup Y, E)$ be a bipartite graph with $|X| = n$, $|Y| = m$, and assume that $|G(x,x')| \sim c^2 m$ for almost all pairs x, $x' \in X$. Then $|G(x)| \sim cm$ for almost all vertices $x \in X$.*

Proof. Indeed, suppose that $|G(x)| \geq (c+\varepsilon)m$ for all vertices $x \in X'$, where $X' \subset X$ is "big". Let G' be the subgraph of G induced on $X' \cup Y$ and let $|G'(y)| = c_y|X'|$ for all $y \in Y$. Note that $|G'(x,x')| = |G(x,x')|$ for all x, $x' \in X'$. Then we have

$$\sum\nolimits_{x\neq x'}^{X'} |G(x,x')| \sim \frac{1}{2}c^2 m|X'|^2,$$

where we write $\sum\nolimits_{x\neq x'}^{X'}$ for the sum over all pairs $\{x,x'\}$ of distinct vertices from X'. On the other hand,

$$\sum\nolimits_{x\neq x'}^{X'} |G(x,x')| = \sum_{y\in Y}\binom{|G'(y)|}{2} \sim \sum_{y\in Y}\frac{1}{2}c_y^2|X'|^2.$$

Hence $\sum_{y\in Y} c_y^2 \sim c^2 m$ which implies, by the Cauchy–Schwarz inequality, that $c_y \sim c$ for almost all $y \in Y$. But then,

$$cm|X'| \sim \sum_{y\in Y} c_y|X'| = E(G') = \sum_{x\in X'}|G'(x)| \geq (c+\varepsilon)m|X'|,$$

which is a contradiction. The same applies to the set of all vertices $x \in X$ for which $|G(x)| \leq (c-\varepsilon)m$. □

Now we give a proof of the implication $C_3 \Rightarrow C_2$.

Proof of $C_3 \Rightarrow C_2$. We proceed by induction on k. For $k = 2$, the statement follows from Claim 5.3.

Let $k > 2$ be given. We shall prove that $C_3 \Rightarrow C_2$ holds for k. Thus, assume that the link $\mathcal{H}(x,y)$ is (ε, d^2)-regular for almost all x, $y \in V_k$. We shall prove that for almost all $x \in V_k$ the link $\mathcal{H}(x)$ is (ε', d)-regular, where $\varepsilon' \to 0$ as $\varepsilon \to 0$.

Consider an auxiliary bipartite graph $G = (X \cup Y, E)$ with bipartition $X = V_k$ and $Y = [V_1]^2 \times \cdots \times [V_{k-1}]^2$ and edge set

$$E = \{(x,y) \in X \times Y : y \text{ spans a copy of } K_{k-1}(2) \text{ in } \mathcal{H}(x)\}.$$

Then $|X| = n$ and $|Y| = \binom{n}{2}^{k-1} \sim n^{2(k-1)}/2^{k-1}$. Let x and x' be such that $\mathcal{H}(x,x')$ is (ε, d^2)-regular. Since $\mathcal{H}(x,x')$ is a $(k-1,k-1)$-cylinder, we may apply the implication $C_3 \Rightarrow C_4$ that we have already proved to deduce that $\mathcal{H}(x,x')$ contains $\sim n^{2(k-1)}(d^2)^{2^{k-1}}/2^{k-1} = n^{2(k-1)}d^{2^k}/2^{k-1}$ copies of $K_{k-1}(2)$. This means that almost all pairs of vertices x, $x' \in X$ have their common neighborhood of size

$|G(x,x')| \sim n^{2(k-1)}d^{2^k}/2^{k-1}$. Setting $m = \binom{n}{2}^{k-1}$ and $c = d^{2^{k-1}}$, one may apply Claim 5.3 to infer that

$$|G(x)| \sim d^{2^{k-1}} \frac{n^{2(k-1)}}{2^{k-1}} \tag{50}$$

for almost all $x \in X$. For each $x \in X$, set $d_x = |\mathcal{H}(x)|/n^{k-1}$. Using Claim 5.2, we get that

$$|G(x)| \gtrsim d_x^{2^{k-1}} \frac{n^{2(k-1)}}{2^{k-1}}$$

for all x and hence $d \gtrsim d_x$ for almost all $x \in X$. However,

$$dn^k = |\mathcal{H}| = \sum_{x\in X} |\mathcal{H}(x)| = \sum_{x\in X} d_x n^{k-1},$$

whence $dn = \sum_{x\in X} d_x$. We may conclude that $d_x \sim d$ for almost all $x \in X$.

This, in view of (50), means that $\mathcal{H}(x)$ satisfies condition C_4 for $(k-1)$-cylinders. Since $\boldsymbol{C_4} \Rightarrow \boldsymbol{C_3}$ holds for $(k-1)$-cylinders (already proved), $\boldsymbol{C_3} \Rightarrow \boldsymbol{C_2}$ holds for $(k-1)$-cylinders (induction assumption), and $\boldsymbol{C_2} \Rightarrow \boldsymbol{C_1}$ holds for $(k-1)$-cylinders (already proved), we conclude that $\mathcal{H}(x)$ is (ε', d)-regular for almost all $x \in X$, as required. □

6. Proof of the subhypergraph counting formula

The heart of the proof of $P_1 \Rightarrow P_5'$ is in proving a *counting lemma*, which we now formulate. We shall need several definitions and further notation.

Definition 6.1. *Let s and k, $s \geq k \geq 2$, be two integers. An (s,k)-complex $\boldsymbol{\mathcal{H}}$ is a system $\{\mathcal{H}^{(i)}\}_{i=1}^{k}$ such that*

(a) *$\mathcal{H}^{(1)}$ is a partition $V_1 \cup \dots \cup V_s$,*
(b) *$\mathcal{H}^{(i)}$ is an (s,i)-cylinder with s-partition $\mathcal{H}^{(1)}$ for every $1 < i \leq k$,*
(c) *$\mathcal{H}^{(i)}$ underlies $\mathcal{H}^{(i+1)}$ for every $1 \leq i < k$, i.e., $\mathcal{H}^{(i+1)} \subset \mathcal{K}_{i+1}(\mathcal{H}^{(i)})$.*

Now we define the notion of *regularity* for a (k,k)-cylinder with respect to an underlying $(k,k-1)$-cylinder.

Definition 6.2. *Let $\mathcal{G}$ be a $(k,k-1)$-cylinder underlying a (k,k)-cylinder $\mathcal{H}$. We say that $\mathcal{H}$ is (ε,d)-regular with respect to $\mathcal{G}$ if the following condition is satisfied: whenever $\mathcal{G}' \subset \mathcal{G}$ is a $(k,k-1)$-cylinder such that*

$$\left|\mathcal{K}_k(\mathcal{G}')\right| \geq \varepsilon\left|\mathcal{K}_k(\mathcal{G})\right|,$$

we have

$$(d-\varepsilon)\left|\mathcal{K}_k(\mathcal{G}')\right| \leq \left|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G}')\right| \leq (d+\varepsilon)\left|\mathcal{K}_k(\mathcal{G}')\right|.$$

Note that this definition coincides with Definition 2.5 if $k = 2$ or if $\mathcal{G}$ is the complete $(k,k-1)$-cylinder on $V_1 \cup \dots \cup V_k$. We extend the above definition to the case of (s,k)-cylinders $\mathcal{H}$.

Definition 6.3. *Let $\mathcal{G}$ be an $(s,k-1)$-cylinder underlying an (s,k)-cylinder $\mathcal{H}$. We say that $\mathcal{H}$ is (ε,d)-regular with respect to $\mathcal{G}$ if $\mathcal{H}\left[\bigcup_{j\in I} V_j\right]$ is (ε,d)-regular with respect to $\mathcal{G}\left[\bigcup_{j\in I} V_j\right]$ for all $I \in [s]^k$.*

Now we are ready to introduce the concept of regularity for an (s,k)-complex $\boldsymbol{\mathcal{H}}$.

Definition 6.4. *Let $\boldsymbol{d} = (d_2, \ldots, d_k)$ be a vector of positive real numbers such that $0 < d_i \le 1$ for all $i = 2, \ldots, k$. We say that the (s,k)-complex $\boldsymbol{\mathcal{H}}$ is $(\delta, \boldsymbol{d})$-regular if $\mathcal{H}^{(i+1)}$ is (δ, d_{i+1})-regular with respect to $\mathcal{H}^{(i)}$ for every $1 \le i < k$.*

Let $\mathcal{H}^{(k)}$ be an (s,k)-cylinder with s-partition $V_1 \cup \cdots \cup V_s$. We say that a copy of a k-uniform hypergraph $\mathcal{G} \subset \mathcal{H}^{(k)}$ is *transversal* in $\mathcal{H}^{(k)}$ if $|V(\mathcal{G}) \cap V_i| \le 1$ for all $1 \le i \le s$. Our key counting result is as follows.

Theorem 6.5. *Fix $2 \le k \le s$. For any $\varepsilon > 0$ and any $d_2, \ldots, d_k > 0$, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. If $\delta < \delta_0$ and $\boldsymbol{\mathcal{H}}$ is a $(\delta, \boldsymbol{d})$-regular (s,k)-complex on $V_1 \cup \cdots \cup V_s$, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $|V_i| = n \ge n_0$ for all i, then the number of transversal $K_s^{(k)}$ in $\mathcal{H}^{(k)}$ is $(1+O_1(\varepsilon)) d_k^{\binom{s}{k}} \ldots d_2^{\binom{s}{2}} n^s$.*

In the proof of this theorem, we shall need the following notions of "link" and "extended link" for complexes.

Definition 6.6. *Let $\boldsymbol{\mathcal{H}}$ be an (s,k)-complex on $V_1 \cup \cdots \cup V_s$, where $s \ge k$, and $x \in V_s$. We define $\boldsymbol{\mathcal{H}}(x) = \{\mathcal{H}^{(i)}(x)\}_{i=2}^{k}$ and, if $s > k$, we also set $\widetilde{\boldsymbol{\mathcal{H}}}(x) = \{\widetilde{\mathcal{H}}_x^{(i)}\}_{i=1}^{k}$, where $\widetilde{\mathcal{H}}_x^{(i)}$ $(i \in [k])$ is the i-uniform hypergraph defined by*

$$\widetilde{\mathcal{H}}_x^{(i)} = \begin{cases} \mathcal{H}^{(i+1)}(x) & \text{if } 1 \le i < k, \\ \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x)) & \text{if } i = k. \end{cases} \tag{51}$$

In (51) above, $\mathcal{H}^{(i)}(x)$ is the usual link of the vertex x in the (s,i)-cylinder $\mathcal{H}^{(i)}$, and $\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x))$ denotes the $(s-1,k)$-cylinder formed by the edges of $\mathcal{H}^{(k)}$ that are cliques in the link $\mathcal{H}^{(k)}(x)$. Note that $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ can be viewed as an extension of $\boldsymbol{\mathcal{H}}(x)$ in a sense that

$$\widetilde{\boldsymbol{\mathcal{H}}}(x) = \boldsymbol{\mathcal{H}}(x) \cup \{\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x))\}.$$

It is easy to see that $\boldsymbol{\mathcal{H}}(x)$ is an $(s-1, k-1)$-complex and $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ is an $(s-1,k)$-complex. Indeed, since $\boldsymbol{\mathcal{H}}$ is an (s,k)-complex, for $1 \le i < k$, we have $\mathcal{H}^{(i+1)} \subset \mathcal{K}_{i+1}(\mathcal{H}^{(i)})$ (cf. Definition 6.1). Hence, for every vertex $x \in V_s$, we have

$$\mathcal{H}^{(i+1)}(x) \subset \mathcal{K}_{i+1}(\mathcal{H}^{(i)}(x)),$$

and, therefore,

$$\widetilde{\mathcal{H}}_x^{(i+1)} \subset \mathcal{K}_{i+1}(\widetilde{\mathcal{H}}_x^{(i)}).$$

For $i = k$, we have

$$\widetilde{\mathcal{H}}_x^{(k)} = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x)) \subset \mathcal{K}_k(\mathcal{H}^{(k)}(x)) = \mathcal{K}_k(\widetilde{\mathcal{H}}_x^{(k-1)})$$

directly from Definition 6.6.

The proof of Theorem 6.5 is based on the following two propositions.

Proposition 6.7. *For any $2 \le k < s$, any $\tilde{\delta} > 0$, and any $d_2, \ldots, d_k > 0$, there are constants $\delta > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. Let $\boldsymbol{\mathcal{H}}$ be a $(\delta, \boldsymbol{d})$-regular (s,k)-complex on $V_1 \cup \cdots \cup V_s$, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $|V_i| = n \ge n_0$ for all $i \in [s]$, and let $\tilde{\boldsymbol{d}} = (d_2 d_3, \ldots, d_{k-1} d_k, d_k)$. Then, for all but at most $\tilde{\delta} n$ vertices $x \in V_s$, the extended link $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ is a $(\tilde{\delta}, \tilde{\boldsymbol{d}})$-regular $(s-1,k)$-complex.*

Proposition 6.8. *For any $k > 2$, any $\delta' > 0$, and any $d_2, \ldots, d_k > 0$, there are constants $\delta > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. Let $\boldsymbol{\mathcal{H}}$ be a $(\delta, \boldsymbol{d})$-regular $(k+1,k)$-complex on $V_1 \cup \cdots \cup V_{k+1}$, where $\boldsymbol{d} = (d_2, \ldots, d_k)$ and*

$|V_i| = n \geq n_0$ for all $i \in [k+1]$, and let $\boldsymbol{d}' = (d_2 d_3, \ldots, d_{k-1} d_k)$. Then, for all but at most $\delta' n$ vertices $x \in V_{k+1}$, the link $\boldsymbol{\mathcal{H}}(x)$ is a $(\delta', \boldsymbol{d}')$-regular $(k, k-1)$-complex.

For our induction to work, we shall prove Theorem 6.5 and Propositions 6.7 and 6.8 simultaneously.

Proof of Theorem 6.5 *and Propositions* 6.7 *and* 6.8. For given $s \geq k \geq 2$, we denote the statement of Theorem 6.5 by $S(s,k)$ and for $s > k \geq 2$, we denote the statement of Proposition 6.7 by $L(s,k)$. We shall prove (i)–(vi) below.

(i) The statement $S(2,2)$ is true.
(ii) The implication $S(k, k-1) \Rightarrow S(k,k)$ holds for every $k \geq 3$.
(iii) The implication $S(s,k)$, $L(s+1,k) \Rightarrow S(s+1,k)$ holds for every $s \geq k \geq 2$.
(iv) The statement $L(3,2)$ is true.
(v) The implication $L(k+1,k) \Rightarrow L(s,k)$ holds for every $s > k \geq 2$.
(vi) The implication $S(k,k-1)$, $L(k+1,k-1) \Rightarrow L(k+1,k)$ holds for every $k > 2$.

From (i)–(vi), one may easily deduce by induction (see the diagram below) that Theorem 6.5 holds for every $s \geq k \geq 2$ and Proposition 6.7 holds for every $s > k \geq 2$.

$$\begin{array}{ccccccc}
\underbrace{S(2,2),\ L(3,2)}_{(i,\ iv)} & \xrightarrow{(iii,\ v)} & S(3,2),\ L(4,2) & \xrightarrow{(iii,\ v)} & S(4,2),\ L(5,2) & \xrightarrow{(iii,\ v)} & \ldots \\
 & & \big\downarrow{\scriptstyle (ii,\ vi)} & & & & \\
 & & S(3,3),\ L(4,3) & \xrightarrow{(iii,\ v)} & S(4,3),\ L(5,3) & \xrightarrow{(iii,\ v)} & \ldots \\
 & & & & \big\downarrow{\scriptstyle (ii,\ vi)} & & \\
 & & & & S(4,4),\ L(5,4) & \xrightarrow{(iii,\ v)} & \ldots
\end{array}$$

The purpose of Proposition 6.8 is to simplify the proof of (vi) (this is also the reason why we prove this proposition for $(k+1,k)$-complexes only). Indeed, if we denote by $L'(k)$ the statement of Proposition 6.8, we shall prove the following:

(vi') The implication $S(k,k-1)$, $L(k+1,k-1) \Rightarrow L'(k)$ holds for every $k > 2$.
(vi'') The implication $S(k,k-1)$, $L'(k) \Rightarrow L(k+1,k)$ holds for every $k > 2$.

Clearly (vi') and (vi'') imply (vi).

Moreover, from (i)–(vi) one can deduce that $L'(k)$ holds for every $k > 2$, that is, Proposition 6.8 holds as well.

Now we prove statements (i)–(v), (vi'), and (vi'').

(i) **(Proof of $\boldsymbol{S(2,2)}$)** Statement $S(2,2)$ follows directly from the definition of regularity: a (δ, d_2)-regular $(2,2)$-cylinder $\mathcal{H}^{(2)}$ contains $(d_2 + O_1(\delta))n^2$ edges.

(ii) **(Proof of $\boldsymbol{S(k,k-1) \Rightarrow S(k,k)}$)** Suppose now that $S(k,k-1)$ is true for some $k \geq 3$ and let $\boldsymbol{\mathcal{H}}$ be a $(\delta, \boldsymbol{d})$-regular (k,k)-complex. Observe first that $\{\mathcal{H}^{(i)}\}_{i=1}^{k-1}$ forms a $(\delta, (d_2, \ldots, d_{k-1}))$-regular $(k, k-1)$-complex; therefore, if $\delta \ll \varepsilon'$, the number of transversal $K_k^{(k-1)}$ in $\mathcal{H}^{(k-1)}$ is $(1 + O_1(\varepsilon'))d_{k-1}^{\binom{k}{k-1}} \ldots d_2^{\binom{k}{2}} n^k$. Furthermore, we know that $\mathcal{H}^{(k)}$ is (δ, d_k)-regular with respect to $\mathcal{H}^{(k-1)}$. In particular, this means that

$$(d_k-\delta)\left|\mathcal{K}_k(\mathcal{H}^{(k-1)})\right| \le \left|\mathcal{H}^{(k)}\cap\mathcal{K}_k(\mathcal{H}^{(k-1)})\right| \le (d_k+\delta)\left|\mathcal{K}_k(\mathcal{H}^{(k-1)})\right|.$$

Since $\left|\mathcal{H}^{(k)}\cap\mathcal{K}_k(\mathcal{H}^{(k-1)})\right|$ counts the number of transversal $K_k^{(k)}$ in $\mathcal{H}^{(k)}$, we conclude that this number is $(1+O_1(\varepsilon))d_k^{\binom{k}{k}}d_{k-1}^{\binom{k}{k-1}}\dots d_2^{\binom{k}{2}}n^k$, if $\varepsilon'\ll\varepsilon$ and $\delta\ll\min\{\varepsilon,d_k\}$.

(iii) **(Proof of $S(s,k)$, $L(s+1,k)\Rightarrow S(s+1,k)$)** Assume that $S(s,k)$ and $L(s+1,k)$ are true for some $s\ge k\ge 2$ and consider a $(\delta,\boldsymbol{d})$-regular $(s+1,k)$-complex $\boldsymbol{\mathcal{H}}$ on $V_1\cup\dots\cup V_{s+1}$, where $|V_i|=n\gg n_0$ for all $i\in[s+1]$ and n_0 is a large positive integer.

From $L(s+1,k)$ we know that $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ is a $(\tilde\delta,\tilde{\boldsymbol{d}})$-regular (s,k)-cylinder for all but $\tilde\delta n$ vertices $x\in V_{s+1}$, as long as $\delta\ll\tilde\delta$. From $S(s,k)$ we immediately have that $\mathcal{H}^{(k)}\cap\mathcal{K}_k(\mathcal{H}^{(k)}(x))$ contains

$$(1+O_1(\varepsilon'))d_k^{\binom{s}{k}}(d_kd_{k-1})^{\binom{s}{k-1}}\dots(d_3d_2)^{\binom{s}{2}}(d_2n)^s$$

transversal $K_s^{(k)}$ for any such 'good' x if $\tilde\delta\ll\varepsilon'$. Each such transversal $K_s^{(k)}$ in $\mathcal{H}^{(k)}\cap\mathcal{K}_k(\mathcal{H}^{(k)}(x))$ together with x span a transversal $K_{s+1}^{(k)}$ in $\mathcal{H}^{(k)}$. Hence, the total number of transversal $K_{s+1}^{(k)}$ in $\mathcal{H}^{(k)}$ is bounded from below by

$$\begin{aligned}(1-\varepsilon')(1-\tilde\delta)nd_k^{\binom{s}{k}}&(d_kd_{k-1})^{\binom{s}{k-1}}\dots(d_3d_2)^{\binom{s}{2}}(d_2n)^s\\ &=(1-\varepsilon')(1-\tilde\delta)d_k^{\binom{s}{k}+\binom{s}{k-1}}\dots d_2^{\binom{s}{2}+\binom{s}{1}}n^{s+1}\\ &\le(1-\varepsilon)d_k^{\binom{s+1}{k}}\dots d_2^{\binom{s+1}{2}}n^{s+1}.\end{aligned}\tag{52}$$

For the upper bound we get

$$\begin{aligned}(1+\varepsilon')nd_k^{\binom{s}{k}}&(d_kd_{k-1})^{\binom{s}{k-1}}\dots(d_3d_2)^{\binom{s}{2}}(d_2n)^s+\tilde\delta n\times n^s\\ &=(1+\varepsilon')d_k^{\binom{s}{k}+\binom{s}{k-1}}\dots d_2^{\binom{s}{2}+\binom{s}{1}}n^{s+1}+\tilde\delta n^{s+1}\\ &\le(1+\varepsilon)d_k^{\binom{s+1}{k}}\dots d_2^{\binom{s+1}{2}}n^{s+1},\end{aligned}\tag{53}$$

provided that $\max\{\varepsilon',\tilde\delta\}\ll\min\{\varepsilon,d_2,\dots,d_k\}$.

(iv) **(Proof of $L(3,2)$)** Let $\mathcal{H}^{(2)}$ be a (δ,d_2)-regular $(3,2)$-cylinder and, for $x\in V_3$ and $i=1,2$, set $V_i'=\mathcal{H}_i^{(2)}(x)$. It follows from Claim 4.1 that $(d_2-\delta)n\le|V_i'|\le(d_2+\delta)n$, $i=1,2$, for all but $2\delta^{1/2}n$ vertices $x\in V_3$. We shall show that $\mathcal{H}^{(2)}[V_1'\cup V_2']$ is $(2\delta^{1/2},d_2)$-regular.

Indeed, let $U_1\subset V_1'$, $U_2\subset V_2'$, such that $|\mathcal{K}_2(U_1\cup U_2)|\ge 2\delta^{1/2}|V_1'||V_2'|$. Note that $2\delta^{1/2}|V_1'||V_2'|\ge 2\delta^{1/2}(d_2-\delta)^2n^2\ge\delta n^2=\delta|V_1||V_2|$, where we used the fact that $\delta\ll d_2$. The (δ,d_2)-regularity of $\mathcal{H}^{(2)}$ concludes the argument.

(v) **(Proof of $L(k+1,k)\Rightarrow L(s,k)$)** This fact follows from the simple observation that every (s,k)-complex $\boldsymbol{\mathcal{H}}$ can be viewed as a union of $\binom{s-1}{k}$ many $(k+1,k)$-complexes that contain V_s.

(vi′) **(Proof of $S(k,k-1)$, $L(k+1,k-1)\Rightarrow L'(k)$)** Assume that statements $S(k,k-1)$ and $L(k+1,k-1)$ are true and let $\boldsymbol{\mathcal{H}}=\{\mathcal{H}^{(i)}\}_{i=1}^k$ be a $(\delta, \boldsymbol{d})$-regular $(k+1,k)$-complex on $V_1 \cup \dots \cup V_{k+1}$, where $|V_i| = n \gg n_0$ for all $i \in [k+1]$ and n_0 is a large positive integer.

To prove $L'(k)$, it suffices to show that for all but at most $\delta' n$ vertices $x \in V_{k+1}$, the link $\boldsymbol{\mathcal{H}}(x)$ is a $(\delta', \boldsymbol{d}')$-regular $(k, k-1)$-complex, where $\boldsymbol{d}' = (d_2 d_3, \dots, d_{k-1} d_k)$ and $\delta' \to 0$ as $\delta \to 0$.

Observe first that $\{\mathcal{H}^{(i)}\}_{i=1}^{k-1}$ is a $(\delta, (d_2, d_3, \dots, d_{k-1}))$-regular $(k+1, k-1)$-complex. Thus, we can apply statement $L(k+1, k-1)$ on $\{\mathcal{H}^{(i)}\}_{i=1}^{k-1}$ and obtain that (cf. (51))

(a) $\{\mathcal{H}^{(i)}(x)\}_{i=2}^{k-1}$ is a $(\tilde{\delta}, (d_2 d_3, \dots, d_{k-2} d_{k-1}))$-regular $(k, k-2)$-complex, and

(b) $\mathcal{H}^{(k-1)} \cap \mathcal{K}_k(\mathcal{H}^{(k-1)}(x))$ is $(\tilde{\delta}, d_{k-1})$-regular with respect to $\mathcal{H}^{(k-1)}(x)$

for all but at most $\tilde{\delta} n$ vertices $x \in V_{k+1}$,where $\tilde{\delta} \to 0$ as $\delta \to 0$.

Hence, the only thing remaining to prove statement $L'(k)$ is the regularity of $\mathcal{H}^{(k)}(x)$ with respect to $\mathcal{H}^{(k-1)}(x)$. We do this by showing that for all but $4k\tilde{\delta}^{1/2} n$ vertices $x \in V_{k+1}$ satisfying (a) and (b), the link $\mathcal{H}^{(k)}(x)$ is $(2\tilde{\delta}^{1/2}, d_{k-1} d_k)$-regular with respect to $\mathcal{H}^{(k-1)}(x)$. Consequently, $\boldsymbol{\mathcal{H}}(x)$ is a $(\delta', \boldsymbol{d}')$-regular for all but at most $\delta' n$ vertices $x \in V_{k+1}$, where $\delta' = \tilde{\delta} + 4k\tilde{\delta}^{1/2} \to 0$ as $\delta \to 0$.

Suppose that there exist $t \geq 2\tilde{\delta}^{1/2} n$ vertices $x_1, \dots, x_t$ satisfying (a) and (b) for which $\mathcal{H}^{(k)}(x_i)$ is not $(2\tilde{\delta}^{1/2}, d_{k-1} d_k)$-regular with respect to $\mathcal{H}^{(k-1)}(x_i)$, $i \in [t]$. More precisely, suppose that for every $i \in [t]$, there exists a $(k-1, k-2)$-cylinder $\mathcal{G}_i \subset \mathcal{H}^{(k-1)}(x_i)$ such that

$$|\mathcal{K}_{k-1}(\mathcal{G}_i)| \geq 2\tilde{\delta}^{1/2} |\mathcal{K}_{k-1}(\mathcal{H}^{(k-1)}(x_i))| \tag{54a}$$

and

$$|\mathcal{H}^{(k)}(x_i) \cap \mathcal{K}_{k-1}(\mathcal{G}_i)| < \big(d_{k-1} d_k - 2\tilde{\delta}^{1/2}\big) |\mathcal{K}_{k-1}(\mathcal{G}_i)|. \tag{54b}$$

Suppose further that these $(k-1, k-2)$-cylinders have $(k-1)$-partition $V_1 \cup \dots \cup V_{k-1}$. We define a $(k, k-1)$-cylinder $\mathcal{G}$ by

$$\mathcal{G} = \mathcal{H}^{(k-1)}[V_1 \cup \dots \cup V_{k-1}] \cup \bigcup_{i=1}^{t} \{x_i \cup e \colon e \in \mathcal{G}_i\}.$$

It is easy to see that

$$|\mathcal{K}_k(\mathcal{G})| = \sum_{i=1}^{t} |\mathcal{H}^{(k-1)} \cap \mathcal{K}_{k-1}(\mathcal{G}_i)| \tag{55a}$$

and

$$|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})| = \sum_{i=1}^{t} |\mathcal{H}^{(k)}(x_i) \cap \mathcal{K}_{k-1}(\mathcal{G}_i)|. \tag{55b}$$

We combine equations (54b) and (55b) and obtain

$$|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})| < \big(d_{k-1} d_k - 2\tilde{\delta}^{1/2}\big) \sum_{i=1}^{t} |\mathcal{K}_{k-1}(\mathcal{G}_i)|.$$

On the other hand, we shall show that

$$|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})| \geq \left(d_{k-1}d_k - 2\tilde{\delta}^{1/2}\right) \sum_{i=1}^{t} |\mathcal{K}_{k-1}(\mathcal{G}_i)|, \tag{56}$$

which will be a contradiction. Thus, $t < 2\tilde{\delta}^{1/2}n$. The same applies to the cases in which all the $(k-1, k-2)$-complexes $\mathcal{G}_i$ have the same $(k-1)$-partition $V_1 \cup \dots \cup V_k \setminus V_j$ for some $j \in [k]$, or when we consider the opposite inequality in (54b).

Now we deduce (56). Using assumption (b), (54a), $\mathcal{G}_i \subset \mathcal{H}^{(k-1)}(x_i)$, and $2\tilde{\delta}^{1/2} \geq \tilde{\delta}$, we obtain that for every $i \in [t]$,

$$\begin{aligned} |\mathcal{H}^{(k-1)} \cap \mathcal{K}_{k-1}(\mathcal{G}_i)| &\geq (d_{k-1} - \tilde{\delta})|\mathcal{K}_{k-1}(\mathcal{G}_i)| \\ &\overset{(54a)}{\geq} 2\tilde{\delta}^{1/2}(d_{k-1} - \tilde{\delta})|\mathcal{K}_{k-1}(\mathcal{H}^{(k-1)}(x_i))|. \end{aligned} \tag{57}$$

Consequently, combining (55a) and (57) yields

$$|\mathcal{K}_k(\mathcal{G})| \geq 2\tilde{\delta}^{1/2}(d_{k-1} - \tilde{\delta}) \sum_{i=1}^{t} |\mathcal{K}_{k-1}(\mathcal{H}^{(k-1)}(x_i))|. \tag{58}$$

For $k > 3$, by (a), the $(k, k-2)$-complex $\{\mathcal{H}^{(i)}(x)\}_{i=2}^{k-1}$ restricted on $V_1 \cup \dots \cup V_{k-1}$ is a $(\tilde{\delta}, (d_2d_3, \dots, d_{k-2}d_{k-1}))$-regular $(k-1, k-2)$-complex, and hence by $S(k-1, k-2)$ we have

$$\begin{aligned} \left|\mathcal{K}_{k-1}\left(\mathcal{H}^{(k-1)}(x_i)\right)\right| &= (1 + O_1(1/4))(d_{k-1}d_{k-2})^{\binom{k-1}{k-2}} \dots (d_3d_2)^{\binom{k-1}{2}}(d_2n)^{k-1} \\ &= (1 + O_1(1/4))d_{k-1}^{\binom{k-1}{k-2}} d_{k-2}^{\binom{k}{k-2}} \dots d_2^{\binom{k}{2}} n^{k-1}, \end{aligned}$$

provided that $\tilde{\delta} \ll \min\{d_2, \dots, d_{k-1}, 1/4\}$. It follows from (δ, d_2)-regularity of $\mathcal{H}^{(2)}$ that this equation holds also for $k = 3$. We may assume that $S(k-1, k-2)$ is true since this has already been verified in our inductive proof of $S(k, k-1)$ (see the proof diagram above). Hence,

$$\begin{aligned} |\mathcal{K}_k(\mathcal{G})| &\geq t \times 2\tilde{\delta}^{1/2}(d_{k-1} - \tilde{\delta}) \times (1 - 1/4)d_{k-1}^{\binom{k-1}{k-2}} d_{k-2}^{\binom{k}{k-2}} \dots d_2^{\binom{k}{2}} n^{k-1} \\ &\geq 2\tilde{\delta} d_{k-1}^{\binom{k}{k-1}} d_{k-2}^{\binom{k}{k-2}} \dots d_2^{\binom{k}{2}} n^k. \end{aligned} \tag{59}$$

Since, $\{\mathcal{H}^{(i)}[V_1 \cup \dots \cup V_{k-1} \cup V_{k+1}]\}_{i=1}^{k-1}$ is a $(\delta, (d_2, \dots, d_{k-1}))$-regular $(k, k-1)$-complex, using $S(k, k-1)$, we obtain

$$\left|\mathcal{K}_k\left(\mathcal{H}^{(k-1)}[V_1 \cup \dots \cup V_{k-1} \cup V_{k+1}]\right)\right| = (1 + O_1(1/4))d_{k-1}^{\binom{k}{k-1}} \dots d_2^{\binom{k}{2}} n^k, \tag{60}$$

provided that $\delta \ll \min\{d_2, \dots, d_{k-1}, 1/4\}$.

Combining (59) and (60) yields

$$|\mathcal{K}_k(\mathcal{G})| \geq \tilde{\delta} \left|\mathcal{K}_k\left(\mathcal{H}^{(k-1)}[V_1 \cup \dots \cup V_{k-1} \cup V_{k+1}]\right)\right|. \tag{61}$$

We apply (δ, d_k)-regularity of $\mathcal{H}^{(k)}$ with respect to $\mathcal{H}^{(k-1)}$ and obtain

$$|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})| \geq (d_k - \delta)|\mathcal{K}_k(\mathcal{G})|. \tag{62}$$

Putting equations (55a), (57), and (62) together yields

$$|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})| \geq (d_k - \delta) \times (d_{k-1} - \tilde{\delta}) \sum_{i=1}^{t} |\mathcal{K}_{k-1}(\mathcal{G}_i)|$$
$$\geq \left(d_k d_{k-1} - 2\tilde{\delta}^{1/2}\right) \sum_{i=1}^{t} |\mathcal{K}_{k-1}(\mathcal{G}_i)|,$$

provided that $\delta \ll \tilde{\delta}$.

(vi″) (Proof of $S(k, k-1)$, $L'(k) \Rightarrow L(k+1, k)$) Assume that statements $S(k, k-1)$ and $L'(k)$ are true and let $\boldsymbol{\mathcal{H}}$ be a $(\delta, \boldsymbol{d})$-regular $(k+1, k)$-complex on $V_1 \cup \cdots \cup V_{k+1}$, where $|V_i| = n \gg n_0$ forl all $i \in [k+1]$ and n_0 is a large positive integer.

To prove $L(k+1, k)$, we need to show that for all but at most $\tilde{\delta}n$ vertices $x \in V_{k+1}$, the extended link $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ is a $(\tilde{\delta}, \tilde{\boldsymbol{d}})$-regular (k, k)-complex, where $\tilde{\boldsymbol{d}} = (d_2 d_3, \ldots, d_{k-1} d_k, d_k)$ and $\tilde{\delta} \to 0$ as $\delta \to 0$.

Our assumption that $L'(k)$ is true means that $\{\widetilde{\mathcal{H}}_x^{(i)}\}_{i=1}^{k-1} = \{\mathcal{H}^{(i)}(x)\}_{i=2}^{k} = \boldsymbol{\mathcal{H}}(x)$ is a $(\delta', (d_2 d_3, \ldots, d_{k-1} d_k))$-regular $(k, k-1)$-complex for all but at most $\delta' n$ vertices $x \in V_{k+1}$, where $\delta' \to 0$ as $\delta \to 0$.

Hence, the only thing remaining to prove $L(k+1, k)$ is the regularity of $\widetilde{\mathcal{H}}_x^{(k)} = \mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x))$ with respect to $\widetilde{\mathcal{H}}_x^{(k-1)} = \mathcal{H}^{(k)}(x)$ for almost all vertices $x \in V_{k+1}$. We prove this by showing that $\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x))$ is $(2{\delta'}^{1/2}, d_k)$-regular with respect to $\mathcal{H}^{(k)}(x)$ for every $x \in V_{k+1}$ for which the link $\boldsymbol{\mathcal{H}}(x)$ is $(\delta', (d_2 d_3, \ldots, d_{k-1} d_k))$-regular. Then, $\widetilde{\boldsymbol{\mathcal{H}}}(x)$ is a $(\tilde{\delta}, \tilde{\boldsymbol{d}})$-regular for all but at most $\tilde{\delta}n$ vertices $x \in V_{k+1}$, where $\tilde{\delta} = 2{\delta'}^{1/2} \to 0$ as $\delta \to 0$.

Suppose that $\mathcal{G}$ is a $(k, k-1)$-cylinder, $\mathcal{G} \subset \mathcal{H}^{(k)}(x)$, such that $|\mathcal{K}_k(\mathcal{G})| \geq 2{\delta'}^{1/2} |\mathcal{K}_k(\mathcal{H}^{(k)}(x))|$. We need to show that

$$(d_k - 2{\delta'}^{1/2}) \, |\mathcal{K}_k(\mathcal{G})| \leq \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x)) \cap \mathcal{K}_k(\mathcal{G})\right| \leq (d_k + 2{\delta'}^{1/2}) \, |\mathcal{K}_k(\mathcal{G})| \, . \quad (63)$$

Since $\mathcal{G} \subset \mathcal{H}^{(k)}(x)$ and, therefore, $\mathcal{K}_k(\mathcal{G}) \subset \mathcal{K}_k(\mathcal{H}^{(k)}(x))$, we have

$$\left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{H}^{(k)}(x)) \cap \mathcal{K}_k(\mathcal{G})\right| = \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})\right| . \quad (64)$$

Consequently, (63) is simply

$$(d_k - 2{\delta'}^{1/2}) \, |\mathcal{K}_k(\mathcal{G})| \leq \left|\mathcal{H}^{(k)} \cap \mathcal{K}_k(\mathcal{G})\right| \leq (d_k + 2{\delta'}^{1/2}) \, |\mathcal{K}_k(\mathcal{G})| \, . \quad (65)$$

Observe first that

- since $\{\mathcal{H}^{(i)}(x)\}_{i=2}^{k}$ is a $(\delta', (d_2 d_3, \ldots, d_{k-1} d_k))$-regular $(k, k-1)$-complex, by $S(k, k-1)$ we have

$$\left|\mathcal{K}_k\left(\mathcal{H}^{(k)}(x)\right)\right| = (1 + O_1(1/4))(d_k d_{k-1})^{\binom{k}{k-1}} \ldots (d_3 d_2)^{\binom{k}{2}} (d_2 n)^k, \quad (66)$$

  provided that $\delta' \ll \min\{d_2, \ldots, d_k, 1/4\}$;
- similarly, $\{\mathcal{H}^{(i)}[V_1 \cup \cdots \cup V_k]\}_{i=1}^{k-1}$ is a $(\delta, (d_2, \ldots, d_{k-1}))$-regular $(k, k-1)$-complex; thus, using $S(k, k-1)$ again,

$$\left|\mathcal{K}_k\left(\mathcal{H}^{(k-1)}[V_1 \cup \cdots \cup V_k]\right)\right| = (1 + O_1(1/4)) d_{k-1}^{\binom{k}{k-1}} \ldots d_2^{\binom{k}{2}} n^k, \quad (67)$$

  provided that $\delta \ll \min\{d_2, \ldots, d_k, 1/4\}$.

Now we use equations (66) and (67) to derive (65). Indeed,

$$|\mathcal{K}_k(\mathcal{G})| \geq 2{\delta'}^{1/2}|\mathcal{K}_k(\mathcal{H}^{(k)}(x))| \overset{(66)}{\geq} 2\delta^{1/2}(1-1/4)(d_k d_{k-1})^{\binom{k}{k-1}} \dots (d_3 d_2)^{\binom{k}{2}} (d_2 n)^k$$
$$\geq \delta(1+1/4) d_{k-1}^{\binom{k}{k-1}} \dots d_2^{\binom{k}{2}} n^k \overset{(67)}{\geq} \delta \left| \mathcal{K}_k\left(\mathcal{H}^{(k-1)}[V_1 \cup \dots \cup V_k]\right)\right|, \quad (68)$$

provided that $d_k^{\binom{k}{k-1}} \dots d_3^{\binom{k}{2}} d_2^k \geq \delta^{1/2}$. This last condition is satisfied since we assume that $\delta \ll \min\{d_2, \dots, d_k\}$. Finally, the (δ, d_k)-regularity of $\mathcal{H}^{(k)}$ with respect to $\mathcal{H}^{(k-1)}$ gives (65), as long as $\delta \ll \delta'$.

□

In Definition 6.3, we assumed that for every $I \in [s]^k$, the restriction $\mathcal{H}\left[\bigcup_{j\in I} V_j\right]$ is (ε, d)-regular with respect to $\mathcal{G}\left[\bigcup_{j\in I} V_j\right]$. In other words, the density d_I of the subgraph $\mathcal{H}\left[\bigcup_{j\in I} V_j\right]$ is roughly the same for every $I \in [s]^k$. Now we allow different values of d_I $(I \in [s]^k)$ and state a straightforward extension of Theorem 6.5. We start with some definitions.

Definition 6.9. *Let $\mathcal{G}$ be an $(s, k-1)$-cylinder underlying an (s,k)-cylinder $\mathcal{H}$ and let $\vec{d} = (d_I)_{I\in[s]^k}$ be a list of $\binom{s}{k}$ positive real numbers d_I, where $0 < d_I \leq 1$. We say that $\mathcal{H}$ is $(\varepsilon, \vec{d})$-regular with respect to $\mathcal{G}$ if $\mathcal{H}\left[\bigcup_{j\in I} V_j\right]$ is (ε, d_I)-regular with respect to $\mathcal{G}\left[\bigcup_{j\in I} V_j\right]$ for all $I \in [s]^k$.*

Definition 6.10. *For every integer i $(2 \leq i \leq k)$ let $\vec{d_i} = (d_I)_{I\in[s]^i}$ be a list of $\binom{s}{i}$ positive real numbers d_I, where $0 < d_I \leq 1$, and put $\boldsymbol{\vec{d}} = (\vec{d_2}, \dots, \vec{d_k})$. We say that the (s,k)-complex $\boldsymbol{\mathcal{H}}$ is $(\delta, \boldsymbol{\vec{d}})$-regular if $\mathcal{H}^{(i+1)}$ is $(\delta, \vec{d}_{i+1})$-regular with respect to $\mathcal{H}^{(i)}$ for every $1 \leq i < k$.*

Now we are ready to state an extension of Theorem 6.5.

Corollary 6.11. *Fix $2 \leq k \leq s$. For any $\varepsilon > 0$ and any $\vec{d_2}, \dots, \vec{d_k}$ as described in Definition 6.10, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. If $\delta < \delta_0$ and $\boldsymbol{\mathcal{H}}$ is a $(\delta, \boldsymbol{\vec{d}})$-regular (s,k)-complex on $V_1 \cup \dots \cup V_s$, where $|V_i| = n \geq n_0$ for all i, then the number of transversal $K_s^{(k)}$ in $\mathcal{H}^{(k)}$ is $(1 + O_1(\varepsilon)) \prod_{i=2}^{k} \prod_{I\in[s]^i} d_I \times n^s$.*

The proof of this corollary follows the lines of the proof of Theorem 6.5 and we omit it here. For us, the most interesting case occurs when all underlying cylinders are complete, that is $d_I = 1$ for every $I \in [s]^i$ and $2 \leq i < k$. In this case, the number of transversal $K_s^{(k)}$ in $\mathcal{H}^{(k)}$ is $(1 + O_1(\varepsilon)) \prod_{I\in[s]^k} d_I \times n^s$. We restate this observation in the following corollary.

Corollary 6.12. *Fix $2 \leq k \leq s$. For any $\varepsilon > 0$ and any list $\vec{d} = (d_I)_{I\in[s]^k}$ of $\binom{s}{k}$ positive real numbers d_I, where $0 < d_I \leq 1$, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. If $\delta < \delta_0$ and $\mathcal{H}$ is a $(\delta, \vec{d})$-regular (s,k)-cylinder on $V_1 \cup \dots \cup V_s$, where $|V_i| = n \geq n_0$ for all $i \in [s]$, then the number of transversal $K_s^{(k)}$ in $\mathcal{H}$ is $(1 + O_1(\varepsilon)) \prod_{I\in[s]^k} d_I \times n^s$.*

Let $\mathcal{G}$ be an arbitrary k-uniform hypergraph on s vertices $v_1, \dots, v_s$. We define an (s,k)-cylinder $\widetilde{\mathcal{H}}$ in the following way. For every $I \in [s]^k$, we set

$$\widetilde{\mathcal{H}}\left[\bigcup_{i\in I} V_i\right] = \begin{cases} \mathcal{H}\left[\bigcup_{i\in I} V_i\right] & \text{if } \{v_i : i \in I\} \in E(\mathcal{G}), \\ \bar{\mathcal{H}} = \left(\prod_{i\in I} V_i\right) \setminus \mathcal{H}\left[\bigcup_{i\in I} V_i\right] & \text{otherwise.} \end{cases}$$

Observe that every transversal copy of $\mathcal{G}$ in $\mathcal{H}$ corresponds to exactly one transversal copy of $\mathcal{K}_s^{(k)}$ in $\widetilde{\mathcal{H}}$. Consequently, applying the the previous corollary on $\widetilde{\mathcal{H}}$, we deduce the following counting formula.

Corollary 6.13 (**Subhypergraph counting formula**). *Fix $2 \le k \le s$. For any $\varepsilon > 0$ and any $0 < d \le 1$, there exist $\delta_0 > 0$ and $n_0 \in \mathbb{N}$ for which the following assertion holds. If the hypergraph $\mathcal{H}$ is a (δ, d)-regular (s,k)-cylinder on $V_1 \cup \dots \cup V_s$, where $|V_i| = n \ge n_0$ for all $i \in [s]$ and $\delta < \delta_0$, and $\mathcal{G}$ is an arbitrary k-uniform hypergraph on s vertices, then the number of transversal $\mathcal{G}$ in $\mathcal{H}$ is $(1 + O_1(\varepsilon))d^{e(\mathcal{G})}(1-d)^{\binom{s}{k}-e(\mathcal{G})}n^s$.*

Clearly, one may generalize Corollary 6.13 above to the case in which the (s,k)-cylinder $\mathcal{H}$ has a non-constant density vector $\vec{d} = (d_I)_{I\in[s]^k}$.

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