Optimal Trading Strategies Under Arbitrage

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ABSTRACT

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This thesis analyzes models of financial markets that incorporate the possibility of arbitrage opportunities. The first part demonstrates how explicit formulas for optimal trading strategies in terms of minimal required initial capital can be derived in order to replicate a given terminal wealth in a continuous-time Markovian context. Towards this end, only the existence of a square-integrable market price of risk (rather than the existence of an equivalent local martingale measure) is assumed. A new measure under which the dynamics of the stock price processes simplify is constructed. It is shown that delta hedging does not depend on the "no free lunch with vanishing risk" assumption. However, in the presence of arbitrage opportunities, finding an optimal strategy is directly linked to the non-uniqueness of the partial differential equation corresponding to the Black-Scholes equation. In order to apply these analytic tools, sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration. The phenomenon of "bubbles," which has been a popular topic in the recent academic literature, appears as a special case of the setting in the first part of this thesis. Several examples at the end of the first part illustrate the techniques contained

therein.

In the second part, a more general point of view is taken. The stock price processes, which again allow for the possibility of arbitrage, are no longer assumed to be Markovian, but rather only Itô processes. We then prove the Second Fundamental Theorem of Asset Pricing for these markets: A market is complete, meaning that any bounded contingent claim is replicable, if and only if the stochastic discount factor is unique. Conditions under which a contingent claim can be perfectly replicated in an incomplete market are established. Then, precise conditions under which relative arbitrage and strong relative arbitrage with respect to a given trading strategy exist are explicated. In addition, it is shown that if the market is quasi-complete, meaning that any bounded contingent claim measurable with respect to the stock price filtration is replicable, relative arbitrage implies strong relative arbitrage. It is further demonstrated that markets are quasi-complete, subject to the condition that the drift and diffusion coefficients are measurable with respect to the stock price filtration.

Contents

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Chapter 1

Outline of Thesis

This thesis consists of two main chapters. Chapter 2 treats Markovian market models. We illustrate how optimal trading strategies can be computed using the classical idea of delta hedging. This chapter generalizes the results in Fernholz and Karatzas (2010) and contains the results of the paper

> Ruf, J. (2011+). Hedging under arbitrage. *Mathematical Finance*, forthcoming.

That paper focuses on replicating European-style contingent claims. Chapter 2 contains additional results related to "optimizing" a given trading strategy.

The models studied in Chapter 2 can be considered complete, meaning that any contingent claim studied here can be replicated. It is also of interest, however, to study conditions under which a contingent claim can be generated in an incomplete model, which is the subject of Chapter 3. More specifically, it is shown that the question of completeness and the question of existence of arbitrage can be addressed separately from one another. Chapter 3 is based on the paper

> Ruf, J. (2011). Completeness and arbitrage.

Both chapters deal with markets that do not satisfy the "no free lunch with vanishing risk" (NFLVR) assumption, which is typically a standard assumption in the literature, and which is discussed in the introductions of the following two chapters. The replicability of contingent claims is also studied in both chapters. Chapter 2 restricts its analysis to Markovian models, for which explicit formulas for the replication can be easily derived. Chapter 3 derives existence results for a more general class of models.

The following two chapters are self-contained and therefore exhibit some redundancies. Notational inconsistencies in the two chapters have been minimized, but not entirely eliminated.

The work presented in this thesis was primarily motivated by a desire to better understand the models studied in Stochastic Portfolio Theory (SPT). SPT is not predicated upon the no-arbitrage assumption, but instead models financial markets and studies the existence of arbitrage opportunities that arise; see the survey paper by Bob Fernholz, who developed the field, and by Ioannis Karatzas, a major contributor, for an overview of the recent developments in Fernholz and Karatzas (2009).

In some sense, because it demonstrates that the concept of a "price" exists even in markets studied by SPT, which may allow for the presence of arbitrage opportunities, this thesis unifies SPT and the classical theory of Financial Mathematics. Furthermore, and as previously mentioned, the characterization of replicable claims and the Second Fundamental Theorem of Asset Pricing, which connects the replicability of any contingent claim in the economy with the uniqueness of some "pricing operator," can be proven without having to exclude arbitrage.

Thus, SPT has clarified which assumptions are necessary and which assumptions are extraneous for relevant tasks in Mathematical Finance, such as the pricing and replication of contingent claims. In particular, the assumption of NFLVR is often, despite its mathematical convenience, too strong an assumption. As discussed in Chapter 2, excluding “unbounded profits with bounded risks” is often a sufficient assumption, and also leads to a pricing measure, but one which may no longer be equivalent to the original one.

On the other hand, the many strong tools developed, in particular by Freddy Delbaen and Walter Schachermayer in the 1990s, for the so-called classical no-arbitrage theory, proved invaluable to this thesis' development. Many of the proofs within this thesis, especially those in Chapter 3, begin by transforming models with possible arbitrage opportunities into the no-arbitrage framework, in order to thereafter apply the powerful tools of the classical theory of Financial Mathematics.

The academic literature has documented that the no-arbitrage condition is not necessary for the existence of well-defined option prices. Karatzas et al. (1991b) were among the first to characterize replicable claims via duality methods, which they accomplish without relying on the existence of equivalent local martingale measures. Eckhard Platen developed the Benchmark Approach to Mathematical Finance, which establishes the “real-world pricing formula,” which also does not require an arbitrage-free market, as one of its important concepts; see Platen (2006). Furthermore, in Section 10 of the previously mentioned survey article by Fernholz and Karatzas (2009), the martingale representation theorem is applied, which yields the result that claims can be replicated under certain conditions. This thesis builds on all of these results and generalizes them to less restrictive assumptions.

Chapter 2

The Markovian Case

2.1 Introduction

In a financial market, an investor usually has several trading strategies at her disposal to obtain a given wealth at a specified point in time. For example, if the investor wanted to cover a short-position in a given stock tomorrow at the cheapest cost today, buying the stock today is generally not optimal, as there may be a trading strategy requiring less initial capital that still replicates the exact stock price tomorrow. In this chapter, we show that optimal trading strategies, in the sense of minimal required initial capital, can be represented as delta hedges. We generalize the results of Fernholz and Karatzas (2010)'s paper "On optimal arbitrage," in which specifically the market portfolio is examined, to a wide class of terminal wealths that can be optimally replicated by delta hedges.

We shall not restrict ourselves only to markets satisfying the "No free lunch with vanishing risk" (NFLVR) or, more precisely, the "No arbitrage for general admissible integrands" (NA) condition.[1] Thus, we cannot rely on the existence

[1] We refer the reader to Delbaen and Schachermayer (2006) for a thorough introduction to NA, NFLVR and other notions of arbitrage. Since we shall assume the existence of a square-integrable market price of risk, we implicitly impose the condition that NFLVR fails if and only if NA fails;

of an equivalent local martingale measure, which we otherwise would have done. However, we shall construct another probability measure to take the place of the "risk-neutral" measure. We do not exclude arbitrage a priori for several reasons. First, we cannot always assume the existence of a statistical test that relies upon stock price observations to determine whether an arbitrage opportunity is present, as illustrated in Example 3.7 of Karatzas and Kardaras (2007). In such a situation, a typical agent, who needs to rely on a path-by-path analysis, would not be aware of an arbitrage opportunity and could consequently not benefit from it. Second, examining possible arbitrage opportunities, rather than excluding them a priori, is of interest in itself. Further arguments and empirical evidence supporting the consideration of models without an equivalent local martingale measure are discussed in Section 0.1 of Kardaras (2008) and Section 1 of Platen and Hulley (2008). A model of economic equilibrium for such models is provided in Loewenstein and Willard (2000a). In the spirit of these papers, we shall impose some restrictions on the arbitrage opportunities and exclude a priori models that imply "unbounded profit with bounded risk," which can be recognized by a typical agent.

This chapter is set in the framework of Stochastic Portfolio Theory. For an overview of this field, we recommend the reader consult the monograph by Fernholz (2002) and the survey paper by Fernholz and Karatzas (2009). This chapter contributes to Stochastic Portfolio Theory a clearer understanding of pricing and hedging and its relation to several other current research directions, such as the Benchmark Approach, developed by Eckhard Platen and co-authors in a series of papers. Indeed, we generalize some of the Benchmark Approach's results here and provide tools to compute the so-called "real-world prices" of contingent claims under that approach. The monograph by Platen and Heath (2006) provides an excellent overview of the Benchmark Approach .

see Proposition 3.2 of Karatzas and Kardaras (2007).

Stochastic Portfolio Theory is a suitable framework for studying so-called "relative arbitrage" opportunities: Given a specific strategy, are there other strategies that outperform the original one? A related important observation is made in Fernholz et al. (2005): If one assumes the market to be diverse, that is, if no company can take over the whole market, and to have a bounded volatility structure, then a relative arbitrage opportunity with respect to the market portfolio exists. The existence of relative arbitrage would not conflict per se with the NFLVR assumption as, for example, the existence of admissible suicide strategies in arbitrage-free markets[2] shows. Another example is a stock price that is a strict local martingale. Then there exists a relative arbitrage opportunity with respect to this stock. From this point of view, it seems artificial that one should exclude relative arbitrage with respect to the money market and we shall also explicitly study some models in which such arbitrage is possible. Here, our analysis extends parts of the work done by Delbaen and Schachermayer (1995a) about the Bessel process and its reciprocal. Depending on which process is chosen to model the stock price, either there is arbitrage possible with respect to the money market or there is arbitrage possible with respect to the stock. Both cases can be treated in a uniform way provided that one abstains from making the NFLVR assumption.

There have been several recent papers treating the subject of "bubbles;" a very incomplete list consists of the work by Loewenstein and Willard (2000b), Cox and Hobson (2005), Heston et al. (2007), Jarrow et al. (2007; 2010), Pal and Protter (2010), and Ekström and Tysk (2009). A bubble is usually defined within a model that guarantees NFLVR as the difference between the market price of a tradeable asset and its smallest hedging price. A given asset has a bubble if and only if there exists a relative arbitrage opportunity with respect to this asset. The analysis here includes the case of bubbles, but is more general, as it also allows for

[2] See Section 6.1 of Harrison and Pliska (1981) or Section 1.2 of Karatzas and Shreve (1998) for an example.

models that imply arbitrage: Models including bubbles rely on an equivalent local martingale measure, which differs from models that allow for arbitrage, in which a more complicated change of measure not relying on Girsanov's theorem is necessary. To wit, while the bubbles literature concentrates on a single stock whose price process is modeled as a strict local martingale, we consider markets with several assets with the stochastic discount factor itself being represented by a (possibly strict) local martingale. In the presence of an asset with a bubble, our contribution is limited to the bubble's representation as a relative arbitrage opportunity and to the explicit representation of the optimal replicating strategy. We also discuss the reciprocal of the three-dimensional Bessel process as the standard example for a bubble.

Two phenomena, in particular, have been repeatedly discussed in the bubbles literature: The lack of a unique solution of the corresponding Black-Scholes PDE for an asset and the failure of the classic put-call parity; see, for example, Cox and Hobson (2005). Both these observations also hold for the more general arbitrage situation. We characterize the hedging price as the minimal nonnegative solution for the Black-Scholes PDE and suggest a modified put-call parity, which generalizes to models with arbitrage opportunities.

We set up our analysis in a continuous-time Markovian context; to wit, we focus on stock price processes whose mean rates of return and volatility coefficients only depend on time and on the current market configuration. Furthermore, we concentrate, on (possibly time-inhomogeneous) strategies that depend only on the current stock prices. This restriction to a Markovian model is certainly not the most general one, but it provides us with a rich setup, which provides valuable insight into the most interesting strategies. For such a model and a given Markovian trading strategy, we find an optimal strategy, by which we mean an investment decision rule that uses minimal initial capital but that, nevertheless, leads to the identical terminal wealth as produced by the original strategy. Since we do not rely on a martingale representation theorem, we can allow for a larger number of driving Brownian motions than the number of stocks, which generalizes the ideas of Fernholz and Karatzas (2010) to not only a larger set of strategies, but also to a broader set of models for the specific case of the market portfolio.

Next, we prove that a classical delta hedge yields the the cheapest hedging strategy for European contingent claims. This is of course well-known in the case where an equivalent local martingale measure exists and is extended here to models that allow for arbitrage opportunities and that are not necessarily complete. In this context, we provide sufficient conditions to ensure the differentiability of the hedging price, generalizing results by Heath and Schweizer (2000), Janson and Tysk (2006), and Ekström and Tysk (2009). This set of conditions is also applicable to models satisfying the NFLVR assumption. Because the computations for the optimal trading strategy under the “real-world” measure are often too involved and because we cannot always rely on an equivalent local martingale measure, we derive a non-equivalent change of measure and formulas based thereon, as illustrated, for instance, by a new generalized Bayes’ rule.

The next section introduces the market model, discusses different notions of arbitrage, and contains an initial result concerning the independence of some price candidates from the choice of the market price of risk. Section 2.3, after defining strategies and their associated wealth processes, concludes the discussion of arbitrage. In Section 2.4, we present some of our first main results, including (1) the precise representation of an optimal strategy designed to either replicate a given wealth process or hedge a non path-dependent European claim and (2) sufficient conditions for the differentiability of the hedging price. A modified put-call parity follows directly. In Section 2.5, we prove the next main result of this chapter, which is a change to a non-equivalent probability measure that simplifies computations.

This section also contains several other corollaries such as a generalized Bayes' rule and a discussion of a change-of-numéraire technique. Section 2.6 then provides several examples that illustrate various aspects of the chapter's results and Section 2.7 draws the conclusions. Finally, Section 2.8 serves as an appendix to the chapter and discusses a sufficient condition for a technical assumption made in Section 2.4.

2.2 Market model and market price of risk

In this section, we introduce the market model, discuss the existence of a "market price of risk" and define the stochastic discount factor. We assume the perspective of a small investor who takes positions in a frictionless financial market with finite time horizon T in order to accumulate wealth or hedge a financial claim. By "small" we mean that the investor's trading activities have no impact on prices. Equivalently, the investor is a "price-taker" and the stock prices are given exogenously.

We use the notation $\mathbb{R}^d_+ := \{s = (s_1, \ldots, s_d)^\mathsf{T} \in \mathbb{R}^d, s_i > 0, \text{ for all } i = 1, \ldots, d\}$ and assume a market in which the stock price processes are modeled as positive continuous Markovian semimartingales. That is, we consider a financial market $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^\mathsf{T}$ of the form

$$dS_i(t) = S_i(t) \left(\mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t) \right) \tag{2.1}$$

for all $i = 1, \ldots, d$ and $t \in [0, T]$ starting at $S(0) \in \mathbb{R}^d_+$ and a money market $B(\cdot)$. Here $\mu : [0, T] \times \mathbb{R}^d_+ \to \mathbb{R}^d$ denotes the mean rate of return and $\sigma : [0, T] \times \mathbb{R}^d_+ \to \mathbb{R}^{d \times K}$ denotes the volatility. We assume that both functions are measurable.

For the sake of convenience we only consider discounted (forward) prices and set the interest rate constant to zero; that is, $B(\cdot) \equiv 1$. The flow of information is modeled as a right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$ such that $W(\cdot) = (W_1(\cdot), \ldots, W_K(\cdot))^\mathsf{T}$ is a K-dimensional Brownian motion with independent components. In Section 2.5, we impose more conditions on the filtration $\mathbb{F}$ and the underlying probability space Ω. For the moment, we assume that all stochastic integrals that appear are measurable with respect to the filtration $\mathbb{F}$. The underlying measure and its expectation shall be denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively. The current state of the market $S(0)$ should be clear from the context and so we shall omit specifying $S(0)$ as an index for measures and expectations in most cases.

We only consider those mean rates of return $\mu(\cdot,\cdot)$ and volatilities $\sigma(\cdot,\cdot)$ that imply the stock prices $S_1(\cdot),\cdots,S_d(\cdot)$ exist and are unique and strictly positive. More precisely, denoting the covariance process of the stocks in the market by $a(\cdot,\cdot) \equiv \sigma(\cdot,\cdot)\sigma^{\mathsf{T}}(\cdot,\cdot)$, that is,

$$a_{i,j}(t,S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t,S(t))\sigma_{j,k}(t,S(t))$$

for all $i,j = 1,\ldots,d$ and $t \in [0,T]$, we impose the almost sure integrability condition

$$\sum_{i=1}^{d} \int_0^T \left(|\mu_i(t,S(t))| + a_{i,i}(t,S(t))\right) dt < \infty.$$

Under this condition, the stock prices $S_1(\cdot),\ldots,S_d(\cdot)$ can be expressed as

$$\begin{aligned} S_i(t) =& S_i(0)\exp\left(\int_0^t \left(\mu_i(u,S(u)) - \frac{1}{2}a_{i,i}(u,S(u))\right) du + \right. \\ &\left. + \sum_{k=1}^{K} \int_0^t \sigma_{i,k}(u,S(u))dW_k(u)\right) > 0 \end{aligned} \tag{2.2}$$

for all $i = 1,\ldots,d$ and $t \in [0,T]$. Furthermore, we assume the existence of a market price of risk, which generalizes the concept of the Sharpe ratio to several dimensions by setting the risk factors $W_k(\cdot)$ in relation to the mean rates of return $\mu_i(\cdot,\cdot)$.

Definition 1 (Market price of risk). A market price of risk is a progressively measurable process $\theta(\cdot)$, which maps the volatility structure $\sigma(\cdot,\cdot)$ onto the mean rate of return $\mu(\cdot,\cdot)$. That is,

$$\mu(t,S(t)) = \sigma(t,S(t))\theta(t) \tag{2.3}$$

holds almost surely for all $t \in [0,T]$.

Furthermore, we assume that $\theta(\cdot)$ is square-integrable, to wit,

$$\int_0^T \|\theta(t)\|^2 dt < \infty \tag{2.4}$$

almost surely. The existence of a market price of risk is a central assumption in both the Benchmark Approach (see Chapter 10 of Platen and Heath 2006) and in Stochastic Portfolio Theory (see Section 6 of Fernholz and Karatzas 2009). This assumption enables us to discuss hedging prices, as we do throughout this thesis. Similar assumptions have been discussed in the economic literature. For example, in the terminology of Loewenstein and Willard (2000a), the existence of a square-integrable market price of risk excludes “cheap thrills” but not necessarily “free snacks.” Theorem 2 of Loewenstein and Willard (2000a) shows that a market with a square-integrable market price of risk is consistent with an equilibrium where agents prefer more to less. We discuss the connection between a market price of risk and its square-integrability with various no-arbitrage notions in Remark 1 below.

Based on the market price of risk, we can now define the stochastic discount factor as

$$Z^\theta(t) := \exp\left(-\int_0^t \theta^\mathsf{T}(u)dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 du\right) \tag{2.5}$$

with dynamics

$$dZ^\theta(t) = -\theta^\mathsf{T}(t)Z^\theta(t)dW(t) \tag{2.6}$$

for all $t \in [0, T]$. In classical no-arbitrage theory, $Z^\theta(\cdot)$ represents the Radon-Nikodym derivative that translates the “real-world” measure into the generic “risk-neutral” measure with the money market as the underlying. Since we do not want to impose NFLVR a priori in this thesis, but are rather interested in situations in which NFLVR does not necessarily hold, we shall not assume that the stochastic discount factor $Z^\theta(\cdot)$ is a true martingale. Thus, we can only rely on a local martingale property of $Z^\theta(\cdot)$. Cases where $Z^\theta(\cdot)$ is only a local martingale have, for example, been discussed by Karatzas et al. (1991b), Schweizer (1992), in the Benchmark Approach starting with Platen (2002) and Heath and Platen (2002a;b) and in Stochastic Portfolio Theory; see, for example, Fernholz et al. (2005) and especially, Fernholz and Karatzas (2010). On the other hand, much effort has been made to strengthen Novikov (1972)'s condition to ensure that the stochastic discount factor $Z^\theta(\cdot)$ be a true martingale, for example by Wong and Heyde (2004), Hulley and Platen (2009), Mijatović and Urusov (2009), and the literature therein.

A market price of risk $\theta(\cdot)$ does not have to be uniquely determined. Uniqueness is intrinsically connected to completeness, as we shall see in Chapter 3, and we need not assume it. In general, infinitely many market prices of risk may exist. To illustrate, think of a model with $d = 1$, $K = 2$, $\mu(\cdot, \cdot) \equiv 0$ and $\sigma(\cdot, \cdot) \equiv (1, 1)$. Then, for any $y \in \mathbb{R}$, the constant process $\theta(\cdot) \equiv (-y, y)^\mathsf{T}$ is a square-integrable market price of risk. Another example of this non-uniqueness follows the next proposition.

We observe that the existence of a square-integrable market price of risk implies the existence of a Markovian square-integrable market price of risk. To see this, we define $\theta(\cdot, \cdot) := \sigma^\mathsf{T}(\cdot, \cdot)(\sigma(\cdot, \cdot)\sigma^\mathsf{T}(\cdot, \cdot))^\dagger\mu(\cdot, \cdot)$, where $\dagger$ denotes the Moore-Penrose pseudo-inverse of a matrix. Given the existence of any market price of risk, we know from the theory of least-squares estimation that $\theta(\cdot, \cdot)$ is also a market price of risk. Furthermore, we have $\|\theta(t, S(t))\|^2 \leq \|\nu(t)\|^2$ for all $t \in [0, T]$ almost surely for any market price of risk $\nu(\cdot)$, which yields the square-integrability of $\theta(\cdot, \cdot)$.

The next proposition shows that any square-integrable Markovian market price of risk maximizes the random variable that will later be a candidate for a hedging price. We denote by $\mathcal{F}^S(\cdot)$ the augmented filtration generated by the stock price process. We emphasize that the next result only holds so long as the "terminal payoff" M is $\mathcal{F}^S(T)$-measurable. We generalize the following proposition in Theorem 6 of Chapter 3.

Proposition 1 (Role of Markovian market price of risk)**.** *Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T) \subset \mathcal{F}(T)$. Let $\nu(\cdot)$ denote any square-integrable market price of risk and $\theta(\cdot,\cdot)$ any square-integrable Markovian market price of risk. Then, with*

$$M^\nu(t) := \mathbb{E}\left[\frac{Z^\nu(T)}{Z^\nu(t)}M\middle|\mathcal{F}(t)\right] \text{ and } M^\theta(t) := \mathbb{E}\left[\frac{Z^\theta(T)}{Z^\theta(t)}M\middle|\mathcal{F}(t)\right]$$

for $t \in [0,T]$, where we take the right-continuous modification[3] *for each process, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely. Furthermore, if both $Z^\nu(\cdot)$ and $Z^\theta(\cdot)$ are $\mathcal{F}^S(T)$-measurable, then $Z^\nu(T) \leq Z^\theta(T)$ almost surely.*

Proof. Due to the right-continuity of $M^\nu(\cdot)$ and $M^\theta(\cdot)$ it suffices to show for all $t \in [0,T]$ that $M^\nu(t) \leq M^\theta(t)$ almost surely. We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$. For the sequence of stopping times

$$\tau_n := T \wedge \inf\left\{t \in [0,T] : \int_0^t c^2(s)ds \geq n\right\},$$

where $n \in \mathbb{N}$, we set $c^n(\cdot) := c(\cdot)\mathbf{1}_{\{\tau_n \geq \cdot\}}$ and observe that

$$\begin{aligned}\frac{Z^\nu(T)}{Z^\nu(t)} =& \frac{Z^c(T)}{Z^c(t)} \cdot \exp\left(-\int_t^T \theta^\mathsf{T}(u, S(u))(dW(u) + c(u)du)\right.\\ &\left.- \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right)\\ =& \lim_{n\to\infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)}\\ &\cdot \exp\left(-\int_t^T \theta^\mathsf{T}(u,S(u))(dW(u) + c^n(u)du) - \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right)\end{aligned}$$

with $Z^c(\cdot)$ and $Z^{c^n}(\cdot)$ defined as in (2.5). The limit holds almost surely since both $v(\cdot)$ and $\theta(\cdot,\cdot)$ are square-integrable, which again yields the square-integrability of

[3] *See Theorem 1.3.13 of Karatzas and Shreve (1991).*

$c(\cdot)$. Since $\int_0^T c_n^2(t)dt \leq n$, Novikov's Condition (see Proposition 3.5.12 of Karatzas and Shreve 1991) yields that $Z^{c^n}(\cdot)$ is a martingale. Now, Fatou's lemma, Girsanov's theorem and Bayes' rule (see Chapter 3.5 of Karatzas and Shreve 1991) yield

$$M^\nu(t) \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[\exp\left(-\int_t^T \theta^\mathsf{T}(u,S(u))dW^n(u) - \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right) M \middle| \mathcal{F}(t)\right], \tag{2.7}$$

where $d\mathbb{Q}^n(\cdot) := Z^{c^n}(T)d\mathbb{P}(\cdot)$ is a probability measure, $\mathbb{E}^{\mathbb{Q}^n}$ its expectation operator, and $W^n(\cdot) := W(\cdot) + \int_0^\cdot c^n(u)du$ a K-dimensional $\mathbb{Q}^n$-Brownian motion. Since $\sigma(\cdot, S(\cdot))c^n(\cdot) \equiv 0$ we can replace $W(\cdot)$ by $W^n(\cdot)$ in (2.1). This yields that the process $S(\cdot)$ has the same dynamics under $\mathbb{Q}^n$ as under $\mathbb{P}$. Furthermore, both $\theta(\cdot, S(\cdot))$ and M have, as functionals of $S(\cdot)$, the same distribution under $\mathbb{Q}^n$ as under $\mathbb{P}$. Therefore, we can replace the expectation operator $\mathbb{E}^{\mathbb{Q}^n}$ by $\mathbb{E}$ in (2.7) and obtain the first part of the statement. The last inequality of the statement follows from setting $M = \mathbf{1}_{\{Z^\nu(T)>Z^\theta(T)\}}$ and observing that M must equal zero almost surely. □

We remark that the inequality $M^\nu(\cdot) \leq M^\theta(\cdot)$ can be strict. As an example, choose $M = 1$ and a market with one stock and two Brownian motions, to wit, $d = 1$ and $K = 2$. We set $\mu(\cdot,\cdot) \equiv 0$, $\sigma(\cdot,\cdot) \equiv (1,0)$ and observe that $\theta(\cdot, S(\cdot)) \equiv (0,0)^\mathsf{T}$ is a Markovian market price of risk. Another market price of risk $\nu(\cdot) \equiv (\nu_1(\cdot), \nu_2(\cdot))^\mathsf{T}$ is defined via $\nu_1(\cdot) \equiv 0$, the stochastic differential equation

$$d\nu_2(t) = -\nu_2^2(t)dW_2(t)$$

for all $t \in [0,T]$ and $\nu_2(0) = 1$. That is, $\nu_2(\cdot)$ is the reciprocal of a three-dimensional Bessel process starting at one. Itô's formula yields $Z^\nu(\cdot) \equiv \nu_2(\cdot)$, which is a strict local martingale (see Exercise 3.3.36 of Karatzas and Shreve 1991), and thus $M^\nu(0) = \mathbb{E}[Z^\nu(T)] < 1 = \mathbb{E}[Z^\theta(T)] = M^\theta(0)$.

Under the assumption that an equivalent local martingale measure exists, Theorem 12 of Jacka (1992), Theorem 3.2 of Ansel and Stricker (1993) or Theorem 16 Delbaen and Schachermayer (1995c) show that a contingent claim can be hedged if and only if the supremum over all expectations of the terminal value of the contingent claim under all equivalent local martingale measures is a maximum. In our setup, we also observe that the supremum over all $M^{\tilde{\nu}}(0)$ in the last proposition is a maximum, attained by any Markovian market price of risk. Indeed, we shall prove in Theorem 2 that, under weak analytic assumptions, claims of the form $M = p(S(T))$ can be hedged. The general theory lets us conjecture that all claims measurable with respect to $\mathcal{F}^S(T)$ can be hedged. Theorem 6 of Chapter 3 confirms this conjecture.

As pointed out by Ioannis Karatzas in a personal communication (2010), Proposition 1 might be related to the "Markovian selection results," as in Krylov (1973), Section 4.5 of Ethier and Kurtz (1986), and Chapter 12 of Stroock and Varadhan (2006). There, the existence of a Markovian solution for a martingale problem is studied. It is observed that a supremum over a set of expectations indexed by a family of distributions is attained and the maximizing distribution is a Markovian solution of the martingale problem. This potential connection needs to be worked out in a future research project.

From this point forward, we shall always assume the market price of risk to be Markovian. As we shall see, this choice will lead directly to the optimal trading strategy.

Remark 1 (Market price of risk and NA, NUPBR, NIA). Proposition 3.6 of Delbaen and Schachermayer (1994) shows (compare also Proposition 3.2 of Karatzas and Kardaras 2007) that NFLVR holds, if and only if NA and "no unbounded profit with bounded risk" (NUPBR) hold. NUPBR is also known as "arbitrage of the first kind" (compare Ingersoll 1987; Kardaras and Platen 2009) and as the "BK property" (compare Kabanov 1997; Kardaras 2010, Proposition 1.2). If NUPBR holds, then, in particular, scalable arbitrage opportunities do not exist.

The existence of a square-integrable market price of risk guarantees the existence of a positive stochastic discount factor, which again ensures that NUPBR holds as it is proven in Theorem 3.12 of Karatzas and Kardaras (2007). Moreover, since it is shown in Lemma 3.1 of Delbaen and Schachermayer (1995b) that NA holds, if and only if "no immediate arbitrage" (NIA) holds and the possibility to make some profit using a credit line is excluded. However, since immediate arbitrage is again scalable we can also conclude that NUPBR implies NIA. Therefore, if NUPBR holds, then NFLVR fails, if and only if the second component of NA fails, to wit, if and only if it is possible to make some profit using a credit line. Indeed, the application of this chapter's results to the optimal hedging problem of a bond serves to quantify exactly how much "some profit" is in a given model.

On the other hand, a careful analysis of Section 10 in Karatzas et al. (1991a) or Theorems 3.5 and 3.6 in Delbaen and Schachermayer (1995b), using the fact that the ranges of $\sigma(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are identical, reveals that a necessary condition for NIA is the existence of a market price of risk that satisfies an integrability condition strictly weaker than the condition in (2.4). Furthermore, Theorem 1 of Levental and Skorohod (1995) and Proposition 1.1 of Lyasoff (2010) motivate the integrability condition in (2.4) to prevent general scalable arbitrage opportunities. □

A toy example for a market without a market price of risk $\mathbb{P} \times$ Lebesgue-almost everywhere (and thus with scalable arbitrage) can be described by a drift $\mu(\cdot,\cdot)$ and a volatility structure $\sigma(\cdot,\cdot)$ such that the set $\mathcal{A} \subset [0,T] \times \mathbb{R}_+^d$ defined as $\mathcal{A} := \{(t,s) : \nexists \theta(t,s) \text{ s.t. } \sigma(t,s)\theta(t,s) = \mu(t,s)\}$ has positive measure, by which we mean $p_{\mathcal{A}} := \mathbb{P}(\text{Lebesgue}(t : (t,S(t)) \in \mathcal{A}) > 0) > 0$. We can decompose $\mu(\cdot,\cdot)$ uniquely into the sum of two vectors $\mu^1(\cdot,\cdot)$ in the range of $\sigma(\cdot,\cdot)$ and $\mu^2(\cdot,\cdot)$ orthogonal to its columns. Then, we have $\mu^2(t,s) \neq 0$ for all $(t,s) \in \mathcal{A}$ and

$\mu^{2\mathsf{T}}(\cdot,\cdot)\sigma(\cdot,\cdot) \equiv 0$ always. Investing according to $\mu^2(\cdot,\cdot)$ would thus switch off the risk factors and lead to nonnegative mean rate of return $\mu^{2\mathsf{T}}(\cdot,\cdot)\mu(\cdot,\cdot) = \|\mu^2(\cdot,\cdot)\|^2$. Investing according to such a strategy (see Section 2.3 for a precise definition) would lead to a wealth process (as in (2.11) below) that is greater than one with probability $p_{\mathcal{A}}$. This arbitrage opportunity could be leveraged arbitrarily by replacing the strategy $\mu^2(\cdot,\cdot)$ with $\mu^2(\cdot,\cdot)$ multiplied by a constant, leading to an immediate and unbounded profit. This line of thought, enriched with deep measure-theoretic results, is the underlying idea for the proof of the existence of a market price of risk under the NIA condition in Theorem 3.5 of Delbaen and Schachermayer (1995b).

2.3 Strategies, wealth processes and arbitrage opportunities

In this section, we introduce trading strategies, describe investors' wealth processes and define relative arbitrage. We denote the proportion of the investor's wealth invested in the i^{th} stock by π_i. The proportion of the wealth that is not invested in stocks gets invested in the money market, which yields zero interest rate. The next definition states this more precisely.

Definition 2 (Markovian trading strategy and associated wealth process). We call a function $\pi : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}^d$ a (Markovian trading) strategy and the process $V^\pi(\cdot)$ with dynamics

$$dV^\pi(t) = \sum_{i=1}^{d} \pi_i(t, S(t)) V^\pi(t) \frac{dS_i(t)}{S_i(t)} \tag{2.8}$$

for all $t \in [0,T]$ and with initial condition $V^\pi(0) = 1$ its associated wealth process. To ensure that $V^\pi(\cdot)$ does not explode and to exclude doubling strategies we restrict ourselves to strategies that satisfy the integrability condition

$$\sum_{i=1}^{d} \int_0^T \left(\pi_i(t, S(t)) V^\pi(t)\right)^2 a_{i,i}(t, S(t)) dt < \infty, \tag{2.9}$$

and the nonnegativity condition $V^\pi(t) \geq 0$ for all $t \in [0, T]$ almost surely. We shall use for any $v > 0$ the notation $V^{v,\pi}(\cdot) \equiv vV^\pi(\cdot)$ and interpret v as the investor's initial capital.

For (2.8), we have used that the strategy is self-financing; that is, no wealth is consumed and no money is added to the wealth process from outside. To wit, the wealth at any point of time is obtained by trading the initial wealth according to the strategy $\pi(\cdot, \cdot)$.

We have assumed that a strategy only depends on the current configuration of the market and not on its past, in order to preserve the Markovian property of the model. This has the economic interpretation that investment decisions are based upon the current market environment only. It would be of interest to extend the here presented results to a more general framework allowing for non-Markovian stochastic processes and strategies that may depend on the past of the market, perhaps relying on the Clark-Ocone formula (compare Karatzas and Ocone 1991). However, we allow the strategies to be time-inhomogeneous. Definition 2 allows for functionally generated portfolios (compare Remark 7 of Section 2.5) and hedging strategies for non-path dependent options (European and American style). Defining for all $i = 1, \ldots, d$ and $t \in [0, T]$ the functions $h_i(t, \omega) := V^{v,\pi}(t)\pi_i(t, S(t))$ as the dollar value and $\eta_i(t, \omega) := h_i(t, \omega)/S_i(t)$ as the number of shares held, (2.8) can be written in the more familiar forms

$$\begin{aligned} dV^{v,\pi}(t) &= \sum_{i=1}^{d} \eta_i(t, \omega) dS_i(t) \\ &= \sum_{i=1}^{d} h_i(t, \omega) \frac{dS_i(t)}{S_i(t)} \end{aligned} \tag{2.10}$$

$$= h^{\mathsf{T}}(t,\omega)\sigma(t,S(t))\left(\theta(t,S(t))dt + dW(t)\right)$$

for all $t \in [0,T]$.

The conditions in (2.4) and (2.9) in conjunction with Hölder's inequality yield that

$$\sum_{i=1}^{d}\int_0^T |\pi_i(t,S(t))V^{\pi}(t)\mu_i(t,S(t))|\, dt < \infty$$

almost surely, which guarantees the existence of a strong solution for $V^{\pi}(\cdot)$. If the condition in (2.9) holds with $\pi(\cdot,\cdot)V^{\pi}(\cdot)$ replaced by $\pi(\cdot,\cdot)$ then $V^{\pi}(\cdot)$ stays strictly positive. In this case, analog to (2.2), the solution of the stochastic differential equation in (2.8) is given as

$$\begin{aligned} V^{\pi}(t) = &\exp\Bigg(\int_0^t \pi^{\mathsf{T}}(u,S(u))\mu(u,S(u))du + \int_0^t \pi^{\mathsf{T}}(u,S(u))\sigma(u,S(u))dW(u) \\ &- \frac{1}{2}\int_0^t \pi^{\mathsf{T}}(u,S(u))a(u,S(u))\pi(u,S(u))du\Bigg) \end{aligned} \tag{2.11}$$

for all $t \in [0,T]$. For example, the strategy $\pi^0(\cdot,\cdot) \equiv 0$ invests only in the money market and its associated wealth process satisfies $V^{\pi^0}(\cdot) \equiv 1$. Usually, trading strategies do not lead to wealth processes that only depend on the current state of the market, as the next remark discusses:

Remark 2 (Markovianness of wealth process and dependence on whole path). Obviously, the wealth process of an investor jointly with the stock price process is Markovian if the investor uses a Markovian trading strategy. Yet, at time $t \in [0,T]$ the wealth process does not only depend on the current stock prices $S(t)$ but in most cases also on past stock prices $\{S(u), u \leq t\}$. Important exceptions from this rule are the market portfolio $\pi_i^m(t,s) := s_i / \sum_{j=1}^d s_j$ and investments in single stocks or the money market only; that is, $\pi_i^j(t,s) := \delta_j(i)$ for some $j \in \{1,\dots,d+1\}$ and for all $(t,s) \in [0,T] \times \mathbb{R}_+^d$, where δ_j represents Kronecker's delta function. However, as we shall see in Theorem 1 of Section 2.3, the dependence of the associated wealth processes on the past does not represent a problem in our setup for finding optimal strategies. □

The change of numéraire, that is, the change of the denomination in which the wealth process is quoted, is one of the most useful techniques in mathematical finance; compare Geman et al. (1995) for a derivation and discussion of the change of numéraire technique. It also plays a fundamental role in this chapter. For every numéraire, a special market price of risk exists:

Definition 3 (π-specific market price of risk). Let $\pi(\cdot,\cdot)$ denote a strategy and $\theta(\cdot,\cdot)$ a market price of risk. Define the corresponding π-specific market price of risk $\theta^\pi(t,s) : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}^K$ as

$$\theta^\pi(t,s) := \theta(t,s) - \sigma^\mathsf{T}(t,s)\pi(t,s). \tag{2.12}$$

The following computations show that the π-specific market price of risk exactly translates the volatilities into the mean rates of return relative to the wealth process of $\pi(\cdot,\cdot)$. Let $\rho(\cdot,\cdot)$ be any other strategy and $V^\pi(\cdot)$ always strictly positive. Then, we have from (2.11)

$$\begin{aligned}\frac{V^\rho(t)}{V^\pi(t)} = \exp\Bigg(&\int_0^t (\rho(u,S(u)) - \pi(u,S(u)))^\mathsf{T}\mu(u,S(u))du \\ &+ \int_0^t (\rho(u,S(u)) - \pi(u,S(u)))^\mathsf{T}\sigma(u,S(u))dW(u) \\ &- \frac{1}{2}\int_0^t \big(\rho^\mathsf{T}(u,S(u))a(u,S(u))\rho(u,S(u)) \\ &- \pi^\mathsf{T}(u,S(u))a(u,S(u))\pi(u,S(u))\big)\,du\Bigg)\end{aligned}$$

and thus after a short calculation,

$$\begin{aligned}d\left(\frac{V^\rho(t)}{V^\pi(t)}\right) = &\frac{V^\rho(t)}{V^\pi(t)}\Big(\rho(t,S(t)) - \pi(t,S(t))\Big)^\mathsf{T}\Big(\big(\mu(t,S(t)) - a(t,S(t))\pi(t,S(t))\big)dt \\ &+ \sigma(t,S(t))dW(t)\Big)\end{aligned}$$

$$= \frac{V^\rho(t)}{V^\pi(t)} \Big(\rho(t, S(t)) - \pi(t, S(t)) \Big)^\mathsf{T} \sigma(t, S(t)) dW^\pi(t), \tag{2.13}$$

where

$$W^\pi(t) := W(t) + \int_0^t \theta^\pi(u, S(u)) du \tag{2.14}$$

for all $t \in [0, T]$. Another short computation yields

$$Z^\theta(t) V^{v,\pi}(t) = v \exp\left(- \int_0^t \theta^{\pi\mathsf{T}}(u, S(u)) dW(u) - \frac{1}{2} \int_0^t \|\theta^\pi(u, S(u))\|^2 du \right) \tag{2.15}$$

for all $t \in [0, T]$.

Remark 3 (Change of numéraire). The expression in (2.15) should be contrasted to one in (2.5). The market price of risk $\theta(\cdot, \cdot)$ is replaced by the π-specific market price of risk $\theta^\pi(\cdot, \cdot)$ when we multiply $Z^\theta(\cdot)$ by a strictly positive wealth process $V^{v,\pi}(\cdot)$. This is a well-known fact in the no-arbitrage theory of change of numéraire; compare for example Chapter 9 of Shreve (2004). However, if $Z^\theta(\cdot) V^{v,\pi}(\cdot)$ is not a true martingale, then $Z^\theta(T) V^\pi(T)$ is not a Radon-Nikodym derivative and the process $W^\pi(\cdot)$ is not necessarily a Brownian motion under an equivalent local martingale measure. Corollary 4 of Section 2.5 will extend the classical change of numéraire to this case. □

Arbitrage has been mentioned several times. We conclude this section by discussing exactly what we mean by it. The next definition goes back to Section 3.3 of Fernholz (2002).

Definition 4 (Arbitrage). We call a strategy $\rho(\cdot, \cdot)$ with $\mathbb{P}(V^\rho(T) \geq V^\pi(T)) = 1$ and $\mathbb{P}(V^\rho(T) > V^\pi(T)) > 0$ a relative arbitrage opportunity with respect to the strategy $\pi(\cdot, \cdot)$. We call $\rho(\cdot, \cdot)$ a classical arbitrage opportunity if $\pi(\cdot, \cdot)$ invests fully in the money market, that is, if $\pi(\cdot, \cdot) \equiv 0$.

For a detailed study of arbitrage, and in particular no-arbitrage conditions, we refer the reader to the monograph by Delbaen and Schachermayer (2006). Jarrow et al. (2007; 2010) discuss these conditions with respect to the existence of bubbles and suggest using the stronger condition of "no dominance" first proposed by Merton (1973). Here, we take the opposite approach. Instead of imposing a new condition, the goal of this analysis is to investigate a general class of models and study how much can be said in this more general framework without relying on the tool of an equivalent local martingale measure.

For the sake of completeness and to put this work into perspective we remind the reader how a bubble is frequently defined in the existing literature.[4] From Theorem 1 below it follows then that the existence of a bubble implies a relative arbitrage opportunity.

Definition 5 (Bubble). We say that a strategy $\pi(\cdot, \cdot)$ contains a bubble if the stochastic discount factor $Z^\theta(\cdot)$ is a true martingale and if $Z^\theta(\cdot)V^\pi(\cdot)$ is a strict local martingale, that is, not a martingale, under the equivalent local martingale measure.

In this context, it is important to remind ourselves that $Z^\theta(\cdot)$ is a true martingale if and only if there exists an equivalent local martingale measure $\mathbb{Q}$, under which the stock price processes are local martingales. The question of whether $\mathbb{Q}$ is a martingale measure or only a local martingale measure is not connected to whether $Z^\theta(\cdot)$ is a strict local or a true martingale.

2.4 Optimal strategies

In this section, we derive the representation of optimal strategies in terms of delta hedges. In Subsection 2.4.1, we start from a given Markovian trading strategy and find an optimal strategy leading to the same terminal wealth. As Remark 2 discusses, this result can be interpreted as a hedging result for a certain class of

[4] In the bubbles literature, an alternative definition appears, based upon the characterization of the pricing operator as a charge, that is, an only finitely additive measure. However, it can be shown that this characterization is equivalent to the one here, which relies on strict local martingales; see Section 8 of Jarrow et al. (2010) for the proof and literature that relies on this alternative characterization.

possibly path-dependent payoffs, namely those which are strictly positive and for which a (possibly suboptimal) Markovian trading strategy is known to replicate them. Subsection 2.4.2 treats the hedging of non path-dependent European claims. Finally, Subsection 2.4.3 provides sufficient conditions under which the hedging price in Subsection 2.4.2 is sufficiently differentiable.

2.4.1 Optimizing a given strategy

Simple examples for strategies that we shall "optimize" are the market portfolio, where the portfolio weights are chosen as the market weights for stocks, or a strategy that invests the whole wealth in the money market. Given such a strategy, we look for a new strategy whose associated wealth at time horizon T exactly replicates the original value. We choose the new strategy to be optimal in the sense of minimal required initial capital. This criterion of optimality is directly related to the criterion of the shortest time to beat a portfolio by a given amount; see Section 6.2 of Fernholz and Karatzas (2009).

If $D \geq 0$ is a nonnegative $\mathcal{F}(T)$-measurable random variable such that $\mathbb{E}[D|\mathcal{F}(t)]$ is a function of $S(t)$ for some $t \in [0,T]$, we use the Markovian structure of $S(\cdot)$ to denote conditioning on the event $\{S(t) = s\}$ by $\mathbb{E}^{t,s}[D]$. For the moment, we assume that the associated wealth process stays strictly positive to avoid notational difficulties. We start by defining the function $U^\pi : [0,T] \times \mathbb{R}^d_+ \to [0,1]$ as

$$U^\pi(t,s) := \mathbb{E}^{T-t,s}\left[\frac{Z^\theta(T)V^\pi(T)}{Z^\theta(T-t)V^\pi(T-t)}\right] \tag{2.16}$$

$$= \mathbb{E}^{T-t,s}\left[\exp\left(-\int_{T-t}^{T} \theta^{\pi\mathsf{T}}(u,S(u))dW(u) - \frac{1}{2}\int_{T-t}^{T} \|\theta^\pi(u,S(u))\|^2 du\right)\right]. \tag{2.17}$$

The last equality follows directly from (2.15). As we show in Theorem 1, U^π can be interpreted as a hedging price. It obviously depends on the strategy $\pi(\cdot,\cdot)$.

Proposition 1 yields that U^π does not depend on the choice of the (Markovian) market price of risk $\theta(\cdot,\cdot)$.

We shall assume throughout this section that U^π solves the PDE

$$\begin{aligned}\frac{\partial}{\partial t}U^\pi(t,s) =& \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} s_i s_j a_{i,j}(T-t,s)D^2_{i,j}U^\pi(t,s)\\ &+\sum_{i=1}^{d}\sum_{j=1}^{d} s_i a_{i,j}(T-t,s)\pi_j(T-t,s)D_iU^\pi(t,s),\end{aligned} \tag{2.18}$$

where D_i, $D^2_{i,j}$ denote the partial derivatives with respect to the variable s. Section 2.8, which serves as an appendix, provides a sufficient condition for this assumption, and Remark 5 illustrates that smoothness of U^π is sufficient for U^π to solve the PDE.

The next theorem is the first key result of this chapter. It shows that U^π can be interpreted as a hedging price for the wealth process $V^\pi(\cdot)$: There exists a strategy that costs $U^\pi(T,S(0))$ and replicates the wealth at time T. Furthermore, there is no other strategy that replicates the wealth for less initial capital. Platen (2008) suggests calling this fact "Law of the Minimal Price" to contrast it to the classical "Law of the One Price," which appears if an equivalent martingale measure exists.

Theorem 1 (Optimal strategy)**.** *Let $\pi(\cdot,\cdot)$ denote any Markovian trading strategy with a strictly positive associated wealth process $V^\pi(\cdot)$ and let U^π solve the PDE in* (2.18)*. Then, a new strategy $\widehat{\pi}(\cdot,\cdot)$ exists such that the associated wealth process $V^{\hat{v},\widehat{\pi}}(\cdot)$ with initial wealth $\hat{v} := U^\pi(T,S(0)) \leq 1$ is always strictly positive and has the same value as $V^\pi(\cdot)$ at time T, that is,*

$$V^{\hat{v},\widehat{\pi}}(T) = V^\pi(T).$$

Thus, whenever $Z^\theta(\cdot)V^\pi(\cdot)$ is a strict local martingale, there exists a relative arbitrage opportunity $\widehat{\pi}$ with respect to $\pi(\cdot,\cdot)$. The strategy $\widehat{\pi}$ can be explicitly repre-

sented as

$$\widehat{\pi}_i(t,s) = s_i D_i \log U^\pi(T-t,s) + \pi_i(t,s) \tag{2.19}$$

for all $(t,s) \in [0,T] \times \mathbb{R}_+^d$*. Furthermore,* $\widehat{\pi}(\cdot,\cdot)$ *is optimal: There exists no strategy* $\rho(\cdot,\cdot)$ *such that*

$$V^{\tilde{v},\rho}(T) \geq V^\pi(T) = V^{\hat{v},\widehat{\pi}}(T) \tag{2.20}$$

almost surely for some $\tilde{v} < \hat{v}$*.*

Proof. Let us start by defining the martingale $N^\pi(\cdot)$ as

$$N^\pi(t) := \mathbb{E}[Z^\theta(T)V^\pi(T)|\mathcal{F}(t)] = Z^\theta(t)V^\pi(t)U^\pi(T-t,S(t)) \tag{2.21}$$

for all $t \in [0,T]$ and denoting by $\mathcal{L}$ the infinitesimal generator of $S(\cdot)$, that is,

$$\mathcal{L} = \sum_{i=1}^d s_i \mu_i(t,s) D_i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d s_i s_j a_{i,j}(t,s) D_{i,j}^2. \tag{2.22}$$

Since

$$\begin{aligned} dU^\pi(T-t,S(t)) = &\left(\mathcal{L}U^\pi - \frac{\partial}{\partial t}U^\pi\right)(T-t,S(t))\,dt \\ &+ \sum_{k=1}^K \sum_{i=1}^d S_i(t)\sigma_{i,k}(t,S(t))D_i U^\pi(T-t,S(t))W_k(t) \end{aligned}$$

holds for all $t \in [0,T]$, the product rule of stochastic calculus and (2.15) yield

$$\begin{aligned} \frac{dN^\pi(t)}{Z^\theta(t)V^\pi(t)} = &dU^\pi(T-t,S(t)) + U^\pi(T-t,S(t))\frac{d(Z^\theta(t)V^\pi(t))}{Z^\theta(t)V^\pi(t)} \\ &- \sum_{k=1}^K \theta_k^\pi(t,S(t)) \sum_{i=1}^d S_i(t)\sigma_{i,k}(t,S(t))D_i U^\pi(T-t,S(t))dt. \end{aligned}$$

We obtain the equality

$$\frac{dN^\pi(t)}{N^\pi(t)} = \sum_{k=1}^K \left(\sum_{i=1}^d S_i(t)\sigma_{i,k}(t,S(t)) \frac{D_i U^\pi(T-t,S(t))}{U^\pi(T-t,S(t))} - \theta_k^\pi(t,S(t)) \right) dW_k(t)$$

$$+ C^\pi(t, S(t))dt,$$

where

$$\begin{aligned}
C^\pi(t,s) :=& \frac{\left(\mathcal{L}U^\pi - \frac{\partial}{\partial t}U^\pi\right)(T-t,s)}{U^\pi(T-t,s)} - \sum_{i=1}^d s_i \frac{D_i U^\pi(T-t,s)}{U^\pi(T-t,s)} \sum_{k=1}^K \theta_k^\pi(t,s)\sigma_{i,k}(t,s) \\
=& \frac{1}{U^\pi(T-t,s)} \left(\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d s_i s_j a_{i,j}(t,s) D_{i,j}^2 U^\pi(T-t,s) \right. \qquad (2.23) \\
&\left. + \sum_{i=1}^d \sum_{j=1}^d s_i a_{i,j}(t,s)\pi_j(t,s) D_i U^\pi(T-t,s) - \frac{\partial}{\partial t} U^\pi(T-t,s) \right) \\
=& 0,
\end{aligned}$$

since U^π solves the PDE in (2.18). Thus, we can write

$$\begin{aligned}
\frac{dN^\pi(t)}{N^\pi(t)} =& \sum_{k=1}^K \left(\sum_{i=1}^d \sigma_{i,k}(t,S(t)) \Big(S_i(t) D_i \log\left(U^\pi(T-t,S(t))\right) + \pi_i(t,S(t)) \Big) \right. \\
&\left. - \theta_k(t,S(t)) \right) dW_k(t) \\
=& \sum_{k=1}^K -\theta_k^{\widehat{\pi}}(t,S(t))dW_k(t) \\
=& \frac{d(Z^\theta(t)V^{\hat{v},\widehat{\pi}}(t))}{Z^\theta(t)V^{\hat{v},\widehat{\pi}}(t)},
\end{aligned}$$

where the last equality follows from (2.15). Then, $N^\pi(0) = \hat{v} = Z^\theta(0)V^{\hat{v},\widehat{\pi}}(0)$ and both processes $N^\pi(\cdot)$ and $Z^\theta(\cdot)V^{\hat{v},\widehat{\pi}}(\cdot)$ have the same dynamics such that

$$Z^\theta(T)V^\pi(T) = N^\pi(T) = Z^\theta(T)V^{\hat{v},\widehat{\pi}}(T);$$

see Theorem 1.4.61 of Jacod and Shiryaev (2003). Since zero is an absorbing state for any nonnegative supermartingale and since $Z^\theta(T)V^\pi(T) > 0$ almost surely, we observe that $V^{\widehat{\pi}}(\cdot)$ is a strictly positive process almost surely.

Optimality comes from the fact that for any strategy $\rho(\cdot,\cdot)$ and for any initial wealth $\tilde{v} \geq 0$ the process $Z^\theta(\cdot)V^{\tilde{v},\rho}(\cdot)$ is bounded from below by zero, further implying that it is a supermartingale. Assume we have some strategy $\rho(\cdot,\cdot)$ such that (2.20) is satisfied. Then, we obtain

$$\tilde{v} \geq \mathbb{E}[Z^\theta(T)V^{\tilde{v},\rho}(T)] \geq \mathbb{E}[Z^\theta(T)V^\pi(T)] = \mathbb{E}[Z^\theta(T)V^{\hat{v},\widehat{\pi}}(T)] = \hat{v}, \tag{2.24}$$

which concludes the proof. □

We obtain from (2.21) and the last proof that

$$V^{\hat{v},\widehat{\pi}}(t) = \frac{N^\pi(t)}{Z^\theta(t)} = V^\pi(t)U^\pi(T-t,S(t)), \tag{2.25}$$

which we can rewrite as

$$U^\pi(T-t,S(t)) = \frac{V^{\hat{v},\widehat{\pi}}(t)}{V^\pi(t)}.$$

Thus, $U^\pi(T-t,S(t))$ can be interpreted as the fraction of two different wealth processes at time t that lead to the same terminal wealth, namely the wealth processes associated with the optimal strategy $\widehat{\pi}(\cdot,\cdot)$ and the original strategy $\pi(\cdot,\cdot)$, respectively.

We would like to emphasize that we have not shown that $\widehat{\pi}(\cdot,\cdot)$ is unique. Indeed, since we have not excluded the case that two stock prices have identical dynamics this is not necessarily true. However, if the strategy $\rho(\cdot,\cdot)$ is also optimal, then (2.24) yields that $Z^\theta(\cdot)V^\rho(\cdot)$ is a martingale, and thus $V^\rho(\cdot) \equiv V^{\widehat{\pi}}(\cdot)$; to wit, the optimal wealth process is unique.

The next remarks discuss various assumptions of the last theorem:

Remark 4 (Completeness of the market). One remarkable feature of the last result is that we have not required the market to be complete. In contrast to Fernholz and Karatzas (2010), we do not rely on the martingale representation theorem but instead directly derive a representation for the conditional expectation process of the final wealth $V^\pi(T)$ in the form of another wealth process $\widehat{\pi}(\cdot,\cdot)$. This means that given the existence of some Markovian trading strategy $\pi(\cdot,\cdot)$ to achieve $V^\pi(T)$, an optimal strategy $\widehat{\pi}(\cdot,\cdot)$ exists to achieve $V^\pi(T)$. The explanation for this phenomenon is that all relevant sources of risk for exploiting the relative arbitrage are completely captured by the tradeable stocks. However, we remind the reader that we live here in a setting in which the mean rates of return and volatilities do not depend on an extra stochastic factor. In a "more incomplete" model, with jumps or additional risk factors in mean rates of return or volatilities, this result can no longer be expected to hold. We revisit this discussion in Chapter 3. □

Remark 5 (PDE in (2.18)). The essential assumption of this section is that U^π solves the PDE in (2.18). Sufficient conditions are existence and differentiability conditions on the function H^π of (2.53) in conjunction with the condition in (2.57) in Section 2.8. Another sufficient condition is differentiability of U^π or, more precisely, that $U^\pi \in C^{1,2}([0,T] \times \mathbb{R}^d_+)$. Then, the proof of Theorem 1 yields that U^π automatically solves the PDE in (2.18), at least at all points $(t,s) \in [0,T] \times \mathbb{R}^d_+$ that can be attained by $S(\cdot)$ at time t. This can be seen from the fact that the process $N^\pi(\cdot)$ of (2.21) is a martingale; thus, its dt-term must disappear. This corresponds exactly to the condition $C^\pi(\cdot,\cdot) \equiv 0$, where C^π is defined in (2.23). Multiplying this equation by $U^\pi(T-t,s)$ we obtain the PDE in (2.18). Alternatively, Remark 3 of Fernholz and Karatzas (2010) briefly discusses general but technical assumptions for the necessary differentiability of U^π. Furthermore, it can be observed that it is sufficient that U^π solves the PDE only on the subset of $[0,T] \times \mathbb{R}^d_+$ where the stock price lives. Example 1 of Section 2.6 illustrates this point.

The condition of differentiability in time t can be slightly weakened to piecewise differentiability. If there are m points $0 < t_1 < \ldots < t_m < T$ where U^π is not differentiable, then we can find an optimal strategy up to time t_1, starting from t_1 to t_2 and so on. This will neither change the optimal strategy $\widehat{\pi}(\cdot,\cdot)$ nor the minimal initial capital $\hat{v}$ in any way. This small modification allows us to include strategies $\pi(\cdot,\cdot)$ with "structural breaks," by which we mean strategies whose arbitrage properties are changed at finitely many time steps. An example is a full investment up to time $t \in (0,T)$ in one strategy that can be arbitraged and afterwards a full investment in another strategy that cannot be arbitraged.

Furthermore, as Example 5 of Section 2.6 illustrates, the differentiability of U^π in the stock price dimension is only a sufficient but not a necessary condition for the existence of an optimal strategy. □

The PDE in (2.18) always has the constant function as a solution. The next result classifies U^π within the class of all PDE solutions as the minimal nonnegative solution. This result generalizes Theorem 1 of Fernholz and Karatzas (2010).

Proposition 2 (Characterization of U^π)**.** *The function U^π is the smallest function that solves the PDE in* (2.18) *and is nonnegative for all $(t,s) \in [0,T] \times \mathbb{R}_+^d$ that can be attained by $S(\cdot)$ at time t with initial condition $U^\pi(0,s) \equiv 1$ for all $s \in \mathbb{R}_+^d$. Furthermore, the PDE*

$$\begin{aligned}\frac{\partial}{\partial t}U(t,s) =& \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} s_i s_j a_{i,j}(T-t,s)D_{i,j}^2 U(t,s)\\ &+\sum_{i=1}^{d}\sum_{j=1}^{d} s_i a_{i,j}(T-t,s)\widehat{\pi}_j(T-t,s)D_i U(t,s),\end{aligned}$$

where we have exchanged $\pi(\cdot,\cdot)$ by $\widehat{\pi}(\cdot,\cdot)$, has $U^{\widehat{\pi}}(\cdot,\cdot) \equiv 1$ as its minimal nonnegative solution.

Proof. Consider any sufficiently smooth function $\widetilde{U}^\pi : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}_+$ that solves the PDE in (2.18) and the initial condition $\widetilde{U}^\pi(0,s) \equiv 1$ for all $s \in \mathbb{R}_+^d$. Define, as in (2.21), the process $\widetilde{N}(\cdot)$ as

$$\widetilde{N}^\pi(t) := Z^\theta(t)V^\pi(t)\widetilde{U}^\pi(T-t,S(t))$$

for all $t \in [0,T]$, which is, as in the proof of Theorem 1, a positive supermartingale. Thus, we have

$$\widetilde{U}^\pi(T-t,S(t)) = \frac{\widetilde{N}^\pi(t)}{Z^\theta(t)V^\pi(t)}$$

$$
\begin{aligned}
&\geq \frac{\mathbb{E}^{t,S(t)}\left[\widetilde{N}^{\pi}(T)\right]}{Z^{\theta}(t)V^{\pi}(t)} \\
&= \frac{\mathbb{E}^{t,S(t)}\left[Z^{\theta}(T)V^{\pi}(T)\right]}{Z^{\theta}(t)V^{\pi}(t)} \\
&= U^{\pi}(T-t, S(t))
\end{aligned}
$$

for all $t \in [0,T]$. The second statement of the proposition comes from the fact that $Z^{\theta}(\cdot)V^{\widehat{\pi}}(\cdot)$ is a martingale, which implies $U^{\widehat{\pi}}(\cdot,\cdot) \equiv 1$, and from the same considerations as above. □

The hedging price for the stock of Example 4 in Section 2.6, for instance, is one of many solutions of polynomial growth of the corresponding Black-Scholes type PDE. For example, consider $h_1(t,s) := s$ and $h_1(t,s)$ times the hedging price of (2.52), that is, $h_2(t,s) := 2s\Phi(1/(s\sqrt{t})) - s < s$, for all $(t,s) \in [0,T] \times \mathbb{R}_+$, where Φ denotes the cumulative normal distribution. Then, both h_1 and h_2 solve the PDE

$$
\frac{\partial}{\partial t} h(t,s) = \frac{1}{2} s^4 D^2 h(t,s)
$$

with the identical boundary condition $h(t,0) = 0$ and $h(0,s) = s$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$.

The reason for non-uniqueness in this case is the fact that the second-order coefficient has super-quadratic growth preventing standard theory from being applied; see, for example, Section 5.7.B of Karatzas and Shreve (1991). Furthermore, the boundary condition at infinity is not specified precisely enough. Both solutions grow polynomially, but clearly h_2 is always smaller than h_1. In this specific example, the corresponding process $1/S(\cdot)$ is a three-dimensional Bessel process and therefore stays away from the boundary. If the drift, however, is removed, it is a Brownian motion, which can hit zero. Thus, boundary conditions need to be precisely specified for a PDE in $1/s$ at zero, which corresponds to the precise boundary condition at infinity for the PDE above. Indeed, as the next section shows, the existence of an arbitrage opportunity is equivalent to the positive probability of some process imploding to zero under some measure $\mathbb{Q}$, which corresponds exactly to the observation that $1/S(\cdot)$ in the above example can hit zero.

For the special case $\mu(\cdot,\cdot) \equiv 0$ in one dimension, and under some assumptions on the volatility parameter $\sigma(\cdot,\cdot)$, Ekström et al. (2009) suggest a numerical algorithm that utilizes this characterization and finds the minimal nonnegative solution of a Black-Scholes type PDE that does not have a unique solution.

2.4.2 Hedging of contingent claims

So far, we have started from a given Markovian trading strategy $\pi(\cdot,\cdot)$ and then "optimized" it. However, one might imagine situations in which one wants to hedge a contingent claim but does not know a possibly suboptimal strategy $\pi(\cdot,\cdot)$ a priori. How can we find, in such a situation, an optimal strategy? In the following we resolve this problem for Markovian claims. We shall also provide weak sufficient conditions for the corresponding hedging price to be differentiable in Subsection 2.4.3. We now explicitly allow the associated wealth processes to hit zero.

To simplify computations later on, we introduce some notation. As before, the expectation operator corresponding to the event $\{S(t) = s\}$ is written as $\mathbb{E}^{t,s}$. Using the Markovian structure of our model, we denote, outside of the expectation operator, by $(S^{t,s}(u))_{u\in[t,T]}$ a stock price process with the dynamics of (2.1) and $S(t) = s$, in particular, $S^{0,S(0)}(\cdot) \equiv S(\cdot)$. We observe that $Z^\theta(u)/Z^\theta(t)$ depends for $u \in (t,T]$ on $\mathcal{F}(t)$ only through $S(t)$ and we write similarly $(\tilde{Z}^{\theta,t,s}(u))_{u\in[t,T]}$ for $(Z^\theta(u)/Z^\theta(t))_{u\in[t,T]}$, with $\tilde{Z}^{\theta,t,s}(t) = 1$ on the event $\{S(t) = s\}$. When we want to stress the dependence of a process on the state $\omega \in \Omega$ we shall write, for example, $S(t,\omega)$.

We emphasize the standing assumptions made in Section 2.2, namely, that the stock price process $S(\cdot)$ with dynamics specified in (2.1) starting in $S(0) \in \mathbb{R}_+^d$ is $\mathbb{R}^d$-valued, unique and stays in the positive orthant. Furthermore, a square-integrable Markovian market price of risk exists almost surely.

For any measurable function $p : \mathbb{R}_+^d \to [0, \infty)$, representing the payoff of the contingent claim, we define a candidate $h^p : [0,T] \times \mathbb{R}_+^d \to [0, \infty)$ for the hedging price of the corresponding European option, similar to the definition of U^π in (2.16) as

$$h^p(t,s) := \mathbb{E}^{T-t,s}\left[\tilde{Z}^\theta(T)p(S(T))\right]. \tag{2.26}$$

The only difference between h^p and U^π is that we do not normalize h^p with a wealth process. Since $S(\cdot)$ is Markovian, h^p is well-defined. The equation in (2.26) appears as the "real-world pricing formula" in the Benchmark Approach; compare Equation (9.1.30) of Platen and Heath (2006).

Let us denote by $\text{supp}(S(\cdot))$ the support of $S(\cdot)$, that is, the smallest closed set in $[0,T] \times \mathbb{R}^n$ such that

$$\mathbb{P}((t, S(t)) \in \text{supp}(S(\cdot)) \text{ for all } t \in [0,T]) = 1.$$

We call $\text{i-supp}(S(\cdot))$ the union of $(0, S(0))$ and the interior of $\text{supp}(S(\cdot))$ and assume that

$$\mathbb{P}((t, S(t)) \in \text{i-supp}(S(\cdot)) \text{ for all } t \in [0,T)) = 1.$$

This assumption is made to exclude degenerate cases, where $S(\cdot)$ can hit the boundary of its support with positive probability.

Definition 6 (Point of support). We call any $(t,s) \in \text{i-supp}(S(\cdot))$ a point of support for $S(\cdot)$.

We remark that each such point (t,s) satisfies $t < T$. For example, if $S(\cdot)$ is a one-dimensional geometric Brownian motion then the set of points of support for $S(\cdot)$ is exactly $(0, S(0)) \cup \{(t,s) \in (0,T) \times \mathbb{R}_+\}$.

Applying Itô's rule to (2.26) yields the following result, which in particular provides a mechanism for pricing and hedging contingent claims under the Benchmark Approach. Its proof is similar to the one of Theorem 1. In order to avoid introduction of extra notation and to be consistent with Theorem 1, we state the optimal trading strategies in terms of proportions of the current wealth. This might formally lead to a division by zero when the wealth process hits zero, but in that case no investments will happen anyway. We refer the reader to Theorem 1 of Ruf (2011), where the theorem and its proof are stated in terms of numbers of shares held.

Theorem 2 (Markovian representation for non path-dependent European claims). *Assume that we have a contingent claim of the form* $p(S(T)) \geq 0$ *and that the function* h^p *of* (2.26) *is sufficiently differentiable or, more precisely, that we have for all points of support* (t,s) *for* $S(\cdot)$ *that* $h^p \in C^{1,2}(\mathcal{U}_{T-t,s})$ *for some neighborhood* $\mathcal{U}_{T-t,s}$ *of* $(T-t,s)$*. Then, with*

$$\pi_i^p(t,s) := s_i D_i \log\left(h^p(T-t,s)\right) \tag{2.27}$$

for all $i = 1, \ldots, d$ *and* $(t,s) \in [0,T] \times \mathbb{R}_+^d$*, and with* $v^p := h^p(T, S(0))$*, we get*

$$V^{v^p,\pi^p}(T-t) = h^p(t, S(T-t))$$

for all $t \in [0,T]$*. Furthermore, the strategy* $\pi^p(\cdot,\cdot)$ *is optimal in the sense of Theorem 1 and* h^p *solves the PDE*

$$\frac{\partial}{\partial t} h^p(T-t,s) = \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t,s) D_{i,j}^2 h^p(T-t,s) \tag{2.28}$$

at all points of support (t,s) *for* $S(\cdot)$*.*

Proof. Let us start by defining the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T) p(S(T)) | \mathcal{F}(t)] = Z^\theta(t) h^p(T-t, S(t))$$

for all $t \in [0,T]$. Although h^p is not assumed to be in $C^{1,2}((0,T]\times\mathbb{R}^d)$ but only to be locally smooth, we can apply a localized version of Itô's formula; see, for example, Section IV.3 of Revuz and Yor (1999). Then, the product rule of stochastic calculus can be used to obtain the dynamics of $N^p(\cdot)$. Since $N^p(\cdot)$ is a martingale, the corresponding dt-term must disappear. This observation, in connection with (2.3) and the positivity of $Z^\theta(\cdot)$, yields the PDE in (2.28). Itô's formula, now applied to $h^p(T-\cdot, S(\cdot))$, and the PDE in (2.28) imply

$$dh^p(T-t,S(t)) = h^p(T-t,S(t))\sum_{i=1}^d \pi_i^p(t,S(t))\frac{dS_i(t)}{S_i(t)}$$

for all $t \in [0,T]$. Then, both $h^p(T-\cdot, S(\cdot))$ and $V^{v^p,\pi^p}(\cdot)$ are stochastic exponentials and solve the same stochastic differential equation. Theorem 1.4.61 of Jacod and Shiryaev (2003) yields $h^p(T-\cdot, S(\cdot)) \equiv V^{v^p,\pi^p}(\cdot)$.

The optimality of $\pi^p(\cdot,\cdot)$ follows exactly as in Theorem 1. □

The last result generalizes Proposition 3 of Platen and Hulley (2008), where the same result is derived for a one-dimensional, complete market with a time-transformed squared Bessel process of dimension four modeling the stock price process.

We remark that, as before, we neither assumed a complete market nor utilized a representation theorem. In particular, at no point did we assume invertibility or full rank of the volatility matrix $\sigma(\cdot,\cdot)$. Under these general assumptions, there is no hope to be able to hedge all contingent claims on the Brownian motion $W(T)$. However, $W(T)$ appears in this class of models only as a nuisance parameter and it is of no economic interest to trade in it directly.

Remark 6 (Delta hedging). Writing (2.27) as

$$\pi_i^p(t,S(t))\frac{V^{v^p,\pi^p}(t)}{S_i(t)} = D_i h^p(T-t,S(t))$$

and observing that the left-hand side is the number of shares invested in stock i at time t shows that the optimal strategy is a delta hedge as in classical Financial Mathematics, when one tries to hedge a contingent claim. Of course, $\widehat{\pi}(\cdot,\cdot)$ of (2.19) can be interpreted in a similar way: U^π is the risk-adjusted expected final wealth relative to the current wealth. Since everything has been expressed with respect to a wealth process $V^\pi(\cdot)$ the associated strategy $\pi(\cdot,\cdot)$ is added to obtain the optimal strategy $\widehat{\pi}(\cdot,\cdot)$. □

Example 9.2.2 of Fernholz et al. (2005) illustrates that the classical put-call parity can fail. Using the machinery of this section, we can directly show that a modified version of the put-call parity holds. An equivalent version in the situation of an equivalent local martingale measure with possible bubbles has already been derived in Lemma 7 of Jarrow et al. (2007). The put-call parity is sometimes applied incorrectly in the literature, see, for example, Emanuel and Macbeth (1982)[5]. In this context, we refer the reader also to the discussion in Madan and Yor (2006).

Corollary 1 (Modified put-call parity). *For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T)-L)^+$ and $(L-S_1(T))^+$, respectively, with strike price L:*

$$\mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(L-S_1(T))^+\right] + s_1 U^{\pi^1}(T-t,s) \\ = \mathbb{E}^{t,s}\left[\tilde{Z}^\theta(T)(S_1(T)-L)^+\right] + L U^{\pi^0}(T-t,s), \quad (2.29)$$

where $\pi^0(\cdot,\cdot) \equiv 0$ denotes the strategy for holding a monetary unit and $\pi^1(\cdot,\cdot) \equiv (1,0,\ldots,0)^\mathsf{T}$ the strategy for holding stock $S_1(\cdot)$.

Proof. The statement follows from the linearity of expectation. □

Due to Theorem 2, under weak differentiability assumptions, optimal strategies exist for the money market, the stock $S_1(T)$, the call and the put. Thus, the

[5]We thank Peter Carr for pointing us to this reference.

left-hand side of (2.29) corresponds to the sum of the hedging prices of a put and the stock, and the right-hand side corresponds to the sum of the hedging prices of a call and L monetary units. The difference between this and the classical put-call parity is that the current stock price and the strike L are replaced by their hedging prices. Section 2.2 of Bayraktar et al. (2010b) have recently observed an another version of the put-call parity. Instead of replacing the current stock price by its hedging price, they replace the European call price by the American call price and restore the put-call parity this way.

2.4.3 Smoothness of hedging price

Next, we shall provide sufficient conditions under which the function h^p of the last subsection is sufficiently smooth. Towards this end, we need the following definition.

Definition 7 (Locally Lipschitz and locally bounded). We call a function $f : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}$ locally Lipschitz and locally bounded on $\mathbb{R}_+^d$ if for all $s \in \mathbb{R}_+^d$ the function $t \to f(t,s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that

$$\sup_{\substack{\frac{1}{M} \leq \|y\|, \|z\| \leq M \\ y \neq z}} \frac{|f(t,y) - f(t,z)|}{\|y - z\|} + \sup_{\frac{1}{M} \leq \|y\| \leq M} |f(t,y)| \leq C(M)$$

for all $t \in [0,T]$.

In particular, if f has continuous partial derivatives, it is locally Lipschitz and locally bounded. We require several assumptions in order to show the necessary differentiability of h^p in Theorem 3 below. It is subject to future research to determine the precise conditions which yield the existence of a delta hedge, possibly without requiring h^p to be the classical solution of a PDE.

(A1) The functions $\theta_k(\cdot,\cdot)$ and $\sigma_{i,k}(\cdot,\cdot)$ are for all $i = 1,\dots,d$ and $k = 1,\dots,K$ locally Lipschitz and locally bounded.

(A2) For all points of support (t,s) for $S(\cdot)$ there exist some $C>0$ and some neighborhood $\mathcal{U}$ of (t,s) such that

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{i,j}(u,y)\xi_i\xi_j \geq C\|\xi\|^2 \tag{2.30}$$

for all $\xi \in \mathbb{R}^d$ and $(u,y) \in \mathcal{U}$.

(A3) The payoff function p is chosen so that for all points of support (t,s) for $S(\cdot)$ there exist some $C>0$ and some neighborhood $\mathcal{U}$ of $(T-t,s)$ such that $h^p(u,y) \leq C$ for all $(u,y) \in \mathcal{U}$.

If h^p is constant for $\tilde{d} \leq d$ coordinates, say the last ones, Assumption (A2) can be weakened to requesting the uniform ellipticity only in the remaining $d-\tilde{d}-1$ coordinates; that is, the sum in (2.30) goes only to $d-\tilde{d}-1$ and $\xi \in \mathbb{R}^{d-\tilde{d}-1}$. Assumption (A3) holds in particular if p is of linear growth; that is, if $p(s) \leq C\sum_{i=1}^{d} s_i$ for some $C>0$ and all $s \in \mathbb{R}_+^d$, since $\tilde{Z}^{\theta,t,s}(\cdot)S_i^{t,s}(\cdot)$ is a nonnegative supermartingale for all $i=1,\ldots,d$.

We emphasize that the conditions here are weaker than the ones used in Section 9 of Fernholz and Karatzas (2010) for the case of the market portfolio, which can be represented as $p(s)=\sum_{i=1}^{d} s_i$. In particular, the stochastic integral component in $Z^\theta(\cdot)$ does not present any technical difficulty in our approach.

We proceed in two steps. In the first step, we use the theory of stochastic flows to derive continuity of $S^{t,s}(T)$ and $\tilde{Z}^{\phi,t,s}(T)$ in t and s. This theory relies on Kolmogorov's lemma, see, for example, Theorem IV.73 of Protter (2003), and studies continuity of stochastic processes as functions of their initial conditions. We refer the reader to Kunita (1984) and Chapter V of Protter (2003) for an introduction to and further references for stochastic flows. We shall prove continuity of $S^{t,s}(\cdot)$ and $\tilde{Z}^{\phi,t,s}(\cdot)$ at once and introduce for that the $d+1$-dimensional process $X^{t,s,z}(\cdot) := ({S^{t,s}}^\mathsf{T}(\cdot), z\tilde{Z}^{\phi,t,s}(\cdot))^\mathsf{T}$.

The following lemma modifies Theorem V.37 of Protter (2003) for our context. This result will be of use below.

Lemma 1 (Stochastic flow, globally Lipzschitz)**.** *Fix* $\tilde{d} \in \mathbb{N}$. *We consider a system of* $\tilde{d}$ *stochastic differential equations of the form*

$$dY_i^{t,y}(u) = \tilde{\mu}_i(u, Y^{t,y}(u))dt + \sum_{k=1}^{K} \tilde{\sigma}_{i,k}(u, Y^{t,y}(u))dW_k(u), \tag{2.31}$$
$$Y_i^{t,y}(t) = y_i$$

for all $u \in [t, T]$, *for all* $(t, y) \in [0, T] \times \mathbb{R}^{\tilde{d}}$, *where* $Y^{t,y}(\cdot) = (Y_1^{t,y}(\cdot), \ldots, Y_{\tilde{d}}^{t,y}(\cdot))^\mathsf{T}$ *denotes a* $\tilde{d}$*-dimensional vector. The drift* $\tilde{\mu} : [0, T] \times \mathbb{R}^{\tilde{d}} \to \mathbb{R}^{\tilde{d}}$ *and volatility coefficient* $\tilde{\sigma} : [0, T] \times \mathbb{R}^{\tilde{d}} \to \mathbb{R}^{\tilde{d} \times K}$ *are assumed to be measurable and to satisfy the global Lipschitz condition*

$$\sum_{i=1}^{\tilde{d}} |\tilde{\mu}_i(u, y_1) - \tilde{\mu}_i(u, y_2)| + \sum_{i=1}^{\tilde{d}} \sum_{k=1}^{K} |\tilde{\sigma}_{i,k}(u, y_1) - \tilde{\sigma}_{i,k}(u, y_2)| \leq C\|y_1 - y_2\|$$

for all $(u, y_1, y_2) \in [0, T] \times \mathbb{R}^{\tilde{d}} \times \mathbb{R}^{\tilde{d}}$ *for some constant* $C > 0$.

Then, the stochastic differential equation in (2.31) *has a unique solution* $Y^{t,s}(\cdot)$. *It has a modification, which we again call* $Y^{t,s}(\cdot)$, *and which satisfies the following continuity property: Fix any countable set of time indices* $\mathbb{T} = \{t_i \in [0, T]\}_{i \in \mathbb{N}}$. *Then, there exists a subset* $\tilde{\Omega} \subset \Omega$ *with* $\mathbb{P}(\tilde{\Omega}) = 1$ *such that for all* $\omega \in \tilde{\Omega}$, $k \in \mathbb{N}$, $t \in \mathbb{T}$, *and for all* $y_1, y_2 \in \mathbb{R}^{\tilde{d}}$ *with* $\|y_1 - y_2\| \leq 2^{-k-2}$ *we have*

$$\sup_{u \in [t,T]} \|Y^{t,y_1}(u) - Y^{t,y_2}(u)\| \leq c_1(\omega) 2^{-c_2(\omega)k}$$

for some constants $c_1(\omega), c_2(\omega) \in \mathbb{R}_+$. *In particular, the constants* $c_1(\omega)$ *and* $c_2(\omega)$ *can be chosen independently of* $t \in \mathbb{T}$.

Proof. The lemma basically states Theorem V.37 of Protter (2003). The explicit continuity comes from an analysis of the arguments in the proof of Kolmogorov's Lemma; compare Theorem IV.73 of Protter (2003). There, we use Chebyshev's inequality simultaneously for all $t \in \mathbb{T}$ and then follow the proof line by line. □

We can now prove the continuity of the process $X^{t,s,1}(\cdot)$ in t and s using a localization technique:

Lemma 2 (Stochastic flow, locally Lipzschitz)**.** *We fix a point $(t,s) \in [0,T] \times \mathbb{R}_+^d$ so that $X^{t,s,1}(\cdot)$ is strictly positive and an $\mathbb{R}_+^{d+1}$-valued process. Then, under Assumption (A1), we have for all sequences $(t_k,s_k)_{k\in\mathbb{N}} \subset [0,T] \times \mathbb{R}_+^d$ with $\lim_{k\to\infty}(t_k,s_k) = (t,s)$ that*

$$\lim_{k\to\infty} \sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| = 0$$

almost surely, where we set $X^{t_k,s_k,1}(u) := (s_k^\mathsf{T}, 1)^\mathsf{T}$ for $u \le t_k$. In particular, for $K(\omega)$ sufficiently large we have that $X^{t_k,s_k,1}(u,\omega)$ is strictly positive and $\mathbb{R}_+^{d+1}$-valued for all $k > K(\omega)$ and $u \in [t,T]$.

Proof. Since the class of locally Lipschitz and locally bounded functions is closed under summation and multiplication, Assumption (A1) yields that the drift and diffusion coefficients of $X^{u,y,z}(\cdot)$ are locally Lipschitz for all $(u,y,z) \in [0,T] \times \mathbb{R}_+^d \times \mathbb{R}_+$. We start by assuming $t_k \ge t$ for all $k \in \mathbb{N}$ and obtain

$$\begin{aligned}\sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| \le & \sup_{u\in[t,t_k]} \|(s_k^\mathsf{T},1)^\mathsf{T} - X^{t,s,1}(u)\| \\ & + \sup_{u\in[t_k,T]} \|X^{t_k,s_k,1}(u) - X^{t_k,s,1}(u)\| \\ & + \sup_{u\in[t_k,T]} \|X^{t_k,s,1}(u) - X^{t_k,S^{t,s}(t_k),\tilde{Z}^{\phi,t,s}(t_k)}(u)\|\end{aligned}$$

for all $k \in \mathbb{N}$. The first term on the right-hand side of the last inequality goes to zero as k increases by the continuity of the sample paths of $X^{t,s,1}(\cdot)$. The arguments and the localization technique in the proof of Theorem V.38 in Protter (2003) in conjunction with Lemma 1 yield that

$$\lim_{k\to\infty} \sup_{u\in[\tilde{t},T]} \|X^{\tilde{t},y_k,z_k}(u) - X^{\tilde{t},s,1}(u)\| = 0$$

for all $\tilde{t} \in \{t, t_1, t_2, \ldots\}$ and any sequence $((y_k^\intercal, z_k)^\intercal)_{k\in\mathbb{N}} \subset \mathbb{R}_+^{d+1}$ with $(y_k^\intercal, z_k)^\intercal \to (s^\intercal, 1)^\intercal$ as $k \to \infty$ almost surely. The convergence is uniformly in $\tilde{t} \in \{t, t_1, t_2, \ldots\}$. We now choose for $(y_k^\intercal, z_k)^\intercal$ the sequences $(s_k^\intercal, 1)^\intercal$ and $(S^{t,s^\intercal}(t_k, \omega), \tilde{Z}^{\phi,t,s}(t_k, \omega))^\intercal$ for all $\omega \in \Omega$. This proves the statement if $t_k \geq t$ for all $k \in \mathbb{N}$. In the case of the reversed inequality $t_k \leq t$, we observe

$$\sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| \leq \sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t_k,s,1}(u)\| + \sup_{u\in[t,T]} \|X^{t_k,s,1}(u) - X^{t,s,1}(u)\|,$$

which again yields continuity, similar to above. □

In the second step, we use a technique from the theory of PDEs to conclude the necessary smoothness of h^p. The following result has been used by Ekström, Janson and Tysk. We present it here on its own to underscore the analytic component of our argument:

Lemma 3 (Schauder estimates and smoothness)**.** *Fix a point $(t, s) \in [0, T) \times \mathbb{R}_+^d$ and a neighborhood $\mathcal{U}$ of (t, s). Suppose Assumption (A1) holds along with inequality in* (2.30) *for all $\xi \in \mathbb{R}^d$ and $(u, y) \in \mathcal{U}$ and some $C > 0$. Let $(f_k)_{k\in\mathbb{N}}$ denote a sequence of solutions of the PDE in* (2.28) *on $\mathcal{U}$, uniformly bounded under the supremum norm on $\mathcal{U}$. If $\lim_{k\to\infty} f_k(t, s) = f(t, s)$ on $\mathcal{U}$ for some function $f : \mathcal{U} \to \mathbb{R}$, then f solves the PDE in* (2.28) *on some neighborhood $\tilde{\mathcal{U}}$ of (t, s). In particular, $f \in C^{1,2}(\tilde{\mathcal{U}})$.*

Proof. We refer the reader to the arguments and references provided in Section 2 of Janson and Tysk (2006) and Theorem 3.2 of Ekström and Tysk (2009). The central idea is to use the interior Schauder estimates by Knerr (1980) in conjunction with Arzelà-Ascoli type of arguments to prove the existence of first- and second-order derivatives of f. □

We can now prove the smoothness of the hedging price h^p:

Theorem 3. *Under Assumptions (A1)-(A3) there exists for all points of support (t, s) for $S(\cdot)$ some neighborhood $\mathcal{U}$ of $(T-t, s)$ such that the function h^p defined in (2.26) is in $C^{1,2}(\mathcal{U})$.*

Proof. We define $\tilde{p} : \mathbb{R}_+^{d+1} \to \mathbb{R}_+$ by $\tilde{p}(s_1, \ldots, s_d, z) := zp(s_1, \ldots, s_d)$ and $\tilde{p}^M : \mathbb{R}_+^{d+1} \to \mathbb{R}_+$ by $\tilde{p}^M(\cdot) := \tilde{p}(\cdot)\mathbf{1}_{\{\tilde{p}(\cdot)\leq M\}}$ for some $M > 0$ and approximate $\tilde{p}^M$ by a sequence of continuous functions $\tilde{p}^{M,m}$ (compare for example Appendix C.4 of Evans 1998) such that $\lim_{m\to\infty} \tilde{p}^{M,m} = \tilde{p}^M$ pointwise and $\tilde{p}^{M,m} \leq 2M$ for all $m \in \mathbb{N}$. The corresponding expectations are defined as

$$\tilde{h}^{p,M}(u, y) := \mathbb{E}^{T-u,y}[\tilde{p}^M(S_1(T), \ldots, S_d(T), \tilde{Z}^\theta(T))]$$

for all $(u, y) \in \tilde{\mathcal{U}}$ for some neighborhood $\tilde{\mathcal{U}}$ of $(T-t, s)$ and equivalently $\tilde{h}^{p,M,m}$.

We start by proving continuity of $\tilde{h}^{p,M,m}$ for large m. For any sequence $(t_k, s_k)_{k\in\mathbb{N}} \subset [0, T] \times \mathbb{R}_+^d$ with $\lim_{k\to\infty}(t_k, s_k) = (t, s)$, Lemma 2, in connection with Assumption (A1), yields

$$\lim_{k\to\infty} \tilde{p}^{M,m}(S^{t_k,s_k}(T), \tilde{Z}^{\theta,t_k,s_k}(T)) = \tilde{p}^{M,m}(S^{t,s}(T), \tilde{Z}^{\theta,t,s}(T)).$$

The continuity of $\tilde{h}^{p,M,m}$ follows then from the bounded convergence theorem.

Now, Lemma 2.6 of Janson and Tysk (2006), in connection with Assumption (A2), guarantees that $\tilde{h}^{p,M,m}$ is a solution of the PDE in (2.28). Lemma 3 then yields that firstly, $\tilde{h}^{p,M}$ and secondly, in connection with Assumption (A3), h^p also solve the PDE in (2.28) on some neighborhood $\mathcal{U}$ of $(T-t, s)$. In particular, h^p is in $C^{1,2}(\mathcal{U})$. □

The last theorem generalizes the results in Ekström and Tysk (2009) to several dimensions and to non-continuous payoff functions p. Chapters 6 and 15 of Friedman (1976) and Janson and Tysk (2006) have related results, but they impose linear growth conditions on $a(\cdot, \cdot)$ so that the PDE in (2.28) has a unique solution of polynomial growth. We are especially interested in the situation in which multiple solutions may exist. Heath and Schweizer (2000) present results in the case when the process corresponding to the PDE in (2.28) does not leave the positive orthant. As Fernholz and Karatzas (2010) observe, this condition does not necessarily hold if there is no equivalent local martingale measure. In the case of $Z^\theta(\cdot)$ being a martingale, our assumptions are only weakly more general than the ones in Heath and Schweizer (2000) by not requiring $a(\cdot,\cdot)$ to be continuous in the time dimension. Further results are also obtained by Section III.7 of Kunita (1984), but under strong continuity assumptions on $a(\cdot,\cdot)$. However, in all these research articles, the authors show that the function h^p indeed solves the PDE in (2.28) not only locally but globally and satisfies the corresponding boundary conditions. We have here abstained from imposing the stronger assumptions these papers rely on and concentrate on the local properties of h^p. For our application, it is sufficient to observe that $h^p(T-t, S(t))$ converges to $p(S(T))$ as t goes to T; compare the proof of Theorem 2.

The next section provides an interpretation of our approach to prove the differentiability of h^p; all problems on the spatial boundary, arising for example from a discontinuity of $a(\cdot,\cdot)$ on the boundary of the positive orthant, have been "conditioned away," so that $S(\cdot)$ can get close to but never actually attains the boundary.

2.5 Change of measure

We obtained in Theorem 1 a precise description of an optimal strategy $\widehat{\pi}(\cdot,\cdot)$ to replicate the wealth $V^\pi(T)$ at time T. However, in order to compute this strategy we need to compute the "deltas" of the expectation U^π of the risk-adjusted wealth $Z^\theta(T)V^\pi(T)$. In Theorem 4, we shall provide a useful representation of U^π by performing a change of measure. To be able to do then the computations, we provide the dynamics of the stock price processes and a formula for conditional expectations under the new probability measure in Corollaries 2 and 3. We end this section by proving a result concerning the change of numéraire in Corollary 4, illustrating in Proposition 3 how a canonical probability space can be constructed to satisfy the technical assumptions of this section, and in several remarks discussing connections of this work to some literature.

Theorem 1.4 of Delbaen and Schachermayer (1995b) shows that NA implies the existence of a local martingale measure which is absolutely continuous with respect to $\mathbb{P}$. On the other side, a consequence of this section is the existence of a local martingale measure under NUPBR, such that $\mathbb{P}$ is absolutely continuous with respect to it. Indeed, as discussed in Remark 1, NA and NUPBR together yield NFLVR, which again yields an equivalent local martingale measure corresponding exactly to the one discussed in this section. Another point of view, which we do not take here, is the recent insight by Kardaras (2010) on the equivalence of NUPBR and the existence of a finitely additive probability measure that is, in some sense, weakly equivalent to $\mathbb{P}$ and under which $S(\cdot)$ has some notion of weak local martingale property.

Our approach via a "generalized change of measure" is in the spirit of the work by Föllmer (1972; 1973), Meyer (1972), Section 2 of Delbaen and Schachermayer (1995a), and Section 7 of Fernholz and Karatzas (2010). They show that for the strictly positive $\mathbb{P}$-local martingale $Z^\theta(\cdot)$, there exists a probability measure $\mathbb{Q}$ such that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ and $d\mathbb{P}/d\mathbb{Q} = 1/Z^\theta(T \wedge \tau^\theta)$, where τ^θ is the first hitting time of zero by the process $1/Z^\theta(\cdot)$. Their analysis has been built upon by several authors, for example by Section 2 of Pal and Protter (2010). We complement this research direction by determining the dynamics of the $\mathbb{P}$-Brownian motion $W(\cdot)$ under the new measure $\mathbb{Q}$. These dynamics do not follow directly from an application of a Girsanov-type argument since $\mathbb{Q}$ need not be absolutely continuous with respect to $\mathbb{P}$. Similar results for the dynamics have been obtained in Lemma 4.2 of Sin (1998) and Section 2 of Delbaen and Shirakawa (2002). However, they rely on additional assumptions on the existence of solutions for some stochastic differential equations. Wong and Heyde (2004) prove the existence of a measure $\tilde{\mathbb{Q}}$ satisfying $\mathbb{E}^{\mathbb{P}}[Z^\theta(T)] = \tilde{\mathbb{Q}}(\tau^\theta > T)$, where $W(\cdot)$ has the same $\tilde{\mathbb{Q}}$-dynamics as we derive, but $\mathbb{P}$ is not necessarily absolutely continuous with respect to $\tilde{\mathbb{Q}}$.

For the results in this section, we do not need that U^π solves the PDE in (2.18). However, we make the technical assumption[6] that the probability space Ω is the space of right-continuous paths $\omega : [0,T] \to \mathbb{R}^m \cup \{\Delta\}$ for some $m \in \mathbb{N}$ with left limits at $t \in [0,T]$ if $\omega(t) \neq \Delta$ and with an absorbing "cemetery" point Δ. By that we mean that $\omega(t) = \Delta$ for some $t \in [0,T]$ implies $\omega(u) = \Delta$ for all $u \in [t,T]$ and for all $\omega \in \Omega$. This point Δ will represent explosions of $Z^\theta(\cdot)$, which do not occur under $\mathbb{P}$, but may occur under a new probability measure $\mathbb{Q}$ constructed below. We further assume that the filtration $\mathbb{F}$ is the right-continuous modification of the filtration generated by the paths ω, or more precisely by the projections $\xi_t(\omega) := \omega(t)$. Concerning the original probability measure we assume that $\mathbb{P}(\omega : \omega(T) = \Delta) = 0$ and that for all $t \in [0,T]$, ∞ is an absorbing state for $Z^\theta(\cdot)$; that is, $Z^\theta(t) = \infty$ implies $Z^\theta(u) = \infty$ for all $u \in [t,T]$. This assumption, consistent with the dynamics of (2.6) specifies $Z^\theta(\cdot)$ only on a set of measure zero and is made for notational convenience.

We emphasize that we have not assumed completeness of the filtration $\mathbb{F}$. Indeed, we shall construct a new probability measure $\mathbb{Q}$ that is not necessarily equivalent to the original measure $\mathbb{P}$ and can assign positive probability to nullsets of $\mathbb{P}$. If we had assumed completeness of $\mathbb{F}$, we could not guarantee that $\mathbb{Q}$ could be consistently defined on all subsets of these nullsets, which had been included in

[6]The results in this section hold for more general probability spaces and filtrations, basically for these filtered spaces that allow for extension theorems. However, for the sake of clarity, we restrict ourselves here to this special version of a probability space and filtration. Compare the appendix of Föllmer (1972) for details.

$\mathbb{F}$ during the completion process. The fact that we need the cemetery point Δ and cannot restrict ourselves to the original canonical space is also not surprising. The point Δ represents events that have under $\mathbb{P}$ probability zero, but under $\mathbb{Q}$ have positive probability. Föllmer and Imkeller (1993) discuss another example where a change of measure needs additional events, and thus extensions of the original probability space.

All these assumptions are needed to prove the existence of a measure $\mathbb{Q}$ with $d\mathbb{P}/d\mathbb{Q} = 1/Z^\theta(T \wedge \tau^\theta)$. After having ensured its existence, one then can take the route suggested by Theorem 5 of Delbaen and Schachermayer (1995a) and start from any probability space satisfying the usual conditions, construct a canonical probability space satisfying the technical assumptions mentioned above, and then perform all necessary computations on this space. We shall detail these technical steps in Proposition 3.

For now, the goal is to construct a measure $\mathbb{Q}$ under which the computation of U^π simplifies. For that, we define the sequence of stopping times

$$\tau_i^\theta := \inf\{t \in [0,T] : Z^\theta(t) \geq i\}$$

with $\inf\emptyset := \infty$ and the sequence of σ-algebras $\mathcal{F}^i := \mathcal{F}(\tau_i^\theta \wedge T)$ for all $i \in \mathbb{N}$. We observe that the definition of $\mathcal{F}^i$ is independent of the probability measure and define the stopping time $\tau^\theta := \lim_{i\to\infty} \tau_i^\theta$ with corresponding σ-algebra $\mathcal{F}^{\infty,\theta} := \mathcal{F}(\tau^\theta \wedge T)$ generated by $\cup_{i=1}^\infty \mathcal{F}^{i,\theta}$.

Within this framework, Meyer (1972) and Example 6.2.2 of Föllmer (1972) rely on an extension theorem (compare Chapter 5 of Parthasarathy 1967) to show the existence of a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}(T))$ satisfying

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}\left[Z^\theta(\tau_i^\theta \wedge T)\mathbf{1}_A\right] \tag{2.32}$$

for all $A \in \mathcal{F}^{i,\theta}$, where we now write $\mathbb{E}^{\mathbb{P}}$ for the expectation under the original measure. We summarize these insights in the following theorem:

Theorem 4 (Generalized change of measure)**.** *There exists a measure $\mathbb{Q}$ such that for all stopping times $\widetilde{T}$ with $\widetilde{T} \leq T$ and for all $\mathcal{F}(\widetilde{T})$-measurable random variables $D \geq 0$ we have*

$$\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)D\right] = \mathbb{E}^{\mathbb{Q}}\left[D\mathbf{1}_{\left\{1/Z^\theta(\widetilde{T})>0\right\}}\right], \tag{2.33}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation with respect to the new measure $\mathbb{Q}$. That is, $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$. Under this measure $\mathbb{Q}$, the process $\widetilde{W}(\cdot) = \left(\widetilde{W}_1(\cdot), \ldots \widetilde{W}_K(\cdot)\right)^\mathsf{T}$ with

$$\widetilde{W}_k(t \wedge \tau^\theta) := W_k(t \wedge \tau^\theta) + \int_0^{t\wedge\tau^\theta} \theta_k(u, S(u))du \tag{2.34}$$

for all $k = 1, \ldots, K$ and $t \in [0, T]$ is a K-dimensional Brownian motion stopped at time τ^θ.

Proof. The existence of a measure $\mathbb{Q}$ satisfying (2.32) follows as in the discussion above. Now, for any set $A \in \mathcal{F}(\widetilde{T})$ we have

$$A = \left(A \cap \left\{\tau^\theta \leq \widetilde{T}\right\}\right) \cup \bigcup_{i=1}^{\infty}\left(A \cap \left\{\tau^\theta_{i-1} < \widetilde{T} \leq \tau^\theta_i\right\}\right).$$

From the fact that $\tau^\theta \leq \widetilde{T}$ holds, if and only if $1/Z^\theta(\widetilde{T}) = 0$ holds and from the identity in (2.32) we obtain

$$\begin{aligned}
\mathbb{Q}\left(A \cap \left\{\frac{1}{Z^\theta\left(\widetilde{T}\right)} > 0\right\}\right) &= \sum_{i=1}^{\infty} \mathbb{Q}\left(A \cap \left\{\tau^\theta_{i-1} < \widetilde{T} \leq \tau^\theta_i\right\}\right) \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{P}}\left[Z^\theta(\tau^\theta_i \wedge T)\mathbf{1}_{A\cap\left\{\tau^\theta_{i-1}<\widetilde{T}\leq\tau^\theta_i\right\}}\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)\mathbf{1}_{A\cap\left\{\tau^\theta_{i-1}<\widetilde{T}\leq\tau^\theta_i\right\}}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)\mathbf{1}_A\right],
\end{aligned}$$

where the last identity holds since $\mathbb{P}\left(\tau^\theta \leq \widetilde{T}\right) = 0$. This yields the representation of (2.33). From Girsanov's theorem (compare Theorem 8.1.4 of Revuz and Yor 1999) we obtain that on $\mathcal{F}^{i,\theta}$ the process $\widetilde{W}(\cdot)$ is under $\mathbb{Q}$ a K-dimensional Brownian motion stopped at $\tau_i^\theta \wedge T$. Since $\cup_{i=1}^{\infty}\mathcal{F}^{i,\theta}$ generates $\mathcal{F}^{\infty,\theta}$ and forms a π-system, we get the dynamics of (2.34). □

Thus, an equivalent local martingale measure exists, if and only if $\mathbb{Q}(1/Z^\theta(T) > 0) = 1$. On the other hand, if no equivalent local martingale measure exists, then valuing a wealth process must include the barrier aspect $1/Z^\theta(T) > 0$. To wit, allowing for arbitrage requires calculating the $\mathbb{Q}$-probability of the reciprocal $1/Z^\theta(\cdot)$ of the stochastic discount factor hitting zero. We emphasize that we need not know $\pi(\cdot,\cdot)$ to calculate the corresponding hedging price U^π, but only its final associated wealth as a function of the stock prices $S(\cdot)$. However, in this case we cannot obtain the strategy $\widehat{\pi}(\cdot,\cdot)$ from Theorem 1.

A further consequence of Theorem 4 is the fact that the dynamics of the stock price process and the reciprocal of the stochastic discount factor simplify under $\mathbb{Q}$ as the next corollary shows.

Corollary 2 (Evolution of important processes under $\mathbb{Q}$)**.** *The stock price process $S(\cdot)$ and the reciprocal $1/Z^\theta(\cdot)$ of the stochastic discount factor evolve until the stopping time τ^θ under $\mathbb{Q}$ according to*

$$dS_i(t) = S_i(t)\sum_{k=1}^{K}\sigma_{i,k}(t,S(t))d\widetilde{W}_k(t)$$

and

$$d\left(\frac{1}{Z^\theta(t)}\right) = \frac{1}{Z^\theta(t)}\sum_{k=1}^{d}\theta_k(t,S(t))d\widetilde{W}_k(t)$$

for all $i = 1,\dots,d$ and $t \in [0,T]$.

Proof. This is a direct consequence of the representation of $\widetilde{W}(\cdot)$ in (2.34) and Definition 1 of the market price of risk. □

The results of the last corollary play an essential role when we do computations, since the first hitting time of the reciprocal of the stochastic discount factor can in most cases be easily represented as a first hitting time of the stock price. This now usually follows some more tractable dynamics, as we shall see in Section 2.6. Theorem 4 also holds for expectations conditioned on $\mathcal{F}(t)$: the next corollary generalizes the well-known Bayes' rule for classical changes of measures; compare Lemma 3.5.3 of Karatzas and Shreve (1991). Similar computations appear already in Proposition 4.2 of Föllmer (1972).

Corollary 3 (Bayes' rule, $\mathbb{Q}$-martingale property of $1/Z^\theta(\cdot)$)**.** *Let $\widetilde{T}$ denote any stopping time with $\widetilde{T} \leq T$. For all $\mathcal{F}(\widetilde{T})$-measurable random variables $D \geq 0$ the representation*

$$\mathbb{E}^{\mathbb{Q}}\left[D\mathbf{1}_{\left\{1/Z^\theta\left(\widetilde{T}\right)>0\right\}}\middle|\mathcal{F}(t)\right] = \mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)D|\mathcal{F}(t)\right]\frac{1}{Z^\theta\left(t\wedge\widetilde{T}\right)}\mathbf{1}_{\left\{1/Z^\theta\left(t\wedge\widetilde{T}\right)>0\right\}} \tag{2.35}$$

holds $\mathbb{Q}$-almost surely (and thus $\mathbb{P}$-almost surely) for all $t \in [0,T]$. Furthermore, for any process $N(\cdot)$, $N(\cdot)\mathbf{1}_{\{1/Z^\theta(\cdot)>0\}}$ is a $\mathbb{Q}$-martingale, if and only if $N(\cdot)Z^\theta(\cdot)$ is a $\mathbb{P}$-martingale. In particular, the process $1/Z^\theta(\cdot)$ is a $\mathbb{Q}$-martingale.

Proof. We observe that for all $\mathcal{F}\left(\tau_i^\theta\wedge\widetilde{T}\right)$-measurable random variables $D \geq 0$, (2.32) can be rewritten as

$$\mathbb{E}^{\mathbb{Q}}[D] = \mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\tau_i^\theta\wedge\widetilde{T}\right)D\right]. \tag{2.36}$$

due to the martingale property of the bounded process $Z^\theta(\cdot\wedge\tau_i^\theta)$ under $\mathbb{P}$. We fix a time $t \in [0,T]$. For any $A \in \mathcal{F}(t)$, (2.33) with D replaced by $D\mathbf{1}_{\tilde{A}}$ where $\tilde{A} = A \cap \left\{\widetilde{T} > t\right\} \in \mathcal{F}\left(\widetilde{T}\wedge t\right)$ and the same techniques, as in the proof of Theorem 4, yield

$$\mathbb{E}^{\mathbb{Q}}\left[D\mathbf{1}_{\left\{1/Z^\theta\left(\widetilde{T}\right)>0\right\}}\mathbf{1}_{\tilde{A}}\right] = \mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)D\mathbf{1}_{\tilde{A}}\right]$$

$$
\begin{aligned}
&= \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right) D \middle| \mathcal{F}(t)\right] \frac{1}{Z^\theta\left(t \wedge \widetilde{T}\right)} \mathbf{1}_{\tilde{A}} Z^\theta\left(t \wedge \widetilde{T}\right)\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right) D \middle| \mathcal{F}(t)\right] \frac{1}{Z^\theta\left(t \wedge \widetilde{T}\right)}\right. \\
&\quad \left. \cdot \mathbf{1}_{\left\{\tau_{i-1}^\theta < t \wedge \widetilde{T} \leq \tau_i^\theta\right\}} \mathbf{1}_{\tilde{A}} Z^\theta\left(\tau_i^\theta \wedge t \wedge \widetilde{T}\right)\right] \\
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right) D \middle| \mathcal{F}(t)\right] \frac{1}{Z^\theta\left(t \wedge \widetilde{T}\right)} \mathbf{1}_{\left\{\tau_{i-1}^\theta < t \wedge \widetilde{T} \leq \tau_i^\theta\right\}} \mathbf{1}_{\tilde{A}}\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right) D \middle| \mathcal{F}(t)\right] \frac{1}{Z^\theta\left(t \wedge \widetilde{T}\right)} \mathbf{1}_{\left\{1/Z^\theta\left(t \wedge \widetilde{T}\right) > 0\right\}} \mathbf{1}_{\tilde{A}}\right],
\end{aligned}
$$

where the second-to-last equality relies on the identity of (2.36). This yields the representation in (2.35). The other statements follow from choosing $\widetilde{T} = T$, $D = N(T)$ and $D = 1/Z^\theta(T)$. □

For the case of strict local martingales the equivalence of the last corollary is generally not true. Take as an example $N(\cdot) \equiv 1$ and $Z^\theta(\cdot)$ a strict local martingale under $\mathbb{P}$. Then, $Z^\theta(\cdot)N(\cdot) \equiv Z^\theta(\cdot)$ is a local $\mathbb{P}$-martingale but $N(\cdot)\mathbf{1}_{\{1/Z^\theta(\cdot)>0\}} \equiv \mathbf{1}_{\{1/Z^\theta(\cdot)>0\}}$ is clearly not a local $\mathbb{Q}$-martingale. The reason for this lack of symmetry is that a sequence of stopping times that converges $\mathbb{P}$-almost surely to T need not necessarily converge $\mathbb{Q}$-almost surely to T.

We have seen that Theorem 4 implies that $1/Z^\theta(\cdot)$ stopped at zero is a martingale under the new measure. As Delbaen and Schachermayer (1995a) and Pal and Protter (2010) have discussed, the other direction holds trivially true: Let $\mathbb{Q}$ denote some measure; $M(\cdot)$ a $\mathbb{Q}$-martingale started at some positive value $M(0) > 0$; and T_0 the first hitting of zero by $M(\cdot)$. Then, under the new measure $d\widetilde{\mathbb{P}} := M(T \wedge T_0)d\mathbb{Q}$, the process $1/M(\cdot)$ is again a local martingale due to Girsanov's theorem and Itô's formula. It is a martingale, if and only if $M(\cdot)$ does not hit zero under the original measure $\mathbb{Q}$.

In order to simplify computations even more, the following change of numéraire for strictly positive wealth processes can be useful.

Corollary 4 (Change of numéraire)**.** *Let $\pi(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ denote two strategies such that $V^\pi(\cdot)$ is strictly positive and $\widetilde{T}$ a stopping time with $\widetilde{T} \leq T$. There exists a measure $\mathbb{Q}^\pi$ such that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}^\pi$ and*

$$\mathbb{E}^{\mathbb{P}}\left[Z^\theta\left(\widetilde{T}\right)V^\rho\left(\widetilde{T}\right)\right] = \mathbb{E}^{\mathbb{Q}^\pi}\left[\frac{V^\rho\left(\widetilde{T}\right)}{V^\pi\left(\widetilde{T}\right)}\mathbf{1}_{\left\{1/\left(Z^\theta(\widetilde{T})V^\pi(\widetilde{T})\right)>0\right\}}\right], \tag{2.37}$$

where $\mathbb{E}^{\mathbb{Q}^\pi}$ denotes the expectation with respect to the new measure $\mathbb{Q}^\pi$. Under this measure $\mathbb{Q}^\pi$, the process $W^\pi(\cdot) = (W_1^\pi(\cdot), \ldots W_K^\pi(\cdot))^\mathsf{T}$ with

$$W_k^\pi(t\wedge\tau^\pi) := W_k(t\wedge\tau^\pi) + \int_0^{t\wedge\tau^\pi}\theta_k^\pi(u,S(u))du \tag{2.38}$$

for all $k = 1,\ldots,K$ and $t\in[0,T]$ is a K-dimensional Brownian motion stopped at time $\tau^\pi := \lim_{i\to\infty}\inf\{t\in[0,T] : Z^\theta(t)V^\pi(t)\geq i\}$. The process $\theta^\pi(\cdot,\cdot)$ here is exactly the π-specific market price of risk from Definition 3. The equality in (2.13) *holds until the stopping time τ^π and, in particular, the processes $S(\cdot)$ and $1/(Z^\theta(\cdot)V^\pi(\cdot))$ evolve until τ^π under $\mathbb{Q}^\pi$ according to*

$$dS_i(t) = S_i(t)\sum_{j=1}^d a_{i,j}(t,S(t))\pi_j(t,S(t))dt + S_i(t)\sum_{k=1}^K\sigma_{i,k}(t,S(t))dW_k^\pi(t) \tag{2.39}$$

and

$$d\left(\frac{1}{Z^\theta(t)V^\pi(t)}\right) = \theta^\pi(t,S(t))\frac{1}{Z^\theta(t)V^\pi(t)}dW^\pi(t)$$

for all $i = 1,\ldots,d$ and $t\in[0,T]$. Furthermore, the statements of Corollary 3 hold with $\mathbb{Q}$ replaced by $\mathbb{Q}^\pi$ and $Z^\theta(\cdot)$ replaced by $Z^\theta(\cdot)V^\pi(\cdot)$. This yields the representation

$$U^\pi(T-t,s) = \mathbb{Q}^\pi\left(\frac{1}{Z^\theta(T)V^\pi(T)} > 0\,\middle|\,\mathcal{F}(t)\right). \tag{2.40}$$

Proof. The proof goes exactly along the lines of Theorem 4 and Corollaries 2 and 3 with the obvious modifications. □

We emphasize the similarity of the $\mathbb{Q}^\pi$-dynamics of $S(\cdot)$ in (2.39) and the PDE in (2.18) for U^π.

The next proposition demonstrates how one can construct a probability space that satisfies the technical conditions of this section:

Proposition 3 (Canonical probability space)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *denote any probability space, equipped with a filtration* $\mathbb{F} = \mathcal{F}(\cdot)$ *that satisfies the usual conditions. There exists a probability space* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$*, equipped with a filtration* $\tilde{\mathcal{F}}(\cdot)$*, which supports a probability measure* $\mathbb{Q}$ *such that* $\tilde{\mathbb{P}}$ *is absolutely continuous with respect to* $\mathbb{Q}$*, and such that* (2.33) *holds for any* $\tilde{\mathcal{F}}(\tilde{T})$*-measurable random variable* $D \geq 0$ *for any* $\tilde{\mathcal{F}}(\cdot)$*-stopping time* $\tilde{T}$*.*

Furthermore, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *has the same distributional properties as* $(\Omega, \mathcal{F}, \mathbb{P})$*; that is, it supports a* K*-dimensional Brownian motion* $W(\cdot)$*, a vector-valued process* $\theta(\cdot, \cdot)$*, a vector-valued process* $\mu(\cdot, \cdot)$*, a matrix-valued process* $\sigma(\cdot, \cdot)$*, a* d*-dimensional progressively measurable stock price process* $S(\cdot)$ *that satisfies* (2.1)*, and a process* $Z^\theta(\cdot)$ *that satisfies* (2.6)*. The filtration* $\tilde{\mathcal{F}}(\cdot)$ *can be assumed to be completed under* $\mathbb{Q}$ *but not necessarily under* $\tilde{\mathbb{P}}$*. However, any process* $\phi(\cdot)$ *that is progressively measurable with respect to the* $\tilde{\mathbb{P}}$*-completed filtration has a modification* $\tilde{\phi}(\cdot)$ *that is progressively measurable with respect to* $\tilde{\mathcal{F}}(\cdot)$ *and that is indistinguishable from* $\phi(\cdot)$ *under* $\tilde{\mathbb{P}}$*. Furthermore, the process* $\widetilde{W}(\cdot)$ *of* (2.34) *is a Brownian motion under* $\mathbb{Q}$*, at least up to some stopping time.*

Proof. We map the probability space Ω on the canonical space $\tilde{\Omega} = C([0, \infty), \mathbb{R}^n \cup \{\Delta\})$ of $\mathbb{R}^n \cup \{\Delta\}$-valued functions which are absorbed in the "cemetery point" Δ and continuous before absorption. We use here $n = 1 + K + d + K + d + dK$. The paths in $\mathbb{R}^n$ are the images of $Z^\nu(\cdot), W(\cdot), S(\cdot), \theta(\cdot, S(\cdot))$, $\mu(\cdot, S(\cdot))$, and $\sigma(\cdot, S(\cdot))$.

As underlying filtration $\tilde{\mathcal{F}}(\cdot)$ we choose the right-continuous version of the canonical one, that is, the filtration generated by the paths but not completed by the null-sets. The mapping from Ω to $\tilde{\Omega}$ induces a measure $\tilde{\mathbb{P}}$.

Although the filtration $\tilde{\mathcal{F}}(\cdot)$ is not completed and stochastic integrals appear, the relations in (2.1) and (2.6) hold since all processes appearing are progressively measurable with respect to $\tilde{\mathcal{F}}(\cdot)$. The limit in the definition of stochastic integrals is therefore a-fortiori $\tilde{\mathcal{F}}(\cdot)$-measurable.

We observe that this probability space satisfies the technical requirements to introduce the measure $\mathbb{Q}$; see Meyer (1972). We now apply Theorem 4.

We finally augment the filtration $\tilde{\mathcal{F}}(\cdot)$ with all $\mathbb{Q}$-nullsets without changing the dynamics of the underlying processes under $\mathbb{Q}$, and therefore nor under $\tilde{\mathbb{P}}$ (see Theorem 2.7.9 of Karatzas and Shreve 1991). We refer the reader to the argument explicated in the first remark of Section 1 in Delbaen and Schachermayer (1995a) for the existence of an indistinguishable modification $\tilde{\phi}(\cdot)$ of some process $\phi(\cdot)$ as in the statement of the proposition. □

The next remarks relate our results to the existing literature:

Remark 7 (Portfolio-generating functions). Fernholz (1999) introduces and discusses strategies $\rho(\cdot,\cdot)$ of the form

$$\rho_i(t,s) = \left(D_i \log(R(\pi^m(t,s))) + 1 - \sum_{j=1}^{d} \pi_j^m(t,s) D_j \log(R(\pi^m(t,s))) \right) \pi_i^m(t,s)$$

for all $i = 1, \ldots, d$ and $(t,s) \in [0,T] \times \mathbb{R}_+^d$ where $\pi^m(\cdot,\cdot)$ denotes the market portfolio with $\pi_i^m(t,s) := s_i / \sum_{j=1}^d s_j$ and R any positive twice differentiable function satisfying some weak boundedness conditions. Then, Fernholz (1999) shows that the pathwise formula

$$\log\left(\frac{V^\rho(T)}{V^{\pi^m}(T)}\right) = \log\left(\frac{R(\pi^m(T,S(T)))}{R(\pi^m(0,S(0)))}\right) + \int_0^T \Theta(t,S(t))dt$$

holds where $\Theta : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}$ is some function that can be written down explicitly. This yields, in connection with Corollary 4, the formula

$$U^\rho(T,s) = \frac{1}{R(\pi^m(0,S(0)))} \mathbb{E}^{\mathbb{Q}^{\pi^m}} \left[R(\pi^m(T,S(T))) \exp\left(\int_0^T \Theta(t,S(t))dt \right) \cdot \mathbf{1}_{\left\{1/\left(Z^\theta(T)V^{\pi^m}(T)\right)>0\right\}} \right],$$

which can be used to compute optimal trading strategies. □

Remark 8 (Perfect balance and optimal growth). Kardaras (2008) discusses in the case of the market portfolio $\pi_i(t,s) = \pi_i^m(t,s) := s_i / \sum_{j=1}^d s_j$ for all $(t,s) \in [0,T] \times \mathbb{R}_+^d$ and $i = 1,\dots,d$ the "perfect balance condition" $\mu(\cdot,\cdot) = a(\cdot,\cdot)\pi(\cdot,\cdot)$, which is exactly the mean rate of return appearing in the dynamics of (2.39). If the "perfect balance condition" holds under the "real-world" measure $\mathbb{P}$, then each component of the market portfolio $\pi^m(\cdot,\cdot)$ is a martingale. If $\pi(\cdot,\cdot)$ is not the market portfolio then this martingale property usually does not hold for the components of $\pi(\cdot,\cdot)$. However, the condition still implies that the strategy $\pi(\cdot,\cdot)$ is growth-optimal in the sense of Problem 4.6 of Fernholz and Karatzas (2009). That means that $\pi(\cdot,\cdot)$ maximizes the mean rate of return $\rho^\mathsf{T}(\cdot,\cdot)\mu(\cdot,\cdot) - 1/2\rho^\mathsf{T}(\cdot,\cdot)a(\cdot,\cdot)\rho(\cdot,\cdot)$ of the logarithm of the associated wealth process over all strategies $\rho(\cdot,\cdot)$. More generally, if

$$\mu(\cdot,\cdot) = a(\cdot,\cdot)\pi(\cdot,\cdot) + \sigma(\cdot,\cdot)c(\cdot,\cdot),$$

for some $c : [0,T] \times \mathbb{R}_+^d \to \mathbb{R}^K$ such that the stochastic exponential of $\theta^\pi(\cdot,\cdot) \equiv c(\cdot,\cdot)$ in (2.12) is a martingale, then there is no arbitrage possible with respect to $\pi(\cdot,\cdot)$. This follows directly from the fact that the martingale property implies that $\mathbb{P}$ and $\mathbb{Q}^\pi$ are equivalent and thus firstly, $W^\pi(\cdot)$ of (2.38) is a true Brownian motion and secondly, the fraction $V^\rho(\cdot)/V^\pi(\cdot)$ of (2.13) is a supermartingale for any strategy $\rho(\cdot,\cdot)$ under $\mathbb{Q}^\pi$; compare Corollary 4. Formally, for $c(\cdot,\cdot) \neq 0$ the perfect balance condition of Kardaras (2008) is satisfied in the case of $\pi(\cdot,\cdot) \equiv \pi^m(\cdot,\cdot)$; however, the interest rates are "out of balance." In the literature, a growth-optimal portfolio is often also called "numéraire portfolio;" compare Section 3.5 of Platen (2002). □

Remark 9 (Connections to the work of Delbaen and Schachermayer (1995a)). Delbaen and Schachermayer (1995a) show that under equivalent technical assumptions, there exists for any strictly positive local martingale $Z(\cdot)$ a measure such that under the new measure $1/Z(\cdot)$ is a martingale that can hit zero. In their work, $Z(\cdot)$ represents the reciprocal of the stock price while in this work we treat the case of a stochastic discount factor. In both situations a positive probability of $1/Z(\cdot)$ hitting zero under the new measure leads to an arbitrage opportunity. In this work, we can additionally compute a strategy that uses minimal initial capital to perform the arbitrage. Furthermore, we do not look only at arbitrage with respect to the money market but also at arbitrage with respect to a much broader class of strategies. □

2.6 Examples

In this section, we discuss several examples for markets that imply arbitrage opportunities. Examples 1, 2 and 3 study different strategies for the three-dimensional Bessel process with drift. Example 4 concentrates on the reciprocal of the three-dimensional Bessel, a standard example in the bubbles literature. Finally, Example 5 illustrates a process that leads to a hedging price that is not differentiable and not even continuous but where the delta hedge still works.

Example 1 (Three-dimensional Bessel process with drift - money market). One of the best known examples for markets with arbitrage is the three-dimensional Bessel process, as discussed in Section 3.3.C of Karatzas and Shreve (1991). A Bessel process starting at some point $x > 0$ is in distribution identical to the Euclidean norm of a three-dimensional Brownian motion with the first component starting at x and the other two components starting at zero. We study here a class of models that contain the Bessel process as special case and generalize the example for arbitrage of A.V. Skorohod in Section 1.4 of Karatzas and Shreve (1998). For that, we begin with defining an auxiliary stochastic process $X(\cdot)$ as a Bessel process with drift $-c$, that is,

$$dX(t) = \left(\frac{1}{X(t)} - c\right) dt + dW(t) \tag{2.41}$$

for all $t \in [0, T]$ with $W(\cdot)$ denoting a Brownian motion on its natural filtration $\mathbb{F} = \mathbb{F}^W$ and $c \in [0, \infty)$ a constant. The process $X(\cdot)$ is strictly positive, since it is a Bessel process, thus strictly positive under the equivalent measure where $\{W(t) - ct\}_{0 \le t \le T}$ is a Brownian motion. The stock price process is now defined via the stochastic differential equation

$$dS(t) = \frac{1}{X(t)} dt + dW(t) \tag{2.42}$$

for all $t \in [0, T]$. Both processes $X(\cdot)$ and $S(\cdot)$ are assumed to start at the same point $S(0) > 0$. From (2.41) and (2.42) we obtain directly the identity $X(t) = S(t) - ct$, which yields the stock price dynamics

$$dS(t) = S(t) \left(\frac{1}{S^2(t) - S(t)ct} dt + \frac{1}{S(t)} dW(t) \right)$$

for all $t \in [0, T]$. Furthermore, since $c \ge 0$ holds, we have strictly positive stock prices $S(t) > ct \ge 0$ for all $t \in [0, T]$. Thus, for $c > 0$, the model allows for an "obvious arbitrage" in the sense of Definition 1.2 in Guasoni et al. (2010). If $c = 0$ then $S(\cdot) \equiv X(\cdot)$ and the stock price process is a Bessel process. Of course, the market price of risk is exactly $1/X(\cdot)$ or, more precisely, we have

$$\theta(t, s) = 1/(s - ct)$$

and

$$Z^\theta(t) = \exp\left(-\int_0^t \frac{1}{S(u) - cu} dW(u) - \frac{1}{2} \int_0^t \frac{1}{(S(u) - cu)^2} du \right)$$

for all $(t, s) \in [0, T] \times \mathbb{R}_+$ with $s > ct$. Thus, the reciprocal $1/Z^\theta(\cdot)$ of the stochastic discount factor hits zero exactly when $S(t)$ hits ct. This follows directly from the $\mathbb{Q}$-dynamics of $1/Z^\theta(\cdot)$ derived in Corollary 2 and a strong law of large numbers as in Lemma A.2 of Kardaras (2008).

Let us start by looking at a general, for the moment not-specified Markovian trading strategy $\pi(\cdot, \cdot)$ whose associated wealth at time T is a function of the stock price, that is, $V^\pi(T) =: p(S(T))$. For all $(t, s) \in [0, T] \times \mathbb{R}_+$ with $s > ct$, by relying on Corollary 3 and changing the measure $\mathbb{Q}$ to $\bar{\mathbb{Q}}$ under which $\left\{\bar{S}(t)\right\}_{0 \le t \le T} := \{S(t) - ct\}_{0 \le t \le T}$ is a Brownian motion, we obtain

$$
\begin{aligned}
U^\pi(T-t, s) =& \mathbb{E}^{t,s}\left[\frac{Z^\theta(T) V^\pi(T)}{Z^\theta(t) V^\pi(t)}\right] \\
=& \frac{1}{V^\pi(t)}\, \mathbb{E}^{\mathbb{Q}}\left[p(S(T)) \mathbf{1}_{\{\min_{t \le u \le T}\{S(u) - cu\} > 0\}} \middle| \mathcal{F}(t)\right]\Big|_{S(t)=s} \\
=& \frac{1}{V^\pi(t)} \mathbb{E}^{\bar{\mathbb{Q}}}\Bigg[\exp\left(-c\left(\bar{S}(T) - \bar{S}(t)\right) - \frac{c^2(T-t)}{2}\right) \\
& \cdot\ p(S(T)) \mathbf{1}_{\left\{\min_{t \le u \le T}\left\{\bar{S}(u)\right\} > 0\right\}} \Bigg| \mathcal{F}(t)\Bigg]\Bigg|_{\bar{S}(t) = s - ct} \\
=& \frac{1}{V^\pi(t)} \int_0^\infty \exp\left(-c(y - s + ct) - \frac{c^2(T-t)}{2}\right) p(y + cT) \\
& \cdot \frac{1}{\sqrt{2\pi(T-t)}} \left(\exp\left(-\frac{(y - s + ct)^2}{2(T-t)}\right) - \exp\left(-\frac{(y + s - ct)^2}{2(T-t)}\right)\right) dy
\end{aligned} \tag{2.43}
$$

$$
\begin{aligned}
=& \frac{1}{V^\pi(t)} \Bigg(\int_{\frac{cT - s}{\sqrt{T-t}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) p(z\sqrt{T-t} + s) dz \\
& - \exp(2cs - 2c^2 t) \\
& \cdot \int_{\frac{cT - 2ct + s}{\sqrt{T-t}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) p(z\sqrt{T-t} - s + 2ct) dz\Bigg),
\end{aligned} \tag{2.44}
$$

where we have plugged in the density of a Brownian motion absorbed at zero (compare Problem 2.8.6 of Karatzas and Shreve 1991) and made use of the substitution $z = (y - s + cT)/\sqrt{T-t}$ and $z = (y + s + cT - 2ct)/\sqrt{T-t}$, respectively.

Let us consider the investment in the money market only, to wit, the strategy $\pi^0(\cdot,\cdot) \equiv 0$ and $V^{\pi^0}(t) \equiv 1 \equiv p(s)$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$. This yields the hedging price of one monetary unit

$$U^{\pi^0}(T-t,s) = \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - \exp(2cs - 2c^2 t)\Phi\left(\frac{-s-cT+2ct}{\sqrt{T-t}}\right), \quad (2.45)$$

where Φ denotes the cumulative standard normal distribution function. In the special case $c = 0$ we have

$$U^{\pi^0}(T-t,s) = 2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1. \quad (2.46)$$

For the first derivative we obtain

$$\begin{aligned} \frac{\partial}{\partial s} U^{\pi^0}(T-t,s) = &- 2c\exp(2cs - 2c^2 t)\Phi\left(\frac{-s-cT+2ct}{\sqrt{T-t}}\right) \\ &+ \sqrt{\frac{2}{\pi(T-t)}}\exp\left(-\frac{(cT-s)^2}{2(T-t)}\right), \end{aligned}$$

which simplifies in the case of $c = 0$ to

$$\frac{\partial}{\partial s} U^{\pi^0}(T-t,s) = \sqrt{\frac{2}{\pi(T-t)}}\exp\left(-\frac{s^2}{2(T-t)}\right). \quad (2.47)$$

It can be easily checked that U^{π^0} solves the PDE in (2.18) for all $(t,s) \in [0,T] \times \mathbb{R}_+$ with $s > ct$. This is sufficient to apply Theorem 1 (compare Remark 5) to find the optimal hedging strategy of one monetary unit:

$$\widehat{\pi}^0(t,s) = s\frac{\partial}{\partial s}\log\left(U^{\pi^0}(T-t,s)\right). \quad (2.48)$$

For $c = 0$ we obtain thus from (2.46) and (2.47) the representation

$$\widehat{\pi}^0(t,s) = \frac{2\frac{s}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right)}{2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1} > 0, \quad (2.49)$$

where ϕ denotes the standard normal density.

It comes at no surprise that, in order to beat the money market, we have to be long the stock. The strategy $\widehat{\pi}^0(\cdot,\cdot)$ has for $c = 0$ another interpretation. To derive it, we observe that $U^{\pi^0}(T-t,s)$ is the probability that a Brownian motion $\widetilde{W}(\cdot)$ starting at s does not hit zero before time $T-t$. Using the density of the hitting time $T_0 := \inf\{t \geq 0 : \widetilde{W}(t) = 0\}$, (compare for example Proposition 2.8.5 of Karatzas and Shreve 1991) yields

$$U^{\pi^0}(T-t,s) = \mathbb{Q}\left(T_0 > T-t\right) = \frac{1}{\sqrt{2\pi}} \int_{T-t}^{\infty} \frac{s}{y^{\frac{3}{2}}} \exp\left(-\frac{s^2}{2y}\right) dy,$$

which gives

$$\frac{\partial}{\partial s} U^{\pi^0}(T-t,s) = \frac{1}{\sqrt{2\pi}} \int_{T-t}^{\infty} \left(\frac{1}{y^{\frac{3}{2}}} - \frac{s^2}{y^{\frac{5}{2}}}\right) \exp\left(-\frac{s^2}{2y}\right) dy$$

and

$$\widehat{\pi}^0(t,s) = 1 - \frac{\frac{1}{\sqrt{2\pi}} \int_{T-t}^{\infty} \frac{s^3}{y^{\frac{5}{2}}} \exp\left(-\frac{s^2}{2y}\right) dy}{U^{\pi^0}(T-t,s)}.$$

This is exactly

$$\widehat{\pi}^0(t,s) = 1 - s^2\, \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T_0}\middle| \min_{0 \leq u \leq T-t} \widetilde{W}(u) > 0\right].$$

It is well-known that a Bessel process allows for arbitrage. Compare for example Example 3.6 of Karatzas and Kardaras (2007) for an ad-hoc strategy that corresponds to a hedging price of $\Phi(1)$ for a monetary unit if $S(0) = T = 1$. We have improved here the existing strategies and found the optimal one, which corresponds in this setup to a hedging price of $U^{\widehat{\pi}^0}(1,1) = 2\Phi(1) - 1 < \Phi(1)$. □

Remark 10 (Multiple solutions for the PDE in (2.18)). We observe that the hedging price U^{π^0} in (2.45) depends on the drift c. Also, U^{π^0} is sufficiently differentiable, thus by Proposition 2 uniquely characterized as the minimal nonnegative solution of the PDE in (2.18), which does not depend on the drift c. The uniqueness of U^{π^0} by Proposition 2 and the dependence of U^{π^0} on c do not contradict each other, since the nonnegativity of U^{π^0} has only to hold for these points $(t,s) \in [0,T] \times \mathbb{R}_+$ that can be attained by $S(\cdot)$ at time t. For a given time $t \in [0,T]$, these are only the points $s > ct$. Thus, as c increases, the nonnegativity condition weakens since it has to hold for fewer points, thus U^{π^0} can become smaller and smaller. Indeed, plugging in (2.45) the point $s = ct$ yields $U^{\pi^0}(T-t, ct) = 0$. In summary, while the PDE itself does only depend on the (more easily observable) volatility structure of the stock price dynamics, the mean rate of return determines where the PDE has to hold and thus, contributes to determining the exact amount of possible arbitrage. □

In the next example, we price and hedge a European call within the same class of models as in the last example:

Example 2 (Three-dimensional Bessel process with drift - stock and European call). Since we do not know a priori any (possibly suboptimal) strategy that leads to the value $(S(T)-L)^+$ at time T for some strike $L \geq 0$ we cannot rely on Theorem 1 and have to tackle this question slightly differently using the results of Theorem 2. Plugging in $p(y) = (y-L)^+$ in (2.44), defining

$$h^p(T-t,s) := \mathbb{E}^{t,s}\left[\frac{Z^\theta(T)}{Z^\theta(t)}(S(T)-L)^+\right]$$

for all $(t,s) \in [0,T] \times \mathbb{R}^d_+$ with $s > ct$, and using the notation $a \vee b := \max\{a,b\}$ we can simplify the expected risk-adjusted value as follows:

$$\begin{aligned}
h^p(T-t,s) = &\int_{\frac{(cT\vee L)-s}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)(z\sqrt{T-t}+s-L)dz - \exp(2cs-2c^2t) \\
&\cdot \int_{\frac{(cT\vee L)-2ct+s}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)(z\sqrt{T-t}-s+2ct-L)dz \\
=&\sqrt{\frac{T-t}{2\pi}}\exp\left(-\frac{(s-(cT\vee L))^2}{2(T-t)}\right) + (s-L)\Phi\left(\frac{s-(cT\vee L)}{\sqrt{T-t}}\right) \\
&-\exp(2cs-2c^2t)\left(\sqrt{\frac{T-t}{2\pi}}\exp\left(-\frac{((cT\vee L)-2ct+s)^2}{2(T-t)}\right)\right.
\end{aligned}$$

$$+ (2ct - s - L)\Phi\left(\frac{-(cT \vee L) + 2ct - s}{\sqrt{T-t}}\right)\Bigg). \tag{2.50}$$

The modified put-call parity of Corollary 1 could now be applied to give us directly the hedging price of a European put. If $L \le cT$, in particular if $L = 0$, the last expression simplifies to

$$\begin{aligned} h^p(T-t,s) =& s\Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) + \exp(2cs - 2c^2t)\Phi\left(\frac{2ct - s - cT}{\sqrt{T-t}}\right) \\ & \cdot (s - 2ct) - LU^{\pi^0}(T-t,s), \end{aligned}$$

where U^{π^0} denotes the hedging price of one monetary unit given in (2.45). It is simply the difference between the hedging price of the stock and L monetary units since $L \le cT$ implies $S(T) > cT \ge L$ almost surely and the call is always exercised. Using $L = 0$ we get the value of the stock.

For $L = c = 0$, the last equality yields $h^p(t,s) = s$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$ and the stock cannot be arbitraged. There are at least two other ways to see this result right away. Simple computations show directly that $Z^\theta(T) = S(0)/S(T)$ if $c = 0$ and thus, for the strategy $\pi^1(\cdot,\cdot) \equiv 1$, which invests fully in the market, we obtain $U^{\pi^1}(\cdot,\cdot) \equiv 1$. Alternatively, using the representation of U^{π^1} implied by (2.33) we see that the hedging price is just the expectation of a Brownian motion stopped at zero, thus the expectation of a martingale started at one. Every method on its own shows the lack of relative arbitrage with respect to the market if $c = 0$.

On the other hand, if $c > 0$, then relative arbitrage with respect to the market is possible. In this case, the representation implied by (2.33) shows that as soon as the Brownian motion is stopped, which is the first time $S(t)$ equals ct, the value of the random variable of which the expectation is taken jumps to zero and thus the stopped process is not a martingale any more. Obviously,

$$U^{\pi^1}(T-t,s) = \mathbb{E}^{t,s}\left[\frac{Z^\theta(T)S(T)}{Z^\theta(t)s}\right] = \frac{1}{s}h^p(T-t,s)$$

is sufficiently differentiable and thus, Remark 5 and Theorem 1 yield the optimal arbitrage opportunity

$$\widehat{\pi}^1(t,s) = \frac{\frac{2ct}{\sqrt{T-t}}\phi\left(\frac{s-cT}{\sqrt{T-t}}\right) + \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) + \exp(2cs-2c^2t)\Phi\left(\frac{2ct-s-cT}{\sqrt{T-t}}\right)(2cs-4c^2t+1)}{U^{\pi^1}(T-t,s)}.$$

We can now find the corresponding strategy for the call price of (2.50). Assuming for the sake of notation that $L \geq cT$, Theorem 2 yields

$$\begin{aligned}\pi^p(t,s) = \Bigg(& s\Phi\left(\frac{s-L}{\sqrt{T-t}}\right) + s\exp(2cs-2c^2t)\Phi\left(\frac{2ct-s-L}{\sqrt{T-t}}\right) \\ & \cdot (2cs+2cK-4c^2t+1) - 2c\sqrt{T-t}\phi\left(\frac{L+s-2ct}{\sqrt{T-t}}\right)\Bigg) / h^p(T-t,s)\end{aligned}$$

as the optimal strategy. If $c = 0$ this simplifies to

$$\pi^p(t,s) = \frac{s\left(\Phi\left(\frac{s-L}{\sqrt{T-t}}\right) + 1 - \Phi\left(\frac{s+L}{\sqrt{T-t}}\right)\right)}{h^p(T-t,s)}.$$

Two notable observations can be made. First, in this model both the money market and the stock simultaneously have a hedging price cheaper than their current price, as long as $c > 0$. Second, in contrast to the classical theory of Financial Mathematics, the mean rate of return under the "real-world" measure does matter in determining the hedging price of calls (or other derivatives) since it influences the possibilities of arbitrage. □

We can now also find a quantile hedge for strategies $\pi(\cdot,\cdot)$ whose associated wealth process $V^\pi(T)$ is only a function of the market $S(T)$. To wit, for such $\pi(\cdot,\cdot)$ and some given $\eta \in [0,1)$ one can compute the value and optimal strategy for the quantity

$$\widetilde{U}^{\pi,\eta}(T,s) := \inf\left\{v > 0 : \exists \text{ strategy } \rho \text{ such that } \mathbb{P}^s\left(V^{v,\rho}(T) > V^\pi(T)\right) \geq 1-\eta\right\}; \tag{2.51}$$

that is, $\widetilde{U}^{\pi,\eta}(T,s)$ represents how much initial capital is needed to obtain the terminal wealth $V^\pi(T)$ with a given probability $1-\eta$. This question has been resolved in the case of the existence of an equivalent local martingale measure by Section 2.4 of Föllmer and Leukert (1999) relying on the Neyman-Pearson lemma. The next example illustrates a possible approach for markets with arbitrage in the case of the market portfolio in the Bessel process setup. Recently, Bayraktar et al. (2010a) solved the problem of finding a quantile hedge by means of formulating a stochastic control problem.

Example 3 (Three-dimensional Bessel process - quantile hedging). In order to compute the quantity in (2.51) it is clearly sufficient to compute the optimal strategy and the initial wealth for the contingent claim $D = S(T)/S(0)\mathbf{1}_{\{S(T)\leq\delta\}}$ for $\delta := \inf\{z > 0 : \mathbb{P}^{S(0)}(S(T) \geq z) \leq \eta\}$. By similar considerations as in Examples 1 and 2, the probability $\mathbb{P}^{S(0)}(S(T) \geq z)$ equals the expectation of a Brownian motion starting at $S(0)$ and stopped at zero to be above z at time T. We obtain, similar to (2.43),

$$\begin{aligned}\mathbb{P}^{S(0)}(S(T) \geq z) =& \mathbb{E}^{\mathbb{Q}^{S(0)}}\left[\mathbf{1}_{\{S(T\wedge T_0)>z\}}\frac{S(T\wedge T_0)}{S(0)}\right]\\ =&\frac{1}{S(0)}\int_z^\infty \frac{y}{\sqrt{2\pi T}}\left(\exp\left(-\frac{(y-S(0))^2}{2T}\right)\right.\\ &\left.-\exp\left(-\frac{(y+S(0))^2}{2T}\right)\right)dy\\ =&\frac{\sqrt{T}}{S(0)\sqrt{2\pi}}\left(\exp\left(-\frac{(z-S(0))^2}{2T}\right)-\exp\left(-\frac{(z+S(0))^2}{2T}\right)\right)\\ &+\Phi\left(\frac{S(0)-z}{\sqrt{T}}\right)+1-\Phi\left(\frac{z+S(0)}{\sqrt{T}}\right),\end{aligned}$$

where $S(\cdot)$ is a Brownian motion starting at $S(0)$ under $\mathbb{Q}^{S(0)}$ and T_0 the first hitting of zero by $S(\cdot)$. The truncation δ and the optimal strategy can now be computed as we have done for calls in Example 2. We omit the computations since they do not contain any new insights. □

Pal and Protter (2010) compute call prices for the reciprocal Bessel process model. This process has appeared several times in the bubbles literature, often called the constant elasticity of variance (CEV) process; see, for example, Section 2.2.2 of Cox and Hobson (2005) or Example 1.2 of Heston et al. (2007). We discuss next how the results of the last examples relate to this model and illustrate that even under the NFLVR condition relative arbitrage is possible.

Example 4 (Reciprocal of the three-dimensional Bessel process). Let the stock price $\tilde{S}(\cdot)$ have the dynamics

$$d\tilde{S}(t) = -\,\tilde{S}^2(t)dW(t)$$

for all $t \in [0,T]$ with $W(\cdot)$ denoting a Brownian motion on its natural filtration $\mathbb{F} = \mathbb{F}^W$. The process $\tilde{S}(\cdot)$ is exactly the reciprocal of the process $S(\cdot)$ of Examples 1 and 2 with $c = 0$, thus strictly positive. We observe that there is no classical arbitrage since the mean rate of return is zero and thus $\mathbb{P}$ is already a local martingale measure. However, there is arbitrage possible with respect to the stock. To wit, if one wants to hold the stock at time T, one should not buy the stock at time zero, but use the strategy $\widehat{\pi}^1(\cdot,\cdot)$ below for a hedging price smaller than $\tilde{S}(0)$ along with the suboptimal strategy $\pi^1(\cdot,\cdot) \equiv 1$. That is, the strategy $\pi^1(\cdot,\cdot)$ contains a bubble according to Definition 5.

We have already observed that $\tilde{S}(T) = 1/S(T)$, which is exactly the stochastic discount factor in Example 1 for $c = 0$ multiplied by $\tilde{S}(t)$. Thus, as in (2.46) the hedging price for the stock is

$$U^{\pi^1}(T-t,s) = 2\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - 1 < 1 \tag{2.52}$$

along with the optimal strategy

$$\widehat{\pi}^1(t,s) = \frac{-2\phi\left(\frac{1}{s\sqrt{T-t}}\right)}{s\sqrt{T-t}\left(2\Phi\left(\frac{1}{s\sqrt{T-t}}\right)-1\right)} + 1 < 1$$

for all $(t,s) \in [0,T) \times \mathbb{R}_+$ similar to (2.49). By Corollary 4, the hedging price U^{π^1} could also be calculated as one minus the probability of explosion (to ∞) of the process $\tilde{S}(\cdot)$ before time T under $\mathbb{Q}^{\pi^1}$, where it has the dynamics

$$d\tilde{S}(t) = \tilde{S}^3(t)dt - \tilde{S}^2(t)dW^{\pi^1}(t).$$

Alternatively, one could also calculate the probability of implosion (to zero) of the process $S(\cdot) = 1/\tilde{S}(\cdot)$ with dynamics

$$dS(t) = dW^{\pi^1}(t)$$

for all $t \in [0,T]$ under $\mathbb{Q}^{\pi^1}$, which is again a Brownian motion as in Example 1 but now starting at $1/S(0)$. For pricing calls, we observe

$$\left(\tilde{S}(T) - L\right)^+ = L\tilde{S}(T)\left(\frac{1}{L} - \frac{1}{\tilde{S}(T)}\right)^+ = \frac{L}{S(t)} \cdot \frac{S(t)}{S(T)}\left(\frac{1}{L} - S(T)\right)^+$$

for $L > 0$. Thus, the price at time t of a call with strike L in the reciprocal Bessel model is the price of $L\tilde{S}(t)$ puts with strike $1/L$ in the Bessel model and can be computed from Example 2 and Corollary 1. For $S(0) = 1$, simple computations will lead directly to Equation (6) of Pal and Protter (2010). The optimal strategy could now be derived with Theorem 2. □

The next example[7] illustrates that U^π need not be differentiable or even smooth in the stock price dimension in order to find an optimal strategy.

Example 5 (U^π not differentiable). Let us consider a market with time horizon $T = 2$ and one stock with dynamics

$$dS(t) = \begin{cases} \mathbf{1}_{\left\{S(t)>\frac{1}{2}\right\}} dW(t) & \text{if } t < 1, \\ \mathbf{1}_{\{S(t)>1\}}\left(\frac{1}{S(t)-1}dt + dW(t)\right) & \text{if } t \geq 1. \end{cases}$$

Thus, up to time $t = 1$ the stock price is either constant or evolves as Brownian motion stopped at $1/2$. If at time $t = 1$ the stock price is less than or equal to

[7]We developed this example after a helpful conversation with Daniel Fernholz.

one, it stays constant and otherwise evolves as a three-dimensional Bessel process shifted by one. From (2.46) the hedging price for the money market is

$$U^{\pi^0}(t,s) = \begin{cases} 1 & \text{if } s \leq 1, \\ 2\Phi\left(\frac{s-1}{\sqrt{t}}\right) - 1 & \text{if } s > 1 \end{cases}$$

for $t \in [0,1]$. Then, the hedging price U^{π^0} is not continuous for $s = 1$, thus not differentiable. However, there always exists an optimal strategy $\widehat{\pi}^0$. For $(t,s) \in [0,1] \times [0,1]$ no arbitrage is possible, which implies that $\widehat{\pi}^0(2-t,s) = 0$ is optimal. For $(t,s) \in [0,1] \times (1,\infty)$ we know that the stock price always stays above one and the optimal strategy is the one given in (2.49) with s replaced by $s-1$ on the right-hand side. For $t \in (1,2]$, the function $U^{\pi^0}(t,s) = \mathbb{E}^{2-t,s}[U^{\pi^0}(1,S(1))]$ is easily shown to be sufficiently differentiable. Therefore, we can apply Theorem 1 to obtain $\widehat{\pi}^0(2-t,s)$. We have illustrated that, although $U^{\pi^0}(t,s)$ is not differentiable in the stock price dimension, namely for $(t,s) \in [0,1] \times \{1\}$ in this example, there can nevertheless exist an optimal strategy $\widehat{\pi}^0(\cdot,\cdot)$. □

2.7 Conclusion

It has been proven that, under weak technical assumptions, there is no equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge. To ensure its existence, weak sufficient conditions have been introduced that guarantee the differentiability of an expectation parameterized over time and over the original market configuration. The dynamics of stochastic processes simplify after a non-equivalent change of measure and a generalized Bayes' rule has been derived. From an analytic point of view, results of Fernholz and Karatzas (2010) concerning non-uniqueness of the Cauchy problem of (2.18) have been generalized to a class of PDEs that allow for a larger set of drifts. With this newly developed machinery, some optimal trading strategies have been computed addressing standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

2.8 Condition that hedging price solves a PDE

In this section, which serves as an appendix to this chapter, we provide a necessary condition for U^π of (2.16) solving the PDE in (2.18) and especially for being sufficiently differentiable.

One way to ensure smoothness of U^π is to follow the arguments in Subsection 2.4.3. To start, we formulate an additional assumption:

(A1') The functions $\theta_k^\pi(\cdot,\cdot)$ are for all $k = 1,\dots,K$ locally Lipschitz and locally bounded.

In particular, this assumption restricts the possible strategies $\pi(\cdot,\cdot)$; however, it allows for the market portfolio $\pi^m(\cdot,\cdot)$, for example.

Then, Assumptions (A1), (A1'), and (A2) guarantee the necessary smoothness of U^π. This can be seen directly from Theorem 3, when we replace $Z^\theta(\cdot)$ by $Z^\theta(\cdot)V^\pi(\cdot)$ and set the payoff function $p(\cdot) \equiv 1$.

An alternative way to show smoothness goes along the lines of Section 9 in Fernholz and Karatzas (2010): First, we remove the stochastic integral by assuming that there exists a real-valued function $H^\pi \in C^{1,2}([0,T] \times \mathbb{R}_+^d)$ such that

$$\sum_{i=1}^{d} \sigma_{i,k}(t,s) s_i D_i H^\pi(t,s) = \theta_k^\pi(t,s) \tag{2.53}$$

for all $k = 1,\dots,K$ and $(t,s) \in [0,T] \times \mathbb{R}_+^d$. That is, if the covariance process $a(t,s)$ has a multiplicative inverse $a^{-1}(t,s)$ on $[0,T] \times \mathbb{R}_+^d$, then H^π has partial derivatives of the form

$$D_i H^\pi(t,s) = \frac{\sum_{j=1}^{d} a_{i,j}^{-1}(t,s)\mu_j(t,s) - \pi_i(t,s)}{s_i}. \tag{2.54}$$

This condition basically means that $\pi(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$ are sufficiently smooth in time and space and have an anti-derivative. As Remark 5 discusses, this assumption can easily be slightly generalized. Applying Itô's formula to H^π yields

$$H^\pi(T,S(T)) - H^\pi(t,S(t)) - \int_t^T \left(\mathcal{L}H^\pi(u,S(u)) - \frac{\partial}{\partial t}H^\pi(u,S(u))\right) du = \int_t^T \theta^{\pi\mathsf{T}}(u,S(u))dW(u),$$

where $\mathcal{L}$ is the infinitesimal generator of $S(\cdot)$ defined in (2.22). Collecting all deterministic terms in a function $k^\pi : [0,T]\times\mathbb{R}_+^d \to \mathbb{R}$, we obtain

$$\begin{aligned} k^\pi(t,s) :=& \mathcal{L}H^\pi(t,s) + \frac{\partial}{\partial t}H^\pi(t,s) - \frac{1}{2}\|\theta^\pi(t,s)\|^2 \\ =& \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_i s_j a_{i,j}(t,s) D_{i,j}^2 H^\pi(t,s) + \sum_{i=1}^d s_i\mu_i(s,t) D_i H^\pi(s,t) \\ &+ \frac{\partial}{\partial t}H^\pi(t,s) - \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_i s_j a_{i,j}(t,s) D_i H^\pi(t,s) D_j H^\pi(t,s) \\ =& \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_i s_j a_{i,j}(t,s) D_{i,j}^2 H^\pi(t,s) + \sum_{i=1}^d\sum_{j=1}^d s_i a_{i,j}(t,s)\pi_j(t,s) D_i H^\pi(t,s) \\ &+ \frac{\partial}{\partial t}H^\pi(t,s) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_i s_j a_{i,j} D_i H^\pi(t,s) D_j H^\pi(t,s), \end{aligned} \tag{2.55}$$

where the last equality follows from the definitions of H^π and $\theta^\pi(\cdot,\cdot)$ in (2.53) and (2.12). Using that, (2.17) can now be written as

$$U^\pi(t,s) = \exp(H^\pi(T-t,s))\mathbb{E}^{T-t,s}\left[\exp(-H^\pi(T,S(T)))\exp\left(\int_{T-t}^T k^\pi(u,S(u))du\right)\right].$$

To proceed, we make the additional assumption that the deterministic function $G^\pi : [0,T]\times\mathbb{R}_+^d \to \mathbb{R}^+$ defined as

$$\begin{aligned} G^\pi(t,s) :=& \exp(-H^\pi(T-t,s))U^\pi(t,s) \\ =& \mathbb{E}^{T-t,s}\left[\exp(-H^\pi(T,S(T)))\exp\left(\int_{T-t}^T k^\pi(u,S(u))du\right)\right], \end{aligned} \tag{2.56}$$

which does not involve a stochastic integral any more, solves the time-inhomogeneous Cauchy problem

$$\frac{\partial}{\partial t}G^\pi(t,s) = \mathcal{L}G^\pi(t,s) + k^\pi(T-t,s)G^\pi(t,s). \tag{2.57}$$

To sum up, this second approach requires the existence of a smooth function H^π satisfying (2.54), and G^π being a solution of the Cauchy problem in (2.57). Chapter 9 of Fernholz and Karatzas (2010) discusses general conditions under which the later assumption is satisfied; however, we feel that these assumptions tend to be more restrictive than assumptions (A1), (A1'), and (A2). Nevertheless, if they hold true, then the next lemma concludes the argument:

Lemma 4 (PDE for U^π)**.** *If G^π defined in* (2.56) *solves the PDE in* (2.57) *then U^π solves the PDE in* (2.18).

Proof. Omitting arguments for the sake of clarity, we have

$$\begin{aligned} D_iU^\pi =& U^\pi D_iH^\pi + \exp(H^\pi)D_iG^\pi, \\ D^2_{i,j}U^\pi =& U^\pi D^2_{i,j}H^\pi + U^\pi D_iH^\pi D_jH^\pi + \exp(H^\pi)D_iH^\pi D_jG^\pi \\ &+ \exp(H^\pi)D_jH^\pi D_iG^\pi + \exp(H^\pi)D^2_{i,j}G^\pi. \end{aligned}$$

Therefore, collecting the U^π terms and comparing them to the representation of k^π in (2.55) and collecting the $\exp(H^\pi)$ terms we obtain

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_is_ja_{i,j}D^2_{i,j}U^\pi + \sum_{i=1}^d\sum_{j=1}^d s_ia_{i,j}\pi_jD_iU^\pi = U^\pi\left(k^\pi - \frac{\partial}{\partial t}H^\pi\right) \\ &+\exp(H^\pi)\left(\sum_{i=1}^d s_iD_iG^\pi\sum_{j=1}^d a_{i,j}\left(\pi_j + s_jD_jH^\pi\right) + \frac{1}{2}\sum_{i=1}^d\sum_{j=1}^d s_is_ja_{i,j}(t,s)D^2_{i,j}G^\pi\right) \\ &\qquad = \exp(H^\pi)(G^\pi k^\pi) - \left(\exp(H^\pi)\frac{\partial}{\partial t}H^\pi\right)G^\pi + \exp(H^\pi)\mathcal{L}G^\pi, \end{aligned}$$

where the last equality follows from the identities

$$\sum_{j=1}^d \sigma_{j,k}(\pi_j + s_jD_jH^\pi) = \theta_k$$

and

$$\sum_{k=1}^{K} \sigma_{i,k}\theta_k = \mu_i$$

for all $i, k = 1, \ldots, d$. That proves the statement since time goes in the reverse direction. □

Chapter 3

Completeness and Relative Arbitrage

3.1 Introduction

This chapter examines conditions under which contingent claims can be replicated by dynamic trading in the stock market. Let $S(\cdot)$ be a continuous, d-dimensional Itô-process (the "stock price process") with respect to a filtration $\mathcal{F}(\cdot)$ and $D \geq 0$ be an $\mathcal{F}(T)$-measurable random variable (the "claim"). The question then is when D can be represented as a stochastic integral of some progressively measurable process (the "trading strategy") with respect to $S(\cdot)$. Replicable claims have been completely characterized if $S(\cdot)$ satisfies the *No Free Lunch with Vanishing Risk (NFLVR)* condition. This notion was introduced by Delbaen and Schachermayer (1994) and is equivalent to the existence of an equivalent measure under which $S(\cdot)$ is a local martingale. If the supremum over the expected values of D under all *equivalent local martingale measures (ELMMs)* is attained, then the claim D can be replicated.

We generalize this characterization for replicable claims to markets that do not necessarily satisfy NFLVR, but allow for a *stochastic discount factor*; this corresponds to a weak structural restriction on the drift of the process. Stochastic discount factors are local martingales and take the place of the Radon-Nikodym derivatives that are used to change the original measure to an ELMM in the case of NFLVR. If the supremum over the expected values of D multiplied by all stochastic discount factors is attained, then the claim D can be replicated.

NFLVR is a mathematical concept introduced to characterize markets that admit an ELMM, and thus exclude arbitrage opportunities. However, from an economic perspective, it is reasonable to consider models which do not satisfy NFLVR. As Loewenstein and Willard (2000a) and Hugonnier (2010) discuss, models without NFLVR may nevertheless lead to an equilibrium where rational agents have an optimum. Thus, although NFLVR is a convenient mathematical assumption, it does not always accurately reflect our economic intuition of arbitrage. In particular, the existence of a stochastic discount factor prevents arbitrage opportunities from being scaled up, thus allowing for the existence of a numéraire portfolio and of solutions to utility maximization problems; see Karatzas and Kardaras (2007).

In a similar vein, the theory of "real world pricing" in the "Benchmark Approach" (see Platen and Heath 2006) acknowledges that no ELMM is needed to have the concept of a price for contingent claims. Models that allow for some kind of arbitrage are furthermore studied in the framework of "Stochastic Portfolio Theory" (see Fernholz 2002; Fernholz and Karatzas 2009), which starts from the premise of realistically describing the evolution of market weights over long time horizons and provides simple testable conditions, such as "diversity" or "sufficient intrinsic volatility," under which arbitrage does exist. These insights and ideas lead to the conclusion that models which impose the existence of a stochastic discount factor, but do not necessarily additionally assume NFLVR, are a natural class of models to study. This chapter is therefore a step to close the gap in the theory between the class of models with and without the assumption of NFLVR.

A nonnegative stock price process $S(\cdot)$ has been dubbed a *bubble*, if the set of ELMMs is nonempty and $S(\cdot)$ is a strict local martingale under an ELMM. Such a stock price models an asset that is overpriced compared with its *intrinsic value*, as measured by its expectation under an ELMM. It behaves locally like a martingale, but in the long run, behaves like a strict supermartingale. Academic literature has recently devoted substantial attention to bubbles, given that they are able to model seemingly overpriced stocks as in the “Internet Bubble” within the framework of NFLVR. We suggest Cox and Hobson (2005), Heston et al. (2007), and Jarrow et al. (2010) as some initial references to this literature. An asymmetry within the class of admissible trading strategies is the reason that this phenomenon of overpriced stock prices appears in models that satisfy NFLVR. Such models allow for the bond to be sold, but usually do not allow for the stock to be sold in order to profit from it being overpriced. For this subtle point, we refer the reader to Yan (1998), where “allowable strategies” are introduced to avoid the asymmetry introduced by admissibility constraints. If one is willing to accept the presence of bubbles, then a natural next step is to allow for some kind of arbitrage, essentially reflecting a bubble in the money market. Such arbitrage arises, for example, after a change of numéraire with an asset that has a bubble.

Having characterized the claims that can be perfectly replicated, it is a natural next step to identify the markets in which all claims can be perfectly replicated. Such a market is then called *complete*. For markets without arbitrage opportunities, the *Second Fundamental Theorem of Asset Pricing (2nd FTAP)* gives a sufficient and necessary condition, stating that a market is complete if and only if the ELMM is unique. This insight regarding the equivalence of the existence of a replicating strategy for any claim and the uniqueness of a pricing measure can be traced back to the seminal papers by Harrison and Kreps (1979) and Harrison and Pliska (1981). For a list of more recent results and references, we point the reader to Section 1.8 of Karatzas and Shreve (1998). Recently, Lyasoff (2010) has studied completeness in markets where capital gains and additional information to the investors are modeled separately.

In this chapter, we show that the 2nd FTAP can be extended to markets that do not proscribe arbitrage. Its generalized version then states that a market is complete if and only if the stochastic discount factor is unique. Clearly, in the case of NFLVR, this condition reduces to the classical one since then any stochastic discount factor generates an ELMM. We conclude that the question regarding the existence of arbitrage and the question regarding the completeness of the market can be addressed separately from one another; see also Jarrow et al. (1999) and Section 10.1 of Fernholz and Karatzas (2009). The proof of the 2nd FTAP in markets that do not proscribe the existence of arbitrage is simple. It relies on a change-of-numéraire technique and an application of the classical 2nd FTAP.

Often, however, completeness is too strong a requirement. We instead introduce the notion of *quasi-completeness*, which only takes into consideration claims measurable with respect to the stock price filtration. We show that markets whose drift and diffusion components are measurable with respect to the stock price filtration are quasi-complete; this generalizes Proposition 1 in Chapter 2, where the Markovian case is studied.

An important element in Stochastic Portfolio Theory is the concept of *(strong) relative arbitrage*. One says that there exists a *relative arbitrage* opportunity with respect to some trading strategy $\pi(\cdot)$ if there exists another trading strategy $\eta(\cdot)$ that outperforms $\pi(\cdot)$; that is, if trading according to $\eta(\cdot)$ yields a higher terminal wealth than trading according to $\pi(\cdot)$. The relative arbitrage is called *strong* if the terminal wealth is strictly dominated almost surely. It has been unclear up until this point whether a relative arbitrage opportunity necessarily implies a strong one.

Having now the characterization of perfect replication and completeness at hand, we can resolve this question: The existence of relative arbitrage does usually not imply that of strong relative arbitrage; however, it does if the market is quasi-complete. We can further state very precise conditions for the existence of both relative arbitrage and strong relative arbitrage opportunities.

We have included several examples of toy markets that illustrate various subtle points of our results in the sections that follow. Example 6 demonstrates that, although a given claim might be measurable with respect to the stock price filtration, the trading strategy to replicate the claim does not necessarily have to be measurable with respect to this filtration. Example 7 illustrates that the drift is important for determining whether a market is quasi-complete or not. In Example 8, we study a stock price with a bubble whose minimal replicating cost is not below its current price. Changing this model slightly then yields Example 9, which treats a model without an ELMM but in which the minimal replicating price for \$1 is again \$1. Finally, Example 10 provides the dynamics of a stock price that implies a strong but diminishing arbitrage opportunity.

We introduce the model and admissible trading strategies in Section 3.2. For a given claim, we provide necessary and sufficient conditions to decide whether it can be replicated in Section 3.3. In Section 3.4, we state and prove a generalized version of the 2nd FTAP and discuss the concept of quasi-completeness. We then apply the tools developed in the previous sections to link relative arbitrage and strong relative arbitrage in Section 3.5 and we conclude in Section 3.6.

3.2 Setup

This section introduces the probabilistic market model and the concepts of *market prices of risk*, *stochastic discount factors*, *trading strategies*, and *(contingent) claims.* Throughout the chapter, we shall assume that all equalities and inequalities only hold in an almost-sure sense.

3.2.1 Market model

We fix a canonical[1] probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\Omega = C([0,\infty), \mathbb{R}^K)$, that is, Ω is the space of all continuous functions $W(\cdot) = (W_1(\cdot), \ldots, W_K(\cdot))^\mathsf{T}$ taking values in $\mathbb{R}^K$ for some fixed $K \in \mathbb{N}$. Furthermore, we fix $\mathbb{P}$ so that the process $W(\cdot)$ has the law of a K-dimensional Brownian motion. We denote by $\mathbb{F} = \{\mathcal{F}(t)\}_{t\geq 0}$ the filtration generated by the paths of $W(\cdot)$, and assume it satisfies the usual assumptions. We further assume for some fixed $d \in \mathbb{N}$ the existence of a vector of d continuous, adapted processes $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^\mathsf{T}$ with values in $(0,\infty)^d$, which represent the price processes of the risky assets in an economy. We assume the existence of a K-dimensional, vector-valued process $\theta(\cdot)$ and of a $d \times K$-dimensional, matrix-valued process $\sigma(\cdot)$, both progressively measurable with respect to the underlying filtration $\mathbb{F}$, such that the dynamics of $S(\cdot)$ can be written as

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t) \Big(\theta_k(t) dt + dW_k(t) \Big) \tag{3.1}$$

for all $t \geq 0$ and $i = 1, \ldots, d$. The process $\sigma(\cdot)$ is not assumed to be of rank d or K. Thus, we do not exclude a-priori stock price models with a non-tradable state variable, such as "stochastic volatility" models.

The strict positivity of $S(\cdot)$ will be guaranteed by imposing $S(0) \in (0,\infty)^d$ and the integrability condition

$$\int_0^T \left(\|\theta(t)\|^2 + \sum_{i=1}^{d} \sum_{k=1}^{K} \sigma_{i,k}^2(t) \right) dt < \infty \tag{3.2}$$

for all $T \geq 0$.

[1]To generalize the results presented here to more general semimartingale models is subject to future research.

We denote by $\mathcal{F}^S(\cdot)$ the with the null sets of $\mathcal{F}$ augmented, right-continuous filtration generated by $S(\cdot)$. More precisely, we define $\mathcal{F}^S(t) := \sigma(S(u), u \leq t)$ for all $t \geq 0$. Since $S(\cdot)$ is a strong solution of (3.1), we have the inclusion $\mathcal{F}^S(t) \subset \mathcal{F}(t)$ for all $t \geq 0$.

3.2.2 Market prices of risk and stochastic discount factors

The special structure of the drift is a standard assumption imposed in order to exclude the possibility of an arbitrage opportunity that could otherwise get scaled unboundedly; see Section 10 of Karatzas et al. (1991a) and the proof of Theorem 3.5 in Delbaen and Schachermayer (1995b). We call the process $\theta(\cdot) = (\theta_1(\cdot), \ldots, \theta_K(\cdot))^\mathsf{T}$ in (3.1), which maps the volatility on the drift, a *market price of risk*.

It is clear that this process is usually not uniquely determined; for example if the number of rows of $\sigma(\cdot)$ is smaller than the number of columns, that is, if $d < K$. In this case, a set of $\mathbb{R}^K$-valued, $\mathbb{F}$-progressively measurable processes $\nu(\cdot)$ exists such that (3.1) is satisfied with any $\nu(\cdot)$ replacing $\theta(\cdot)$. To make this precise, we define

$$\begin{aligned}\Theta &:= \left\{ \nu : [0,\infty) \times \Omega \to \mathbb{R}^K \text{ progressively measurable } \right| \\ &\qquad \left. \int_0^T \|\nu(t)\|^2 dt < \infty \text{ for all } T \geq 0 \right\}, \\ \Theta' &:= \{\nu(\cdot) \in \Theta \mid \sigma(\cdot)\nu(\cdot) \equiv \sigma(\cdot)\theta(\cdot)\}. \end{aligned} \tag{3.3}$$

We call Θ', which as a direct consequence of (3.2) contains $\theta(\cdot)$, the set of all market prices of risk. If any $\nu(\cdot) \in \Theta'$ replaces $\theta(\cdot)$ in (3.1), the dynamics of $S(\cdot)$ are unchanged. We observe that the stochastic process $\theta^m(\cdot)$ defined as

$$\theta^m(\cdot) := \sigma(\cdot)^\dagger \sigma(\cdot)\theta(\cdot), \tag{3.4}$$

where $\dagger$ denotes the Moore-Penrose pseudo-inverse of a matrix, is again a market price of risk and therefore is also an element of Θ'; see Corollary 1 of Penrose (1955).

For any process $\nu(\cdot)$ in Θ, we define $Z^\nu(\cdot)$ as its stochastic exponential; that is,

$$Z^\nu(t) := \exp\left(-\int_0^t \nu^\mathsf{T}(u)dW(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2 du\right) \tag{3.5}$$

for all $t \geq 0$. For $\nu(\cdot) \in \Theta'$, we call $Z^\nu(\cdot)$ a *stochastic discount factor.*

3.2.3 Trading strategies and claims

We shall consider a small investor who can trade dynamically in both a risk-free asset, which pays zero interest rate, and in d stocks with price processes given by $S(\cdot)$. The assumption that the risk-free interest rate equals zero is made here for convenience. We shall assume that the investor can trade in the market without any frictions. In particular, we assume that the investor faces no transaction costs, is allowed to trade continuous fractions of shares, and does not influence market prices. However, the investor shall be restricted to always having nonnegative wealth, as specified in the next paragraph.

We call any progressively measurable vector $\eta(\cdot) = (\eta_1(\cdot), \dots, \eta_d(\cdot))^\mathsf{T}$, where each component of $\eta(\cdot)$ specifies the number of shares held by the investor following that trading strategy, an *(admissible) trading strategy* for initial capital $\tilde{p} \geq 0$ if the corresponding wealth process $V^{\tilde{p},\eta}(\cdot)$ with $V^{\tilde{p},\eta}(0) = \tilde{p}$ and dynamics

$$dV^{\tilde{p},\eta}(t) = \sum_{i=1}^{d} \eta_i(t)dS_i(t) \tag{3.6}$$

for all $t \geq 0$ stays nonnegative.

For any $T > 0$, we call any nonnegative $\mathcal{F}(T)$-measurable random variable $D \geq 0$ a *(contingent) claim.* A claim represents a certain monetary payoff at time T. Even without the existence of a traded asset $S_i(\cdot)$ with $S_i(T) = D$ for some $i = 1, \dots, d$, there might still exist some trading strategy $\eta(\cdot)$ and some $\tilde{p} > 0$ such that $V^{\tilde{p},\eta}(T) = D$ (respectively, $V^{\tilde{p},\eta}(T) \geq D$), in which case the claim is said to be *replicated* (respectively, *superreplicated*) by $\eta(\cdot)$.

Remark 11 (On the admissibility constraint). In the classical theory of Financial Mathematics, one also has to introduce a notion of admissibility for trading strategies in order to prevent the investor from following the notorious "doubling strategies;" see the discussion in Section 6 of Harrison and Kreps (1979). Usually, a more general condition than the nonnegativity of the corresponding wealth process is assumed. However, any such condition implies that the risk-adjusted wealth process $Z^{\nu}(\cdot)V^{\tilde{p},\eta}(\cdot)$ is a supermartingale; see Strasser (2003). This is no longer true when one abstains from imposing the no-arbitrage condition. For example, if $Z^{\nu}(\cdot)$ is a strict local martingale and the wealth process is only restricted to stay above some constant $-\alpha < 0$, then $Z^{\nu}(\cdot)V^{\tilde{p},\eta}(\cdot)$ is usually no longer a supermartingale. This motivates the admissibility constraint made here, which mandates that the wealth process stay nonnegative. We observe that under NFLVR, due to the supermartingale property, any wealth process of a (super-)replicating strategy for some nonnegative claim $D \geq 0$ is again nonnegative, independently from the admissibility constraint. This fact will be used in the proofs of Section 3.3, where we apply results of the no-arbitrage theory to obtain a characterization for claims that can be replicated in markets that do not proscribe arbitrage. We shall revisit these observations in Remark 12. □

3.3 Existence of (super-)replicating trading strategies

Given a specific claim, it is of interest to specify conditions under which its payoff can be obtained by means of dynamic trading in the stocks. Theorem 8.5 of Karatzas et al. (1991b) provides a sufficient condition for the replicability of strictly positive claims. The authors allow for markets that are incomplete, as well as for markets that admit arbitrage opportunities. Here, we extend their result to more general volatility matrices $\sigma(\cdot)$ and to claims that are only nonnegative, and further provide a minimal superreplicating price for general nonnegative claims. By relying on duality methods, the question regarding the existence of superreplicating strategies has been answered in full generality for markets satisfying NFLVR. We refer the reader to Jacka (1992), Ansel and Stricker (1993), El Karoui and Quenez (1995), and Delbaen and Schachermayer (1995c) for more on this topic; see also Kramkov (1996) and Föllmer and Kabanov (1998) for a more general class of models. In the following, we show that these results also extend to markets without an ELMM.

Throughout this section, we fix an horizon $T > 0$ and an $\mathcal{F}(T)$-measurable random variable $D \geq 0$, which represents the claim. We define

$$p := D_0 := \sup_{\nu(\cdot) \in \Theta'} \mathbb{E}\left[Z^\nu(T) D\right] \in [0, \infty], \tag{3.7}$$

with Θ' as in (3.3). We shall see in Theorem 5 that p represents the minimal superreplicating price for D. We now introduce the sequence of stopping times $\tau_0 := 0$,

$$\tau_n := T \wedge \inf\left\{t \in [0, T] \;\middle|\; Z^{\theta^m}(t) \geq n\right\}$$

for all $n \in \mathbb{N}$, where $\theta^m(\cdot)$ has been defined in (3.4) and $\wedge$ denotes the minimum. For any stopping time τ_n and any $\mathcal{F}(\tau_n)$-measurable random variable $\tilde{D} \geq 0$, we set

$$p_{\tau_n}(\tilde{D}) := \sup_{\nu(\cdot) \in \Theta'} \mathbb{E}\left[Z^\nu(\tau_n) \tilde{D}\right]. \tag{3.8}$$

By analogy with p of (3.7), $p_{\tau_n}(\tilde{D})$ can be considered the minimal price for superreplicating $\tilde{D}$ at time τ_n, as we shall demonstrate below. Towards this end, we define

$$p^e_{\tau_n}(\tilde{D}) := \sup_{\nu(\cdot) \in \Theta^e_{\tau_n}} \mathbb{E}\left[Z^\nu(\tau_n) \tilde{D}\right], \tag{3.9}$$

where we denote

$$\Theta^e_{\tau_n} := \{\nu(\cdot) \in \Theta \mid \sigma(\cdot \wedge \tau_n)\nu(\cdot \wedge \tau_n) \equiv \sigma(\cdot \wedge \tau_n)\theta(\cdot \wedge \tau_n), \mathbb{E}[Z^\nu(T)] = 1\} \neq \emptyset.$$

for all $n \in \mathbb{N}$. The next definition is in the spirit of Delbaen and Schachermayer (1995c):

Definition 8 (Maximal trading strategy). We call a trading strategy $\eta(\cdot)$ *maximal* if the supremum is a maximum in (3.7) with $D = V^{p,\eta}(T)$.

Theorem 8 (b) below will motivate the word “maximal,” since it shows that no trading strategy which outperforms a maximal one exists. It is already now clear that $\eta(\cdot)$ is maximal if some $\nu(\cdot) \in \Theta'$ exists such that $Z^\nu(\cdot)V^{p,\eta}(\cdot)$ is a martingale up to time T.

We can now resolve the question regarding the existence of a (super-)replicating strategy for claims in models that do not proscribe arbitrage. As the next lemmas clarify, our argument utilizes the fact that the existence of a square-integrable market price of risk guarantees that the market is basically, up to a stopping time, free of arbitrage. Thus, we shall be able to find a sequence of time-consistent trading strategies, which eventually lead, path-by-path, to a superreplicating strategy.

Lemma 5 (Localized (super-)replication)**.** *Assume $\tilde{D} \geq 0$ is $\mathcal{F}(\tau_n)$-measurable for some $n \in \mathbb{N}$. Then, the equality $\tilde{p} := p_{\tau_n}(\tilde{D}) = p^e_{\tau_n}(\tilde{D})$ holds, and the supremum in* (3.8) *is attained if and only if it is attained in* (3.9)*. The supremum $\tilde{p}$ is the minimal superreplicating price for $\tilde{D}$ at time τ_n. More precisely, if $\tilde{p} < \infty$ then an admissible trading strategy $\tilde{\eta}(\cdot)$ exists such that*

$$V^{\tilde{p},\tilde{\eta}}(\tau_n) \geq \tilde{D}.$$

If $\tilde{p} < \infty$, then there exist an $\mathcal{F}(\tau_n)$-measurable claim $\hat{D} \geq \tilde{D}$, a trading strategy $\hat{\eta}(\cdot)$, and a market price of risk $\hat{\nu}(\cdot) \in \Theta^e_{\tau_n}$, such that $Z^{\hat{\eta}}(\cdot)V^{\tilde{p},\hat{\eta}}(\cdot)$ is a martingale up to time τ_n and $V^{\tilde{p},\hat{\eta}}(\tau_n) = \hat{D}$.

Furthermore, no trading strategy $\eta(\cdot)$ *exists for which* $V^{c,\eta}(\tau_n) \geq \tilde{D}$*, for any* $c \in [0, p_{\tau_n}(\tilde{D}))$*.*

Proof. First, assume that there exist $\eta(\cdot)$ and $c \in [0, p_{\tau_n}(\tilde{D}))$ such that $V^{c,\eta}(\tau_n) \geq \tilde{D}$. We observe that $Z^{\nu}(\cdot)V^{c,\eta(\cdot)}(\cdot)$ is a supermartingale for any $\nu(\cdot) \in \Theta'$. Thus, we obtain

$$\mathbb{E}\left[Z^{\nu}(\tau_n)\tilde{D}\right] \leq \mathbb{E}\left[Z^{\nu}(\tau_n)V^{c,\eta}(\tau_n)\right] \leq c < p_{\tau_n}(\tilde{D})$$

for all $\nu(\cdot) \in \Theta'$, which leads to a contradiction with the definition of $p_{\tau_n}(\tilde{D})$ as a supremum in (3.8).

Now, observe that $\tilde{Z}(\cdot) \equiv Z^{\theta^m}(\cdot \wedge \tau_n)$ is a martingale since it is bounded by n. In particular, it defines a new measure $\mathbb{Q}$ on $\mathcal{F}(T)$, which is equivalent to $\mathbb{P}$, by $d\mathbb{Q}/d\mathbb{P} = \tilde{Z}(T)$. We introduce a fictional market $\tilde{S}(\cdot)$ by $\tilde{S}(\cdot) \equiv S(\cdot \wedge \tau_n)$. Then, $\tilde{S}_i(\cdot)$ is a $\mathbb{Q}$-local martingale for all $i \in \{1, \ldots d\}$. Thus, NFLVR holds for the new market. Since the probability space Ω is the canonical one, any measure $\tilde{\mathbb{Q}}$ on $\mathcal{F}(T)$ under which $\tilde{S}(\cdot)$ is a local martingale and which is equivalent to $\mathbb{P}$ has a representation $d\tilde{\mathbb{Q}}/d\mathbb{P} = Z^{\nu}(T)$ for some $\nu(\cdot) \in \Theta^e_{\tau_n}$. Then, by the classical theory for arbitrage-free markets, a trading strategy $\tilde{\eta}(\cdot)$ exists such that $\tilde{V}^{p^e_{\tau_n}(\tilde{D}),\tilde{\eta}}(\tau_n) \geq \tilde{D}$, where $\tilde{V}^{p^e_{\tau_n}(\tilde{D}),\tilde{\eta}}(\cdot)$ is defined as in (3.6) with $S(\cdot)$ replaced by $\tilde{S}(\cdot)$; see Theorem 9 of Delbaen and Schachermayer (1995c). However, we have $S(\cdot \wedge \tau_n) \equiv \tilde{S}(\cdot \wedge \tau_n)$, and therefore $V^{p^e_{\tau_n}(\tilde{D}),\tilde{\eta}}(\tau_n) \geq \tilde{D}$.

Together with the first part of the proof, where we have shown that any $c \geq 0$ that satisfies $V^{c,\eta}(\tau_n) \geq 0$ for some trading strategy $\eta(\cdot)$ also satisfies $c \geq p_{\tau_n}(\tilde{D})$, this also yields $p^e_{\tau_n}(\tilde{D}) \geq p_{\tau_n}(\tilde{D})$. The inequality in the other direction follows from the fact that for any $\nu(\cdot) \in \Theta^e_{\tau_n}$ there exists $\tilde{\nu}(\cdot) \in \Theta'$ with $Z^{\nu}(\tau_n) = Z^{\tilde{\nu}}(\tau_n)$. To see this, set $\tilde{\nu}(t) = \nu(t)\mathbf{1}_{\{t \leq \tau_n\}} + \theta^m(t)\mathbf{1}_{\{t > \tau_n\}}$ for all $t \geq 0$.

Corollaries 10 and 14 of Delbaen and Schachermayer (1995c) yield the existence of a claim $\hat{D} \geq \tilde{D}$, a trading strategy $\hat{\eta}(\cdot)$, and a market price of risk

$\hat{\nu}(\cdot) \in \Theta^e_{\tau_n}$, such that $Z^{\hat{\eta}}(\cdot)V^{\tilde{p},\hat{\eta}}(\cdot)$ is a martingale up to time τ_n and $V^{\tilde{p},\hat{\eta}}(\tau_n) = \hat{D}$. Assume now that the supremum in (3.8) is attained, say by $\tilde{\nu}(\cdot) \in \Theta'$, but that the supremum in (3.9) is not. Then, again by Corollaries 14 and 10 of Delbaen and Schachermayer (1995c), there exists a claim $\hat{D} \geq \tilde{D}$ with $\mathbb{P}(\hat{D} > \tilde{D}) > 0$ and a trading strategy $\hat{\eta}(\cdot)$, such that $V^{\tilde{p},\hat{\eta}}(\tau_n) = \hat{D}$. However, the supermartingale property of $Z^{\tilde{\nu}}(\cdot)V^{\tilde{p},\hat{\eta}}(\cdot)$ leads directly to a contradiction. □

The next lemma will be of use in Theorem 5, when we need to prove time consistency of strategies. It generalizes the equality $p_{\tau_n}(\tilde{D}) = p^e_{\tau_n}(\tilde{D})$ of Lemma 5. The measurability of the essential suprema in the following lemma is guaranteed as in Lemma 7 below.

Lemma 6 (Sufficiency of local martingale measures)**.** *Fix $n \in \mathbb{N}$. Assume again that $\tilde{D} \geq 0$ is $\mathcal{F}(\tau_n)$-measurable for some $n \in \mathbb{N}$. Then, the equality*

$$\operatorname*{ess\,sup}_{\nu(\cdot)\in\Theta'} \mathbb{E}\left[\left.\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})}\tilde{D}\right| \mathcal{F}(\tau_{n-1})\right] = \operatorname*{ess\,sup}_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\left.\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})}\tilde{D}\right| \mathcal{F}(\tau_{n-1})\right] \tag{3.10}$$

holds.

Proof. As in the proof of Lemma 5, each $\nu(\cdot) \in \Theta^e_{\tau_n}$ corresponds to some $\tilde{\nu}(\cdot) \in \Theta'$. So, we need only show that the left-hand side is less than or equal to the right-hand side in (3.10). Towards this end, we fix $n \in \mathbb{N}$ and introduce the process $W^{\mathbb{Q}}(\cdot) = (W_1^{\mathbb{Q}}(\cdot), \ldots, W_K^{\mathbb{Q}}(\cdot))^\mathsf{T}$ with

$$W_k^{\mathbb{Q}}(\cdot) := W_k(\cdot) + \int_0^\cdot \theta_k^m(u)du$$

for all $k = 1, \ldots, K$. Analogously to (3.5), we define

$$Z^{\nu,\mathbb{Q}}(\cdot) := \exp\left(-\int_0^\cdot \nu^\mathsf{T}(u)dW^{\mathbb{Q}}(u) - \frac{1}{2}\int_0^\cdot \|\nu(u)\|^2 du\right)$$

and observe

$$Z^\nu(\cdot) \equiv Z^{\theta^m}(\cdot)Z^{\nu-\theta^m,\mathbb{Q}}(\cdot) \tag{3.11}$$

for all $\nu(\cdot) \in \Theta$. Fix any $\nu(\cdot) \in \Theta'$ and denote by $\{\tilde{\tau}_i\}_{i \in \mathbb{N}}$ a sequence of stopping times defined as

$$\tilde{\tau}_i := \tau_n \wedge \inf\left\{t \geq \tau_{n-1} \;\middle|\; Z^{\nu-\theta^m,\mathbb{Q}}(t) \geq i Z^{\nu-\theta^m,\mathbb{Q}}(\tau_{n-1})\right\}$$

for all $i \in \mathbb{N}$. The equality in (3.11) and Fatou's lemma yield

$$\begin{aligned}
\mathbb{E}\left[\frac{Z^{\nu}(\tau_n)}{Z^{\nu}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right] &= \mathbb{E}\left[\frac{Z^{\theta^m}(\tau_n)}{Z^{\theta^m}(\tau_{n-1})}\frac{Z^{\nu-\theta^m,\mathbb{Q}}(\tau_n)}{Z^{\nu-\theta^m,\mathbb{Q}}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right]\\
&= \mathbb{E}\left[\frac{Z^{\theta^m}(\tau_n)}{Z^{\theta^m}(\tau_{n-1})}\cdot\lim_{i\to\infty}\frac{Z^{\nu-\theta^m,\mathbb{Q}}(\tilde{\tau}_i)}{Z^{\nu-\theta^m,\mathbb{Q}}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right]\\
&\leq \liminf_{i\to\infty}\mathbb{E}\left[\frac{Z^{\theta^m}(\tau_n)}{Z^{\theta^m}(\tau_{n-1})}\frac{Z^{\nu-\theta^m,\mathbb{Q}}(\tilde{\tau}_i)}{Z^{\nu-\theta^m,\mathbb{Q}}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right]\\
&= \liminf_{i\to\infty}\mathbb{E}\left[\frac{Z^{\nu^{(i)}}(\tau_n)}{Z^{\nu^{(i)}}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right]\\
&\leq \operatorname*{ess\,sup}_{\nu'(\cdot)\in\Theta^e_{\tau_n}}\mathbb{E}\left[\frac{Z^{\nu'}(\tau_n)}{Z^{\nu'}(\tau_{n-1})}\tilde{D}\middle|\mathcal{F}(\tau_{n-1})\right],
\end{aligned}$$

since

$$\nu^{(i)} := \Big(\theta^m(\cdot) + \mathbf{1}_{\{\tilde{\tau}_i \geq t\}}\Big(\nu(\cdot) - \theta^m(\cdot)\Big)\Big) \in \Theta^e_{\tau_n}$$

for all $i \in \mathbb{N}$; this proves the statement. □

We continue by introducing the sequence of random variables

$$D_n := \operatorname*{ess\,sup}_{\nu(\cdot)\in\Theta'}\mathbb{E}\left[\frac{Z^{\nu}(T)}{Z^{\nu}(\tau_n)}D\middle|\mathcal{F}(\tau_n)\right] =: \operatorname*{ess\,sup}_{\nu(\cdot)\in\Theta'} D_n^{\nu} \geq 0 \tag{3.12}$$

for all $n \in \mathbb{N}$. If $p = D_0 < \infty$, then $D_n < \infty$ for all $n \in \mathbb{N}$. We discuss in the next lemma the measurability of each D_n; in particular, we show that D_n represents a claim:

Lemma 7 (Measurability of D_n)**.** *For any $n \in \mathbb{N}$, the essential supremum D_n of (3.12) is $\mathcal{F}(\tau_n)$-measurable.*

Proof. Fix $n \in \mathbb{N}$. Theorem A.3 of Karatzas and Shreve (1998) yields that D_n exists and is $\mathcal{F}(\tau_n)$-measurable. This is due to the observation that for any $\nu^{(1)}(\cdot), \nu^{(2)}(\cdot) \in \Theta'$, there exists $\nu^{(3)}(\cdot) \in \Theta'$, which is defined by $\nu^{(3)}(\cdot \wedge \tau_n) \equiv \nu^{(1)}(\cdot \wedge \tau_n)$ and

$$\nu^{(3)}(t) := \nu^{(1)}(t)\mathbf{1}_{\{D_n^{\nu^{(1)}} \geq D_n^{\nu^{(2)}}\}} + \nu^{(2)}(t)\mathbf{1}_{\{D_n^{\nu^{(1)}} < D_n^{\nu^{(2)}}\}} \tag{3.13}$$

for all $t > \tau_n$ and therefore satisfies the "fork" property

$$D_n^{\nu^{(3)}} = D_n^{\nu^{(1)}} \vee D_n^{\nu^{(2)}}, \tag{3.14}$$

where $\vee$ denotes the maximum. □

The next lemma proves a *dynamic programming principle (DPP)*. It is essential for the results that follow below.

Lemma 8 (Multiplicative DPP)**.** *The sequence of random variables $(D_n)_{n\in\mathbb{N}}$ satisfies the equalities*

$$D_{n-1} = \operatorname{ess} \sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} D_n \middle| \mathcal{F}(\tau_{n-1})\right]$$

if $D_n < \infty$ for all $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ such that $D_n < \infty$. In conjunction with the "fork" property of (3.13) and (3.14), Theorem A.3 of Karatzas and Shreve (1998) yields the existence of a sequence of random variables $(\nu^{(i)})_{i\in\mathbb{N}}$ such that $D_n^{\nu^{(i)}} \uparrow D_n$ as $i \to \infty$. Fix $\epsilon > 0$ and define i^* as

$$i^* := \min\left\{i \in \mathbb{N} \;\middle|\; D_n^{\nu^{(i)}} \geq D_n - \epsilon\right\}.$$

Then, $D_n^{\nu^{(i^*)}}$ is $\mathcal{F}(\tau_n)$-measurable and we obtain

$$\begin{aligned}\operatorname{ess} \sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} D_n \middle| \mathcal{F}(\tau_{n-1})\right] &\leq \operatorname{ess} \sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} D_n^{\nu^{(i^*)}} \middle| \mathcal{F}(\tau_{n-1})\right] + \epsilon \\ &\leq D_{n-1} + \epsilon.\end{aligned}$$

Since the choice of ϵ was arbitrary, this yields one inequality; the other direction follows from

$$D_{n-1} \leq \operatorname{ess}\sup_{\nu(\cdot)\in\Theta'} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} \cdot \operatorname{ess}\sup_{\nu'(\cdot)\in\Theta'} \mathbb{E}\left[\frac{Z^{\nu'}(T)}{Z^{\nu'}(\tau_n)} D \middle| \mathcal{F}(\tau_n)\right] \middle| \mathcal{F}(\tau_{n-1})\right]$$
$$= \operatorname{ess}\sup_{\nu(\cdot)\in\Theta'} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} D_n \middle| \mathcal{F}(\tau_{n-1})\right]$$

and Lemma 6. □

Next, we set $\mathfrak{p}_n := p_{\tau_n}(D_n)$ for all $n \in \mathbb{N}$, as in (3.8). The following time consistency follows from the same argument as the DPP of Lemma 8: The sequence $(\mathfrak{p}_n)_{n\in\mathbb{N}}$ satisfies

$$\mathfrak{p}_n = p \tag{3.15}$$

for all $n \in \mathbb{N}$, with p as in (3.7). We can now state and prove the main result of this section:

Theorem 5 ((Super-)replicating strategy)**.** *There exists no trading strategy $\eta(\cdot)$ for which $V^{c,\eta}(T) \geq D$, for any $c \in [0, p)$. If $p < \infty$, then a trading strategy $\tilde{\eta}(\cdot)$ exists such that $V^{p,\tilde{\eta}}(T) \geq D$. If the supremum in* (3.7) *is attained, then one can choose $\tilde{\eta}(\cdot)$ so that $V^{p,\tilde{\eta}}(T) = D$, that is, the claim D can be exactly replicated by dynamic hedging. If an ELMM exists, then the supremum in* (3.7) *can be replaced by the supremum over all $\nu(\cdot) \in \Theta'$ for which $Z^\nu(\cdot)$ is a martingale.*

Proof. The first part of the statement follows as in Lemma 5. Assume in the following that $p < \infty$, thus $D_n < \infty$ for all $n \in \mathbb{N}$. Now, we inductively construct for each $n \in \mathbb{N}$ trading strategies $\eta^{(n)}(\cdot)$ that satisfy

$$V^{p,\eta^{(n)}}(\tau_{n-1}) = V^{p,\eta^{(n-1)}}(\tau_{n-1}) \tag{3.16}$$

and $V^{p,\eta^{(n)}}(\tau_n) \geq D_n$. According to Lemma 5 and due to (3.15), there exist a contingent claim $\hat{D}_1 \geq D_1$, a trading strategy $\eta^{(1)}(\cdot)$, and a market price of risk

$\nu^{(1)}(\cdot) \in \Theta^e_{\tau_1}$, such that $V^{p,\eta^{(1)}}(\tau_1) = \hat{D}_1$ and $Z^{\nu^{(1)}}(\cdot)V^{p,\eta^{(1)}}(\cdot)$ is a martingale up to time τ_1. Assume that we have determined $\eta^{(n-1)}(\cdot)$, $\nu^{(n-1)}(\cdot)$, and $\hat{D}_{n-1} := V^{p,\eta^{(n-1)}}(\tau_{n-1}) \geq D_{n-1}$, such that $Z^{\nu^{(n-1)}}(\cdot)V^{p,\eta^{(n-1)}}(\cdot)$ is a martingale up to time τ_{n-1}. We observe that $p_{\tau_n}(D_n + \hat{D}_{n-1} - D_{n-1}) = \mathfrak{p}_n = p$, since by Lemma 5

$$\begin{aligned}
p &\leq p^e_{\tau_n}(D_n + \hat{D}_{n-1} - D_{n-1}) \\
&\leq \sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[Z^\nu(\tau_{n-1}) \left(\operatorname{ess}\sup_{\nu'(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\frac{Z^{\nu'}(\tau_n)}{Z^{\nu'}(\tau_{n-1})} D_n \middle| \mathcal{F}(\tau_{n-1}) \right] + \hat{D}_{n-1} - D_{n-1} \right) \right] \\
&\leq \sup_{\nu(\cdot)\in\Theta'} \mathbb{E}\left[Z^\nu(\tau_{n-1})\hat{D}_{n-1} \right] \\
&= p
\end{aligned}$$

due to the DPP of Lemma 8. By Lemma 5 again, there exist a contingent claim $\hat{D}_n \geq D_n + \hat{D}_{n-1} - D_{n-1} \geq D_n$, a trading strategy $\eta^{(n)}(\cdot)$, and a market price of risk $\nu^{(n)}(\cdot) \in \Theta^e_{\tau_n}$ such that $V^{p,\eta^{(n)}}(\tau_n) = \hat{D}_n$ and $Z^{\nu^{(n)}}(\cdot)V^{p,\eta^{(n)}}(\cdot)$ is a martingale up to time τ_n.

Now, the DPP of Lemma 8 yields

$$\begin{aligned}
V^{p,\eta^{(n)}}(\tau_{n-1}) =& \mathbb{E}\left[\frac{Z^{\nu(n)}(\tau_n)}{Z^{\nu(n)}(\tau_{n-1})} V^{p,\eta^{(n)}}(\tau_n) \middle| \mathcal{F}(\tau_{n-1}) \right] \\
\geq& \operatorname{ess}\sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} (D_n + \hat{D}_{n-1} - D_{n-1}) \middle| \mathcal{F}(\tau_{n-1}) \right] \\
\geq& \operatorname{ess}\sup_{\nu(\cdot)\in\Theta^e_{\tau_n}} \left(\mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} D_n \middle| \mathcal{F}(\tau_{n-1}) \right] \right. \\
&\left. + \mathbb{E}\left[\frac{Z^\nu(\tau_n)}{Z^\nu(\tau_{n-1})} \middle| \mathcal{F}(\tau_{n-1}) \right] (\hat{D}_{n-1} - D_{n-1}) \right) \\
=& D_{n-1} + \hat{D}_{n-1} - D_{n-1} \\
=& \hat{D}_{n-1},
\end{aligned}$$

and thus $V^{p,\eta^{(n)}}(\tau_{n-1}) \geq V^{p,\eta^{(n-1)}}(\tau_{n-1})$. Assume that the event $\{V^{p,\eta^{(n)}}(\tau_{n-1}) > V^{p,\eta^{(n-1)}}(\tau_{n-1})\}$ has positive probability. Since $Z^{\nu^{(n-1)}}(\cdot)V^{p,\eta^{(n-1)}}(\cdot)$ is a martingale, this implies that the event $\{V^{p,\eta^{(n)}}(\tau_{n-1}) < V^{p,\eta^{(n-1)}}(\tau_{n-1})\}$ should also have positive probability, leading to a contradiction. Thus, this inductive procedure yields trading strategies $\eta^{(n)}(\cdot)$ that satisfy (3.16) and $V^{p,\eta^{(n)}}(\tau_n) \geq D_n$ for all $n \in \mathbb{N}$.

We define a new trading strategy $\tilde{\eta}(\cdot)$ as

$$\tilde{\eta}(t) = \eta^{(1)}(0)\mathbf{1}_{\{0\}}(t) + \sum_{n=1}^{\infty} \eta^{(n)}(t)\mathbf{1}_{\{t\in(\tau_{n-1},\tau_n]\}}$$

for all $t \geq 0$. We observe that

$$V^{p,\tilde{\eta}}(\tau_n) = \hat{D}_n \geq D_n$$

holds for all $n \in \mathbb{N}$. We now fix any $\omega \in \Omega$ such that $\tau_{n(\omega)} = T$ for some $n(\omega)$. Then, we obtain

$$V^{p,\tilde{\eta}}(T)(\omega) = V^{p,\tilde{\eta}}(\tau_{n(\omega)})(\omega) \geq D_{n(\omega)}(\omega) = D(\omega)$$

with equality if the supremum in (3.7) is attained, due to the observation that $\hat{D}_n = D$ for all $n \in \mathbb{N}$ in that case. Since for almost all $\omega \in \Omega$ such an $n(\omega)$ exists, $\tilde{\eta}(\cdot)$ (super-)replicates D.

If an ELMM exists, we are in the context of the classical theory of Financial Mathematics and then it is sufficient to take the supremum in (3.7) over all ELMMs to obtain the minimal superreplicating price; see Delbaen and Schachermayer (1995c). □

The previous theorem proves, in particular, a conjecture in Chapter 2. There, the Markovian case is discussed and it is demonstrated that the supremum in (3.7) is always attained, as long as D is measurable with respect to $\mathcal{F}^S(\cdot)$, the filtration generated by the stock price processes $S(\cdot)$. For path-independent European-style claims, an explicit trading strategy for the exact replication is constructed, but the question of whether path-dependent claims could be hedged is not resolved. For a more precise statement of these results, see Theorem 6 below.

We wish to draw the reader's attention to a few subtle points concerning the previous theorem. First of all, even if the supremum in (3.7) is not attained, there might nevertheless exist a trading strategy $\eta(\cdot)$ that replicates D, that is, a trading strategy such that $V^{p,\eta}(T) = D$. The claim $D = S(2)$ in Example 8 below illustrates this point. However, such a trading strategy $\eta(\cdot)$ is not maximal in the sense of Definition 8. Second, the replicating price p in (3.7) depends strongly on the admissibility constraint $V^{p,\eta}(\cdot) \geq 0$, as the next remark discusses:

Remark 12 (Relevance of the admissibility constraint). We have observed in Remark 11 that the precise choice of the admissibility constraint is not relevant for determining the costs of replicating a nonnegative claim in markets without arbitrage. *This is no longer true in markets that do not proscribe arbitrage.* Indeed, if we allow for strategies $\eta(\cdot)$ whose associated wealth process is only required to stay above a constant $-\alpha < 0$, then the minimal nonnegative price p^α to (super-)replicate a claim D can be computed as

$$p^\alpha := \sup_{\nu(\cdot)\in\Theta'} \mathbb{E}\left[Z^\nu(T)(D+\alpha)\right] - \alpha \leq p.$$

In particular, it is possible that $p^\alpha < 0$. The fact that p^α is the minimal (super-)replicating price can be seen as in Lemma 5. The strategy $\eta(\cdot)$ that (super-)replicates D under these weaker admissibility condition is exactly the same strategy that (super-)replicates $D + \alpha$ in Theorem 5. □

A further subtle point that we want to emphasize is that the trading strategy $\eta(\cdot)$ which replicates some $\mathcal{F}^S(\cdot)$-measurable claim D in Theorem 5 for the price p is, in general, progressively measurable only with respect to $\mathcal{F}(\cdot)$, but not necessarily with respect to $\mathcal{F}^S(\cdot)$. The next example illustrates this point. To determine sufficient conditions that imply the measurability of the replicating trading strategy with respect to $\mathcal{F}^S(\cdot)$ is a future research project.

Example 6 (Measurability of trading strategies). We set $d = K = 2$, $S_1(0) = S_2(0) = 2$, $\sigma_{1,2}(\cdot) \equiv \sigma_{2,1}(\cdot) \equiv 0$, and $\theta_i(\cdot) \equiv 1/S_i(\cdot)$ for $i = 1, 2$. Furthermore, we define $I_1 := \mathbf{1}_{\{W_1(1)\geq 0\}}$, $I_2 := \mathbf{1}_{\{W_1(1)<0\}}$ and

$$\sigma_{1,1}(t) = \frac{1}{S_1(t)}\mathbf{1}_{[1,\infty)}(t)\left(1 - I_1\mathbf{1}_{\{t>\tau_1\}}\right),$$
$$\sigma_{2,2}(t) = \frac{1}{S_2(t)}\mathbf{1}_{[1,\infty)}(t)\left(1 - I_2\mathbf{1}_{\{t>\tau_2\}}\right),$$

for all $t \geq 0$, where we set $\tau_i := \inf\{t \geq 0 \mid S_i(t) \leq 1\}$ for $i = 1, 2$. Thus, up to time $t = 1$, the market does not move. Then, one of the stock price processes has the dynamics of the reciprocal of a three-dimensional Bessel process, while the other one has the same dynamics only until it hits 1. The sign of $W_1(1)$ decides which of the two processes has which dynamics.

We observe that I_1 is not measurable with respect to the stock price filtration $\mathcal{F}^S(\cdot)$ up to the stopping time $\tau_1 \wedge \tau_2 > 1$. More precisely, for any stopping time $\tilde{\tau} < \tau_1 \wedge \tau_2$, any event $A \in \mathcal{F}^S(\tilde{\tau})$ is independent of the event $\{W_1(1) \geq 0\}$; thus $\{W_1(1) \geq 0\} \notin \mathcal{F}^S(\tilde{\tau})$.

If $\nu(\cdot) \in \Theta'$ denotes any market price of risk, then $\nu_1(t) = 1/S_1(t)$ for $t \geq 1$ ($t \in [1, \tau_1]$) if $I_1 = 0$ ($I_1 = 1$) and $\nu_2(t) = 1/S_2(t)$ for $t \geq 1$ ($t \in [1, \tau_2]$) if $I_2 = 0$ ($I_2 = 1$). We thus obtain from Itô's formula

$$Z^\nu(t) = \frac{S_1(0)}{S_1(t)}\frac{S_2(0)}{S_2(t)}Z^\nu(1)\prod_{i=1}^{2}\Big(1 + I_i\big(\mathcal{E}_i(\nu, t\wedge\tau_i, t) - 1\big)\Big)$$

for all $t \geq 1$ with

$$\mathcal{E}_i(\nu, t_0, t_1) := \exp\left(-\int_{t_0}^{t_1}\nu_i(u)dW_i(u) - \frac{1}{2}\int_{t_0}^{t_1}\nu_i^2(u)du\right)$$

for all $i = 1, 2$ and $t_0, t_1 \geq 0$.

We now fix $T = 2$ and $D = 1$ and obtain

$$\begin{aligned} p &= \sup_{\nu(\cdot)\in\Theta'}\mathbb{E}[Z^\nu(2)] \\ &= \mathbb{E}\left[\frac{S_1(0)S_2(0)}{S_1(2)S_2(2)}\right] \end{aligned}$$

$$= \frac{1}{2}\left(\mathbb{E}\left[\frac{S_2(0)}{S_2(2)}\middle| \{W_1(1) \geq 0\}\right] + \mathbb{E}\left[\frac{S_1(0)}{S_1(2)}\middle| \{W_1(1) < 0\}\right]\right)$$
$$= 2\Phi(2) - 1,$$

where Φ denotes the cumulative standard normal distribution function and the last equality is derived from the expectation of the reciprocal of a three-dimensional process, starting at 2; see, for example, (2.46). Furthermore, the supremum is attained, for example by $\theta(\cdot)$. Thus, there exists a trading strategy, $\eta(\cdot) = (\eta_1(\cdot), \eta_2(\cdot))^\mathsf{T}$, which exactly replicates $D = 1$ and which can be explicitly represented as

$$\eta_i(t) = \frac{2}{\sqrt{2-t}}(1 - I_i)1_{[1,2]}(t)\phi\left(\frac{S_i(t)}{\sqrt{2-t}}\right)$$

for $i = 1, 2$, where ϕ denotes the standard normal density, and the corresponding (unique) wealth process

$$V^{p,\eta}(t) = 1_{[0,1)}(t)p + 1_{[1,2]}(t)\sum_{i=1}^{2}(1 - I_i)\left(2\Phi\left(\frac{S_i(t)}{\sqrt{2-t}}\right) - 1\right)$$

for all $t \in [0, 2]$; compare (2.49) and (2.46).

We observe that $V^{p,\eta}(\tilde{\tau})$ depends for all stopping times $\tilde{\tau} > 1$ on I_1, thus is not measurable with respect to the stock price filtration $\mathcal{F}^S(\tilde{\tau})$ for all stopping times $\tilde{\tau} \in (1, \tau_1 \wedge \tau_2)$. Therefore, there exists no trading strategy $\eta(\cdot)$ that is measurable with respect to the stock price filtration $\mathcal{F}^S(\cdot)$ and that replicates $D = 1$ for initial costs p. □

The last example can easily be adapted to an example for an arbitrage-free market with a claim that is $\mathcal{F}^S(T)$-measurable for some $T > 1$ and that can be replicated by a maximal $\mathcal{F}(\cdot)$-measurable trading strategy, but not by a maximal $\mathcal{F}^S(\cdot)$-measurable trading strategy. Towards this end, we introduce a new market with two stocks $\tilde{S}_i(\cdot) := 1/S_i(\cdot)$ for $i = 1, 2$; both of them are now local martingales, one of them stopped at τ_i. Now, we consider the claim $D = S_1(2)I_2 + S_2(2)I_1$. In order to ensure the measurability of D with respect to $\mathcal{F}^S(2)$, we replace τ_i by $\tau_i \wedge 1.5$. Then, we proceed with the argument of the previous example.

3.4 Completeness and Second Fundamental Theorem of Asset Pricing

In this section, we extend the Second Fundamental Theorem of Asset Pricing to include markets that do not proscribe arbitrage opportunities. Furthermore, we discuss two notions of completeness. To start, we formally introduce the concept of a complete market in the next definition:

Definition 9 (Complete market). A market is called *complete* if for all $T > 0$ and all bounded $\mathcal{F}(T)$-measurable random variables $D \geq 0$ there exist $\tilde{p} > 0$ and a maximal trading strategy $\eta(\cdot)$ that replicates D; that is, there exists a maximal trading strategy $\eta(\cdot)$ such that $V^{\tilde{p},\eta}(T) = D$. Alternatively, if there exist some $T > 0$ and some $\mathcal{F}(T)$-measurable random variable $D \geq 0$ for which no maximal replication exists, then the market is called *incomplete* on $[0, T]$.

In particular, by the martingale representation theorem, a market is complete if $d = K$ and $\sigma(t)$ is invertible for all $t > 0$; see Section 10.1 of Fernholz and Karatzas (2009). As previously noted, the notion of completeness is often too strong and we therefore introduce the notion of quasi-completeness, a slight generalization:

Definition 10 (Quasi-complete market). We call a market *quasi-complete* if for every $T > 0$ and every bounded $\mathcal{F}^S(T)$-measurable random variable $D \geq 0$, there exists a maximal trading strategy $\eta(\cdot)$ that replicates D.

It follows directly from this definition that any complete market is necessarily quasi-complete but not vice versa. We call a function $g : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d_+) \to \mathbb{R}$ *non-anticipative functional* if $g(t, x(\cdot)) = g(t, x(\cdot \wedge t))$ for all $t \in \mathbb{R}_+$ and all $x(\cdot) \in C(\mathbb{R}_+, \mathbb{R}^d_+)$; that is g is non-anticipative if $g(t, x(\cdot))$ depends on the path of x only up to time t. We have the following result, which generalizes Pagès (1987), Duffie (1988)[2], and Proposition 1 in Chapter 2:

[2]We thank Martin Schweizer for pointing us to this reference.

Theorem 6 (Sufficient conditions for quasi-completeness)**.** *If $S(\cdot)$ of* (3.1) *can be represented as the unique solution of*

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \tilde{\sigma}_{i,k}(t, S(\cdot)) \Big(\tilde{\theta}_k(t, S(\cdot)) dt + dW_k(t) \Big),$$

where $\tilde{\sigma}_{i,k}$ and $\tilde{\theta}_k$ are non-anticipative functionals for all $i = 1, \cdots, d$ and $k = 1, \cdots, K$, then the market is quasi-complete. Furthermore, for any $T > 0$ and $\mathcal{F}^S(T)$-measurable random variable $D \geq 0$, $\theta^m(\cdot)$ maximizes the expression in (3.7).

Proof. The proof of Proposition 1 in Chapter 2 carries through with only minor modifications. □

We emphasize that we have not assumed that the volatility matrix $\sigma(\cdot)$ has full rank in the previous theorem. As demonstrated in the next example, an incomplete market is generally not quasi-complete if $\theta^m(\cdot)$ is not progressively measurable with respect to $\mathcal{F}^S(\cdot)$:

Example 7 (Relevance of drift for quasi-completeness). We set $K = 1$, $d = 1$, $S(0) = 1$, $\sigma(t) = 0$, $\theta^m(t) = 0$ for all $t \in [0, 1]$, and $\sigma(t) = 1/S(t)$,

$$\theta^m(t) = \frac{1}{S(t) - \frac{t-1}{|W(1)|}}$$

for all $t > 1$. This market is a slight extension of Example 1 in Chapter 2. We consider $D = 1$, which is $\mathcal{F}^S(2)$-measurable. For any $\nu(\cdot) \in \Theta'$, where Θ' is introduced in (3.3), we obtain

$$\mathbb{E}\left[\frac{Z^\nu(2)}{Z^\nu(1)} D \,\middle|\, \mathcal{F}(1) \right] = \Big(\Phi(1-c) - \exp(2c)(1 - \Phi(1+c)) \Big)\Big|_{c = \frac{1}{|W(1)|}},$$

where Φ denotes the standard normal cumulative distribution function; see (2.45). The last function is decreasing in $c \geq 0$. We thus obtain

$$\sup_{\nu(\cdot) \in \Theta'} \mathbb{E}\left[Z^\nu(2) D \right] = 2\Phi(1) - 1;$$

however, the supremum is not attained. Thus, the model is not quasi-complete. □

The stock price process in the previous example is sometimes called a "bubble," since under the ELMM its price tends to decrease in expectation due to its strict local martingality. We refer the reader to Jarrow et al. (2010) for a definition, further references, and a thorough discussion regarding bubbles in incomplete markets. The next lemma prepares the proof of the Second Fundamental Theorem of Asset Pricing:

Lemma 9 (Rank of volatility matrix in complete market)**.** *If a market is complete, then* $\mathrm{rank}(\sigma(t)) = K$ *Lebesgue-almost everywhere. In particular,* $d \geq K$*.*

Proof. Fix some $T > 0$ and assume a complete market. We now show $\mathrm{rank}(\sigma(t)) = K$ Lebesgue-almost everywhere on $[0, T]$. We introduce a new, fictional market with $d + 1$ stocks

$$\left(\frac{S_1(\cdot)}{V^{p,\tilde{\eta}}(\cdot)}, \ldots, \frac{S_d(\cdot)}{V^{p,\tilde{\eta}}(\cdot)}, \frac{1}{V^{p,\tilde{\eta}}(\cdot)} \right),$$

where p is defined in (3.7) with $D = 1$ and $\tilde{\eta}(\cdot)$ is the corresponding maximal trading strategy, as for example determined in the proof of Theorem 5, such that $V^{p,\tilde{\eta}}(T) \geq 1$. Then, Theorems 11 and 4 of Delbaen and Schachermayer (1995c) yield together that NFLVR holds for the new market. If we denote the volatility matrix of the new market by $\tilde{\sigma}(\cdot)$, then a simple computation shows that $\mathrm{rank}(\sigma(\cdot)) \equiv \mathrm{rank}(\tilde{\sigma}(\cdot))$, and hence, that the new market is also complete. Although we have not assumed $d + 1 \leq K$, the argument of Theorem 1.6.6 in Karatzas and Shreve (1998) works and proves the result. □

We can now formulate and prove the Generalized Second Fundamental Theorem of Asset Pricing:

Theorem 7 (Generalized Second Fundamental Theorem of Asset Pricing)**.** *A market is complete if and only if any process* $\nu(\cdot) \in \Theta'$ *satisfies* $\nu(t) = \theta^m(t)$ *Lebesgue-almost everywhere.*

Proof. If the market is complete, then we have $\text{rank}(\sigma(t)) = K$ Lebesgue-almost everywhere by Lemma 9. This is equivalent to the Lebesgue-almost everywhere uniqueness of $\nu(\cdot)$ in Θ'. For the reverse direction, we observe that the supremum in (3.7) is always taken over a singleton, and is thus trivially attained. □

We remark that any complete market implies $Z^{\nu}(\cdot) \equiv Z^{\theta^m}(\cdot)$ for all $\nu(\cdot) \in \Theta'$. Thus, in the no-arbitrage framework, this directly translates into the uniqueness of the ELMM. However, it is important to note that the question regarding the completeness of the market can be addressed separately from the question regarding the existence of arbitrage; see also Jarrow et al. (1999).

3.5 Relative arbitrage and strong relative arbitrage

In this section, we analyze the interplay of relative arbitrage and strong relative arbitrage opportunities. The concept of relative arbitrage traces back to Merton (1973), where the term "dominant" portfolio is used. He writes:

> "Security (portfolio) A is *dominant* over security (portfolio) B, if on some known date on the future, the return on A will exceed the return on B for some possible states of the world, and will be at least as large as on B, in all possible states of the world."

We also refer to Jarrow et al. (2007; 2010) for a thorough discussion of Merton's no-dominance principle in connection with the existence of bubbles. Delbaen and Schachermayer (1994; 1995c) coined the term "maximal element" for a terminal wealth $V^{p,\pi}(T)$ that cannot be dominated by another terminal wealth $V^{p,\eta}(T)$. In the following, we use the terminology of Stochastic Portfolio Theory; see Fernholz and Karatzas (2009). This line of research does not focus on finding the "right conditions" to exclude arbitrage opportunities, but instead studies these opportunities; see, for example, Fernholz and Karatzas (2010), where relative arbitrage with respect to the market portfolio is studied. We now provide the precise definition on which we shall rely:

Definition 11 (Relative and classical arbitrage). We say that there exists *relative arbitrage* with respect to a trading strategy $\pi(\cdot)$ over the time horizon $[0,T]$ if there exists a trading strategy $\eta(\cdot)$ such that $\mathbb{P}(V^{\tilde{p},\eta}(T) \geq V^{\tilde{p},\pi}(T)) = 1$ and $\mathbb{P}(V^{\tilde{p},\eta}(T) > V^{\tilde{p},\pi}(T)) > 0$. We say that $\eta(\cdot)$ is a *strong relative arbitrage* if $\mathbb{P}(V^{\tilde{p},\eta}(T) > V^{\tilde{p},\pi}(T)) = 1$. If $\pi(\cdot) \equiv 0$, which corresponds to holding the risk-free money market, then we sometimes substitute the word "relative" by "classical."

Obviously, the existence of strong relative arbitrage necessarily implies that of relative arbitrage. However, the converse is less obvious. Using the insights developed in the previous sections, we shall discuss conditions under which the existence of relative arbitrage implies that of strong relative arbitrage in Theorem 8. We start by giving an example showing that this implication does not always hold:

Example 8 (Relative arbitrage without strong relative arbitrage). Let $K = 1$, $d = 1$, $S(0) = 2$, $\theta(\cdot) \equiv 0$ and $\sigma(t) = 0$ for $t \in [0,1] \cup [2,\infty)$. Set

$$\sigma(t) = \frac{1}{S(t)} \mathbf{1}_{\{W(1) \geq 0\} \cap \{\varrho > t\}} \frac{1}{\sqrt{2-t}}$$

for $t \in (1,2)$, where

$$\varrho := \inf \left\{ t \geq 1 : \int_1^t \frac{1}{\sqrt{2-s}} dW(s) = -1 \right\}. \tag{3.17}$$

Then we have $\varrho < 2$, which yields $S(2) = 2$ on the event $\{W(1) < 0\}$, $S(2) = 1$ on the event $\{W(1) \geq 0\}$, and $S(\cdot)$ being a strictly positive, local martingale.

We consider the "buy-and-hold" trading strategy $\pi(\cdot) \equiv 1$, such that $D := V^{2,\pi}(2) = S(2)$. Since NFLVR is satisfied here, it is sufficient to take the supremum in (3.7) over

$$\tilde{\Theta} := \{\nu(\cdot) \in \Theta' : \mathbb{E}[Z^\nu(2)] = 1\},$$

to wit, the subset of Θ' that generates the ELMMs. For any $\nu(\cdot) \in \tilde{\Theta}$ we have $\mathbb{Q}^\nu(W(1) \geq 0) > 0$, where $\mathbb{Q}^\nu$ is defined by $d\mathbb{Q}^\nu/d\mathbb{P} = Z^\nu(2)$, such that $S(\cdot)$ is a strict local martingale under any ELMM. However,

$$p = \sup_{\nu(\cdot)\in\tilde{\Theta}} \mathbb{E}[Z^\nu(2)S(2)] = 2 - \inf_{\nu(\cdot)\in\tilde{\Theta}} \mathbb{Q}^\nu(W(1) \geq 0) = 2.$$

That is, the cheapest trading strategy to superreplicate one share $S(2)$ at time $T = 2$ costs $p = 2$. Fix any trading strategy $\eta(\cdot)$. Then, on the event $\{W(1) < 0\}$ we always have $V^{2,\eta}(2) = 2 = S(2)$. This shows that no strong relative arbitrage exists with respect to $\pi(\cdot)$ over the time horizon $[0, 2]$.

However, relative arbitrage exists. The trading strategy $\eta(\cdot) \equiv 0$ yields $V^{2,\eta}(2) = 2$. Thus, $\mathbb{P}(V^{2,\eta}(2) > S(2)) = \mathbb{P}(W(1) \geq 0) = 1/2 > 0$. To conclude, although the cheapest superreplicating price of a given terminal wealth $V^{\tilde{p},\pi}(T)$ might be $\tilde{p}$, the trading strategy $\pi(\cdot)$ might nevertheless be dominated in the sense of Merton (1973). □

Delbaen and Schachermayer (1998) discuss a model in which the stock price process is a strict local martingale under one measure, but actually a true martingale under an equivalent measure. The previous example exhibits a stock price process such that $Z^\nu(\cdot)S(\cdot)$ is a strict local martingale for all $\nu(\cdot) \in \Theta'$, but where the cheapest price to replicate the stock price is the current stock price itself. This example can be easily modified to obtain a market that allows for arbitrage, but where the cheapest superreplicating price, to pay at time 0, for \$1 at time $T > 0$ is again \$1:

Example 9 (Free lunch with vanishing risk but without strong classical arbitrage). We again set $K = 1$ and $d = 1$. We now consider the stock price process

$\hat{S}(\cdot) := 1/S(\cdot)$ with $S(\cdot)$ defined as in Example 8, which corresponds to a change of numéraire. Itô's formula yields the dynamics

$$d\hat{S}(t) = \hat{S}(t)\sigma(t)\Big(\sigma(t)dt - dW_t\Big)$$

with $\sigma(\cdot)$ as in Example 8. Corollary 15 of Delbaen and Schachermayer (1995c) directly yields that this market does not allow for an ELMM. Any stochastic discount factor $\hat{Z}^{\hat{\nu}}(\cdot)$ in the new model can be written as $\hat{Z}^{\hat{\nu}}(\cdot) = Z^{\nu}(\cdot)S(\cdot)/S(0)$, where $Z^{\nu}(\cdot)$ denotes a stochastic discount factor in the original model of Example 8.

We now set $T = 2$ and consider the claim $D = 1$, which corresponds to holding exactly \$1 at time 2. and obtain $\sup_{\hat{\nu}(\cdot)\in\Theta'} \mathbb{E}[\hat{Z}^{\hat{\nu}}(2)D] = 1$. Thus, despite the existence of arbitrage opportunities, the cheapest price to hold \$1 is again \$1 and no strong classical arbitrage exists, due to reasoning similar to that in Example 8. However, starting with \$1, one can achieve a terminal wealth that is larger than \$1 with positive probability by following the trading strategy $\eta(\cdot) \equiv 1$. □

We can now state precise conditions for the existence of relative arbitrage and strong relative arbitrage opportunities:

Theorem 8 (Conditions for relative arbitrage and strong relative arbitrage)**.** *Fix $T > 0$ and a trading strategy $\pi(\cdot)$ admissible for some initial capital $\tilde{p} > 0$.*

(a) There exists a strong relative arbitrage opportunity with respect to $\pi(\cdot)$ over the time horizon $[0, T]$ if

$$p := \sup_{\nu(\cdot)\in\Theta'} \mathbb{E}[Z^{\nu}(T)V^{\tilde{p},\pi}(T)] < \tilde{p}. \tag{3.18}$$

The converse holds if a trading strategy $\eta(\cdot)$ and a constant $\delta > 1$ exist such that $V^{\tilde{p},\eta}(T) \geq \delta V^{\tilde{p},\pi}(T) \neq 0$.

(b) There exists a relative arbitrage opportunity with respect to $\pi(\cdot)$ over the time horizon $[0, T]$, if and only if

$$\mathbb{E}[Z^{\nu}(T)V^{\tilde{p},\pi}(T)] < \tilde{p} \tag{3.19}$$

for all $\nu(\cdot) \in \Theta'$.

(c) *In particular, the existence of relative arbitrage implies that of strong relative arbitrage over the time horizon* $[0, T]$ *if the market is quasi-complete and* $V^{\tilde{p},\pi}(T)$ *is* $\mathcal{F}^S(T)$*-measurable.*

Proof. We prove (a), (b), and (c) separately:

(a) Assume (3.18) holds. According to Theorem 5, a trading strategy $\eta(\cdot)$ exists such that

$$V^{\tilde{p},\eta}(T) \geq V^{\tilde{p},\pi}(T) + \tilde{p} - p > V^{\tilde{p},\pi}(T),$$

which shows the existence of strong relative arbitrage.

We observe that for any $\nu(\cdot) \in \Theta'$ and for any trading strategy $\eta(\cdot)$, admissible with respect to the initial capital $\tilde{p}$, the process $Z^\nu(\cdot)V^{\tilde{p},\eta}(\cdot)$ is a supermartingale. Thus, if a strong relative arbitrage $\eta(\cdot)$ and some $\delta > 1$ as in the statement of the theorem exist, then

$$\sup_{\nu(\cdot)\in\Theta'} \mathbb{E}[Z^\nu(T)V^{\tilde{p},\pi}(T)] \leq \sup_{\nu(\cdot)\in\Theta'} \mathbb{E}[Z^\nu(T)V^{\tilde{p},\eta}(T)]\frac{1}{\delta} \leq \frac{\tilde{p}}{\delta} < \tilde{p},$$

which implies (3.18).

(b) In a similar vein, assume that a relative arbitrage $\eta(\cdot)$ with respect to $\pi(\cdot)$ exists. Then,

$$\mathbb{E}[Z^\nu(T)V^{\tilde{p},\pi}(T)] < \mathbb{E}[Z^\nu(T)V^{\tilde{p},\eta}(T)] \leq \tilde{p}.$$

for all $\nu(\cdot) \in \Theta'$, yielding (3.19). For the reverse direction, let us introduce, as in Lemma 9, a fictional market with $d + 1$ stocks

$$\left(\frac{S_1(\cdot)}{V^{\tilde{p},\pi}(\cdot)}, \ldots, \frac{S_d(\cdot)}{V^{\tilde{p},\pi}(\cdot)}, \frac{1}{V^{\tilde{p},\pi}(\cdot)}\right).$$

Then, (3.19) yields that no ELMM exists for the new market. Thus, the fictional market allows for classical arbitrage and Theorems 4 and 11 of Delbaen and Schachermayer (1995c) yield the existence of a relative arbitrage opportunity.

(c) If the market is quasi-complete, then the supremum in (3.18) is always a maximum, and consequently, relative arbitrage implies strong relative arbitrage in the case of quasi-completeness. □

The next example illustrates the fact that $p = \tilde{p}$ in (3.18) does not necessarily exclude a strong relative arbitrage opportunity:

Example 10 (Diminishing strong relative arbitrage). We use the same setting as in Example 8 with $\sigma(\cdot)$ modified as follows:

$$\sigma(t) = \frac{1}{S(t)} \sum_{i=1}^{\infty} \frac{1}{i} \mathbf{1}_{\{|W(1)| \in [i-1,i)\} \cap \{\varrho > t\}} \frac{1}{\sqrt{2-t}}$$

for $t \in (1,2)$, where the stopping ϱ is defined as in (3.17). This yields $S(2) = 1 - 1/i$ on the event $\{|W(1)| \in [i-1,i)\}$ for all $i \in \mathbb{N}$. We obtain

$$p = \sup_{\nu(\cdot) \in \tilde{\Theta}} \mathbb{E}[Z^{\nu}(2) S(2)] = 2 - \inf_{\nu(\cdot) \in \tilde{\Theta}} \left(\sum_{i=1}^{\infty} \frac{1}{i} \mathbb{Q}^{\nu}(|W(1)| \in [i-1,i)) \right) = 2,$$

where $\tilde{\Theta}$ and $\mathbb{Q}^{\nu}$ are defined in Example 8. However, the trading strategy $\eta(\cdot) \equiv 0$ yields $V^{2,\eta}(2) = 2 > S(2)$ and is thus a strong relative arbitrage. This example shows that (3.18) is sufficient, but not necessary for the existence of strong relative arbitrage. □

It is clear that we need to assume in part (c) of Theorem 8 that $V^{\tilde{p},\pi}(T)$ be $\mathcal{F}^S(T)$-measurable. To see this, one could, for example, construct a wealth process $V^{\tilde{p},\pi}(\cdot)$ in a quasi-complete model that has exactly the same dynamics as $S(\cdot)$ in Example 8 and allows for relative arbitrage but not strong relative arbitrage.

3.6 Conclusion

In this chapter, we have illustrated that in general the concepts of arbitrage and completeness can be considered separately from each other. More precisely, we have proven a version of the Second Fundamental Theorem of Asset Pricing for markets that do not proscribe arbitrage. We have also provided necessary and sufficient conditions for claims in incomplete markets to be exactly replicable. We have introduced the concept of quasi-complete markets to generalize the idea of complete markets and have further shown that relative arbitrage implies strong relative arbitrage in quasi-complete markets.

We have assumed that the agent can trade dynamically and without any constraints in the market. It is an open question for markets that allow for the presence of arbitrage opportunities, in which manner trading constraints interfere with the replication of claims. In particular, it is not clear under which trading constraints certain arbitrage opportunities disappear. This is subject to future research. A good starting point is the theory for markets that satisfy NFLVR, as developed in Cvitanić and Karatzas (1993) and Föllmer and Kramkov (1997).

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