A REMARK ON $\mathscr{H}^1$ MARTINGALES

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Abstract. The space of $\mathscr{H}^1$ martingales is interesting because of its duality with the space of BMO martingales. It is straightforward to show that every $\mathscr{H}^1$ martingale is a uniformly integrable martingale. However, the converse is not true. That is to say, some uniformly integrable martingales are not $\mathscr{H}^1$ martingales. This brief note provides a template for systematically constructing such processes.

1. Introduction

Let $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$ be a filtered probability space, whose filtration $\mathfrak{F} = (\mathscr{F}_t)_{t\geq 0}$ is right-continuous. By assumption, all processes are defined on $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$ and have càdlàg sample paths. The families of local martingales and uniformly integrable martingales are denoted by $\mathscr{M}_{\text{loc}}$ and $\mathscr{M}$, respectively. Given $p \geq 1$, let $\mathscr{H}^p$ be the family of local martingales $M \in \mathscr{M}_{\text{loc}}$, for which $\mathsf{E}([M]_\infty^{p/2}) < \infty$. Then $\mathscr{H}^p$ forms a Banach space, when endowed with the norm $\|\cdot\|_p$, defined by

$$\|M\|_p := \Big(\mathsf{E}\big([M]_\infty^{p/2}\big)\Big)^{1/p},$$

for all $M \in \mathscr{H}^p$. Let $\mathscr{H}^p_{\text{loc}}$ denote the family of local martingales that are locally in $\mathscr{H}^p$.

The space $\mathscr{H}^1$ has several interesting features. For example, $\mathscr{H}^1$ contains $\mathscr{H}^2$ as well as all local martingales with integrable variation (see e.g Protter 2005, Theorem IV.49). It can also be shown that $\mathscr{H}^2$ and the family of bounded (uniformly integrable) martingales are both dense in $\mathscr{H}^1$ (see e.g Protter 2005, Theorem IV.50). In addition, $\mathscr{M}_{\text{loc}} \subseteq \mathscr{H}^1_{\text{loc}}$ (see e.g Protter 2005, Theorem IV.51). However, the most significant result is the duality between $\mathscr{H}^1$ and the space of BMO martingales. In detail, let

$$\mathrm{BMO} := \Big\{M \in \mathscr{H}^2 \,|\, \mathsf{E}\big((M_\infty - M_{\tau-})^2\big) \leq c^2, \text{ for all } \tau \in \mathfrak{S} \text{ and some } c \in \mathbb{R}_+\Big\},$$

where $\mathfrak{S}$ denotes the family of stopping times defined on $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$ and subject to the convention $M_{0-} := 0$, for all $M \in \mathscr{M}_{\text{loc}}$. Then BMO becomes a Banach space, when endowed with the norm $\|\cdot\|_{\mathrm{BMO}}$, defined by

$$\|M\|_{\mathrm{BMO}} := \sup_{\tau\in\mathfrak{S}} \sqrt{\frac{\mathsf{E}\big((M_\infty - M_{\tau-})^2\big)}{\mathsf{P}(\tau < \infty)}},$$

for all $M \in \mathrm{BMO}$. It follows that $(\mathscr{H}^1)^* \simeq \mathrm{BMO}$ (see e.g Protter 2005, Theorem IV.55).[1]

Now, suppose $M \in \mathscr{H}^1$, in which case the Burkholder-Davis-Gundy inequalities (see e.g Protter 2005, Theorem IV.48) imply that $\mathsf{E}\big(\sup_{t\geq 0} |M_t|\big) < \infty$. An application of the dominated convergence theorem then gives $M \in \mathscr{M}$. This establishes that $\mathscr{H}^1 \subseteq \mathscr{M}$.

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[1]This result is the probability-theoretic analogue of the classical H^1–BMO duality, due to Fefferman and Stein (1972).

However, the reverse inclusion does not hold. We demonstrate this fact below, by providing a recipe for constructing non-negative uniformly integrable martingales in $\mathscr{M} \setminus \mathscr{H}^1$.

2. Construction of Processes in $\mathscr{M} \setminus \mathscr{H}^1$

Given a local martingale $M \in \mathscr{M}_{\mathrm{loc}}$, observe that

$$\mathsf{E}\Big(\sup_{t\geq 0}|M_t|\Big) = \int_0^\infty \mathsf{P}\Big(\sup_{t\geq 0}|M_t| > u\Big)\,\mathrm{d}u$$

and

$$\sum_{n=1}^\infty \mathsf{P}\Big(\sup_{t\geq 0}|M_t| > n\Big) \leq \int_0^\infty \mathsf{P}\Big(\sup_{t\geq 0}|M_t| > u\Big)\,\mathrm{d}u \leq 1 + \sum_{n=1}^\infty \mathsf{P}\Big(\sup_{t\geq 0}|M_t| > n\Big).$$

Since $M \in \mathscr{H}^1$ if and only if $\mathsf{E}(\sup_{t\geq 0}|M_t|) < \infty$, by virtue of the Burkholder-Davis-Gundy inequalities, it follows that $M \in \mathscr{H}^1$ if and only if

$$\sum_{n=1}^\infty \mathsf{P}\Big(\sup_{t\geq 0}|M_t| > n\Big) < \infty. \tag{2.1}$$

This condition plays a key role in our construction.

Fix a non-negative local martingale $M \in \mathscr{M}_{\mathrm{loc}} \setminus \mathscr{M}$ that is not a uniformly integrable martingale, and define the non-decreasing sequence $(c_n)_{n\in\mathbb{N}} \subset (1,\infty)$, by setting

$$c_n := \ln\bigg(\mathrm{e} + \sum_{k=1}^n \mathsf{P}\Big(\sup_{t\geq 0} M_t > k\Big)\bigg), \tag{2.2}$$

for each $n \in \mathbb{N}$. Since $M \notin \mathscr{H}^1$, it follows that $\lim_{n\uparrow\infty} c_n = \infty$. Next, suppose that $(\Omega, \mathscr{F}, \mathsf{P})$ accommodates a discrete $\mathscr{F}_0$-measurable random variable $Y \in \mathbb{N}$ that is independent of M, and whose distribution satisfies $\mathsf{P}(Y > n) = 1/c_n$, for each $n \in \mathbb{N}$, and let

$$\sigma := \inf\{t \geq 0 \,|\, M_t > Y\} \tag{2.3}$$

denote the first time M exceeds Y. It follows that

$$\mathsf{P}\Big(\sup_{t\geq 0} M_t^\sigma > n\Big) \geq \mathsf{P}\Big(\sup_{t\geq 0} M_t > n\Big)\mathsf{P}(Y > n) = \frac{1}{c_n}\mathsf{P}\Big(\sup_{t\geq 0} M_t > n\Big),$$

for each $n \in \mathbb{N}$. Consequently,

$$\sum_{n=1}^\infty \mathsf{P}\Big(\sup_{t\geq 0} M_t^\sigma > n\Big) \geq \lim_{m\uparrow\infty} \frac{1}{c_m}\sum_{n=1}^m \mathsf{P}\Big(\sup_{t\geq 0} M_t > n\Big) = \lim_{m\uparrow\infty}\frac{\mathrm{e}^{c_m} - \mathrm{e}}{c_m} = \infty,$$

since $(c_n)_{n\in\mathbb{N}}$ is non-decreasing and $\lim_{n\uparrow\infty} c_n = \infty$. This implies that $M^\sigma \notin \mathscr{H}^1$. On the other hand, the almost sure limit $M_\infty^\sigma := M_{\infty-}^\sigma \in \mathbb{R}_+$ exists, since M^σ is a non-negative local martingale, and hence also a non-negative supermartingale. Moreover,

$$\begin{aligned}\mathsf{E}(M_\infty^\sigma) &= \sum_{n=1}^\infty \mathsf{E}(M_\infty^\sigma \,|\, Y = n)\mathsf{P}(Y = n) = \sum_{n=1}^\infty \mathsf{E}(M_\infty^{\tau_n} \,|\, Y = n)\mathsf{P}(Y = n)\\ &= \sum_{n=1}^\infty \mathsf{E}(M_\infty^{\tau_n})\mathsf{P}(Y = n) = \sum_{n=1}^\infty \mathsf{E}(M_0)\mathsf{P}(Y = n) = \mathsf{E}(M_0^\sigma),\end{aligned}$$

where

$$\tau_n := \inf\{t \geq 0 \,|\, M_t > n\}$$

for each $n \in \mathbb{N}$. Here, the penultimate equality follows from the fact that $M^{\tau_n} \in \mathscr{M}$, for each $n \in \mathbb{N}$. Consequently, $M^\sigma \in \mathscr{M}$.

Example 2.1. Consider a non-negative local martingale $M \in \mathscr{M}_{\text{loc}}$ that belongs to Class $(\mathcal{C}_0)$, according to the terminology of Nikeghbali and Yor (2006), and suppose that $M_0 = 1$. In that case, M is a strictly positive local martingale without any positive jumps, for which $M_\infty := M_{\infty-} = 0$. The construction above is then applicable, since $\mathsf{E}(M_\infty) = 0 < 1 = \mathsf{E}(M_0)$ implies that $M \in \mathscr{M}_{\text{loc}} \setminus \mathscr{M}$. Moreover, an application of Doob's maximal identity (see Nikeghbali and Yor 2006, Lemma 2.1) provides the following concrete representation for the non-decreasing sequence $(c_n)_{n\in\mathbb{N}}$, defined by (2.2):

$$c_n = \ln\left(\mathrm{e} + \sum_{k=1}^{n} \frac{1}{k}\right),$$

for each $n \in \mathbb{N}$. It is then straightforward to see that $\lim_{n\uparrow\infty} c_n = \infty$, which is the crucial ingredient for showing that $M^\sigma \notin \mathscr{H}^1$, where the stopping time σ is given by (2.3).

Note that the previous construction requires $\mathscr{F}_0$ to be sufficiently large to admit an $\mathscr{F}_0$-measurable random variable Y, independent of M and such that $\mathsf{P}(Y > n)$ can be defined appropriately, for each $n \in \mathbb{N}$. Example 2.2 below demonstrates that the construction above does always not work without this requirement. That is to say, given a non-negative local martingale $M \in \mathscr{M}_{\text{loc}} \setminus \mathscr{M}$, there may be no stopping time $\tau \in \mathfrak{S}$, such that $M^\tau \in \mathscr{M} \setminus \mathscr{H}^1$.

Example 2.2. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables satisfying $\mathsf{P}(\theta_1 = 0) = \mathsf{P}(\theta_1 = 1) = 1/2$. Define the process $M = (M_t)_{t\geq 0}$, by setting $M_t := 2^{[t]} \prod_{i=1}^{[t]} \theta_i$, where $[\,\cdot\,]$ are the Gauss brackets. Note that M is a martingale under its own filtration $\mathfrak{F}^M = (\mathscr{F}_t^M)_{t\geq 0}$. However, it is not uniformly integrable, since $M_\infty = 0$. Consequently, $M \in \mathscr{M}_{\text{loc}} \setminus \mathscr{M}$. Note that $\mathscr{F}_t^M = \sigma(\theta_1, \theta_1\theta_2, \cdots, \prod_{i=1}^{[t]} \theta_i)$, for all $t \geq 0$. In particular, $\mathscr{F}_0^M = \{\emptyset, \Omega\}$. We now claim that $M^\tau \in (\mathscr{M}_{\text{loc}} \setminus \mathscr{M}) \cup \mathscr{H}^1$, for all $\mathfrak{F}^M$-stopping times τ. This implies that the above construction does not work unless $\mathscr{F}_0^M$ is enriched. Indeed, fix an arbitrary $\mathfrak{F}^M$-stopping time τ, and assume that $M^\tau \in \mathscr{M}$. We then have to show that $M \in \mathscr{H}^1$. To see this, note that

$$n_* := \min\{n \in \mathbb{Z}_+ \,|\, \mathsf{P}(\tau \leq n \text{ and } M_\tau > 0) > 0\} < \infty,$$

otherwise, $M^\tau = M \notin \mathscr{M}$, contradicting the assumption. Hence we have

$$\mathsf{P}\left(\tau \leq n_* \text{ and } \prod_{i=1}^{n_*} \theta_i = 1\right) > 0 \qquad \text{and} \qquad \{\tau \leq n_*\} \cap \left\{\prod_{i=1}^{n_*} \theta_i = 1\right\} \in \mathscr{F}_{n_*}^M.$$

This again implies that $\{\prod_{i=1}^{n_*} \theta_i = 1\} \subset \{\tau \leq n_*\}$, yielding $M^\tau \leq 2^{n_*}$, whence $M \in \mathscr{H}^1$.

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