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Projections of Scaled Bessel Processes

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Abstract

Let X and Y denote two independent squared Bessel processes of dimension m and $n-m$, respectively, with $n \geq 2$ and $m \in [0, n)$, making $X+Y$ a squared Bessel process of dimension n. For appropriately chosen function s, the process $s(X+Y)$ is a local martingale. We study the representation and the dynamics of $s(X+Y)$, projected on the filtration generated by X. This projection is a strict supermartingale if, and only if, $m < 2$. The finite-variation term in its Doob-Meyer decomposition only charges the support of the Markov local time of X at zero.

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Introduction

Optional projections of martingales are martingales; however, optional projections of local martingales are not necessarily local martingales. If the local martingale is nonnegative, Fatou's lemma only yields that these optional projections are supermartingales.

Due to their analytic tractability, scaled Bessel processes of dimension two or higher are ideal to study this phenomenon. A first important step has been taken by [6] and [10], who consider the three-dimensional Bessel process, namely the modulus of a three-dimensional Brownian motion started away from zero, in the filtration generated by its components. The reciprocal of the three-dimensional Bessel process is a local martingale; in [6] and [10], it is observed that its optional projection becomes a strict supermartingale when projecting on the first component of the three-dimensional Brownian motion. However, when projecting on the first two components, the optional projection preserves the local martingale property.

In this article, we investigate these surprising observations further by providing a systematic study of optional projections of scaled Bessel processes of any dimension greater than or equal to two. The arguments here are mostly analytic; an alternative approach, involving more probabilistic quantities such as the Laplace transform of the inverse Markov local time of the observed component, can be found in the extended arXiv version [8].

We need to point out the deep work of [4] on intertwining two related Markov processes. In particular, two squared Bessel processes of different dimensions are

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intertwined by an appropriate use of the expectation operator. As already mentioned above, in this article, we complement their insights by focusing on the distinction between strict and non-strict supermartingales.

1 Main Result

Consider a probability space $(\Omega, \mathcal{G}, \mathsf{P})$, equipped with two independent Brownian motions B^X and B^Y. Fix $n \geq 2$ and $m \in [0, n)$ and consider the two stochastic differential equations

$$X_t = 1 + mt + 2\int_0^t \sqrt{X_u}\mathrm{d}B_u^X, \qquad t \geq 0;$$
$$Y_t = (n-m)t + 2\int_0^t \sqrt{Y_u}\mathrm{d}B_u^Y, \qquad t \geq 0.$$

These stochastic differential equations have unique strong solutions, called squared Bessel process of dimension m and $n-m$, respectively; see [11, Section XI.1]. Lévy's characterisation of Brownian motion yields that $X+Y$ is also a squared Bessel process, now of dimension n. Feller's test for explosions yields that $X+Y$ is strictly positive since $n \geq 2$. We shall use $\mathcal{G}_\cdot$ throughout to denote the natural filtration generated by the pair (X, Y).

Next, consider the function

$$s : (0,\infty) \ni w \mapsto \begin{cases} w^{1-n/2}, & \text{if } n > 2; \\ -\log(w), & \text{if } n = 2. \end{cases}$$

Itô's formula yields that $s(X+Y)$ is a local martingale. Let $\mathcal{F}_\cdot$ now denote the smallest right-continuous filtration that makes X adapted. For future reference, note that the process $\int_0^\cdot \sqrt{X_u}\mathrm{d}B_u^X$ is adapted to the filtration $\mathcal{F}_\cdot$. We are interested in the $\mathcal{F}_\cdot$–optional projection Z of $s(X+Y)$, which is the unique $\mathcal{F}_\cdot$–optional process Z such that

$$Z_\tau = \mathsf{E}\left[s(X_\tau + Y_\tau)|\mathcal{F}_\tau\right]$$

holds for all bounded $\mathcal{F}_\cdot$ stopping times τ.

Remark 1.1. In order to ensure that Z above exists, it suffices that $\mathsf{E}\left[|s(X_\tau + Y_\tau)|\right] < \infty$ holds for a fixed bounded $\mathcal{F}_\cdot$ stopping time τ. When $n > 2$, $\mathsf{E}\left[|s(X_\tau + Y_\tau)|\right] < \infty$ holds from the optional sampling theorem because $s(X+Y)$ is a nonnegative local martingale, thus a supermartingale, under $\mathcal{G}_\cdot$. For $n = 2$, we claim that $\mathsf{E}[|\log(J_\tau)|] < \infty$ for all bounded stopping times τ when J is two-dimensional squared Bessel process with $J_0 = 1$. Indeed, first note that $\mathsf{E}[J_\tau] \leq 1 + 2\mathsf{E}[\tau] < \infty$ holds from the dynamics of J, localisation, Fatou's lemma and monotone convergence. Therefore, $\mathsf{E}[\log_+(J_\tau)] \leq \mathsf{E}[J_\tau] < \infty$ holds. Furthermore, since $\log J$ is a local martingale and $J_0 = 1$, we have $\mathsf{E}[\log_-(J_{\tau\wedge\tau_m})] = \mathsf{E}[\log_+(J_{\tau\wedge\tau_m})] \leq 1 + 2\mathsf{E}[\tau \wedge \tau_m]$ along a localising sequence $(\tau_m)_{m\in\mathbb{N}}$, giving $\mathsf{E}[\log_-(J_\tau)] \leq 1 + 2\mathsf{E}[\tau] < \infty$ by Fatou's lemma and monotone convergence.

In order to set the stage for the statement of our main result, recall the Gamma function

$$(0,\infty) \ni k \mapsto \Gamma(k) := \int_0^\infty w^{k-1}\mathrm{e}^{-w}\mathrm{d}w.$$

Furthermore, define the stopping time

$$\rho := \inf\left\{t \geq 0 \mid X_t = 0\right\}, \tag{1.1}$$

which is P–almost surely finite when $0 \leq m < 2$.

In the case $0 < m < 2$, note that X allows for Markov local time process Λ at zero, defined via

$$\Lambda_t := \lim_{\varepsilon \downarrow 0} m\varepsilon^{-m/2} \int_0^t \mathbf{1}_{\{X_u < \varepsilon\}} \mathrm{d}u, \qquad t \geq 0. \tag{1.2}$$

References for existence and properties of Λ are provided in Section 2 below; in particular, it will also be shown there that Λ coincides with the semimartingale local time at zero of the scaled process $X^{1-m/2}/(2-m)$.

With the above notation, we now present the main result of this note.

Theorem 1.2. *The $\mathcal{F}_\cdot$–optional projection Z of $s(X+Y)$ exists and satisfies $Z_t = f(t, X_t)$ for all $t > 0$, where*

$$f(t,x) := \frac{1}{\Gamma\left((n-m)/2\right)} \times \begin{cases} \int_0^\infty (x+2tw)^{1-n/2} w^{(n-m)/2-1} \mathrm{e}^{-w} \mathrm{d}w, & \text{if } n > 2; \\ \int_0^\infty -\log(x+2tw) w^{-m/2} \mathrm{e}^{-w} \mathrm{d}w, & \text{if } n = 2 \end{cases} \tag{1.3}$$

for all $t > 0$ and $x \geq 0$. Furthermore, the following statements hold:

- *If $m \geq 2$ (thus, $n > 2$), then*

$$Z = 1 + 2\int_0^\cdot f'_x(u, X_u)\sqrt{X_u} \mathrm{d}B^X_u; \tag{1.4}$$

  hence Z is a strict local martingale.
- *If $m \in (0,2)$, then Z is a strict supermartingale, that is, not a local martingale. With Λ given by (1.2), the* Doob-Meyer *decomposition of Z is*

$$Z = \begin{cases} 1 + 2\int_0^\cdot f'_x(u, X_u)\sqrt{X_u}\mathrm{d}B^X_u - \frac{\Gamma(m/2)}{\Gamma(n/2-1)} \int_0^\cdot (1/2u)^{(n-m)/2} \mathrm{d}\Lambda_u, & \text{if } n > 2; \\ 2\int_0^\cdot f'_x(u, X_u)\sqrt{X_u}\mathrm{d}B^X_u - \Gamma\left(m/2\right) \int_0^\cdot (1/2u)^{1-m/2} \mathrm{d}\Lambda_u, & \text{if } n = 2. \end{cases} \tag{1.5}$$

- *If $m = 0$, then Z is again a strict supermartingale of the form*

$$Z = \begin{cases} 1 + 2\int_0^{\rho\wedge\cdot} f'_x(u, X_u)\sqrt{X_u}\mathrm{d}B^X_u - \frac{2}{\Gamma(n/2-1)} \int_0^\cdot (1/2u)^{n/2} \mathbf{1}_{\{\rho \leq u\}} \mathrm{d}u, & \text{if } n > 2; \\ 2\int_0^{\rho\wedge\cdot} f'_x(u, X_u)\sqrt{X_u}\mathrm{d}B^X_u - \int_0^\cdot (1/u)\, \mathbf{1}_{\{\rho \leq u\}} \mathrm{d}u, & \text{if } n = 2. \end{cases} \tag{1.6}$$

Section 3 contains a mostly analytic proof of Theorem 1.2. The extended arXiv version [8] contains an alternative proof, using more probabilistic arguments, for the case $n > 2$. This alternative route provides further intuition on the appearance of the local time in the Doob-Meyer decomposition of Z. (Furthermore, this alternative route helped us to formulate the precise statements of Theorem 1.2.) Lemma 2.1 in Section 2 below summarises some results concerning the Markov local time process Λ, appearing in (1.5).

Remark 1.3. Here is a quick argument why Z is a strict supermartingale if $m \in [0,2)$ and $n > 2$. In general, the strict supermartingale property of Z will follow from the non-constant finite-variation terms in (1.5) and (1.6) in the Doob-Meyer decomposition of Z. All these assertions shall be argued in the proof of Theorem 1.2. Assume now that $0 \leq m < 2 < n$, and suppose (as we shall see, by way of contradiction) that Z is a local martingale. Since X and Y are independent and since the function s is decreasing, we have

$$Z_t \leq \mathsf{E}[s(Y_t)] = f(t,0) < \infty, \qquad t > 0.$$

Since Z is additionally strictly positive (recall that $n > 2$ is assumed), hence bounded, it would then follow that $(Z_t)_{t>0}$ is an actual martingale. This would imply by Fatou's lemma (note that $t = 0$ was not covered) that Z is an actual martingale. But this is impossible, since it would have constant expectation, meaning that $s(X+Y)$ also has constant expectation, contradicting the fact that it is a strict local martingale; see (2.3) below. Therefore, we obtain that Z fails to be a local martingale whenever $0 \le m < 2 < n$.

Remark 1.4. The special cases $n = 3$ and $m \in \{1, 2\}$ in Theorem 1.2 are studied in [6] and [10]. When $n = 3$ and $m = 1$, using (1.3) we obtain

$$\begin{aligned} Z_t &= \int_0^\infty \frac{1}{2t\sqrt{X_t + y}} \exp\left(-\frac{y}{2t}\right) \mathrm{d}y \\ &= \exp\left(\frac{X_t}{2t}\right) \int_{X_t}^\infty \frac{1}{2t\sqrt{w}} \exp\left(-\frac{w}{2t}\right) \mathrm{d}w \\ &= \frac{1}{\sqrt{t}} \exp\left(\frac{X_t}{2t}\right) \int_{\sqrt{X_t/t}}^\infty \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y, \\ &= \sqrt{\frac{2\pi}{t}} \exp\left(\frac{X_t}{2t}\right)\left(1 - \Phi\left(\sqrt{\frac{X_t}{t}}\right)\right), \qquad t \ge 0, \end{aligned}$$

where Φ denotes the cumulative normal distribution. Recall the discussion after (1.2), note that Λ in (1.5) is the semimartingale local time of $\sqrt{X}$. In contrast, [6] uses Brownian local time. These local times differ by a factor of 2; see [11, Exercise VI.1.17]. This explains the slight difference in the presentation of the finite-variation part in (1.5) from its representation in [6].

When $n = 3$ and $m = 2$, we obtain

$$\begin{aligned} Z_t &= \frac{1}{\sqrt{2\pi t}} \int_0^\infty \frac{1}{\sqrt{y(X_t + y)}} \exp\left(-\frac{y}{2t}\right) \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{X_t}{4t}\right) \int_1^\infty \frac{1}{\sqrt{w^2 - 1}} \exp\left(-\frac{wX_t}{4t}\right) \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{X_t}{4t}\right) K_0\left(\frac{X_t}{4t}\right), \qquad t \ge 0, \end{aligned}$$

where

$$(0, \infty) \ni k \mapsto K_0(k) := \int_1^\infty \frac{1}{\sqrt{y^2 - 1}} \mathrm{e}^{-yk} \mathrm{d}y$$

denotes the modified Bessel function of the second kind of order zero.

Remark 1.5. As pointed out by [12], if X and Y are appropriately chosen squared radial Ornstein-Uhlenbeck processes (of which squared Bessel processes are special cases) then so is $X + Y$. While it should be possible to extend the arguments below to the case that X, Y, and $X + Y$ are squared radial Ornstein-Uhlenbeck processes (such that $X + Y$ converted to natural scale is a local martingale), the notation would get unnecessarily complicated. We choose to sacrifice this bit of generality for more transparent formulas.

Remark 1.6. The function f of (1.3) satisfies the partial differential equation

$$f'_t(t, x) + m f'_x(t, x) + 2x f''_{x,x}(t, x) = 0, \qquad (t, x) \in (0, \infty)^2. \tag{1.7}$$

This partial differential equation is derived from the assertion of Theorem 1.2 via an application of Itô's formula to the local martingale $f(\cdot, X_{\rho\wedge\cdot})$—see Step 2 of the theorem's proof. The required derivatives of f in (1.7) exist due dominated convergence.

2 Squared Bessel Processes and Their Markov Local Time

We keep all notation from Section 1, and discuss here some useful properties of squared Bessel processes and their Markov local time.

2.1 Facts concerning squared Bessel processes

According to [11, Corollary XI.1.4], the process Y has a density (with respect to Lebesgue measure), given by

$$\mathsf{P}[Y_t \in \mathrm{d}y] = \frac{1}{\Gamma\left((n-m)/2\right)(2t)^{(n-m)/2}} y^{(n-m)/2-1} \exp\left(-\frac{y}{2t}\right) \mathrm{d}y, \qquad t > 0, \quad y \geq 0. \tag{2.1}$$

By Feller's test of explosions, for $m \geq 2$, the process X is strictly positive. For $m \in (0,2)$, X visits level zero, but is instantaneously reflected there, i.e.,

$$\int_0^{\cdot} \mathbf{1}_{\{X_t=0\}} \mathrm{d}t = 0. \tag{2.2}$$

For $m = 0$, the process X is absorbed when it hits zero. For a proof of these facts, see, for example, [11, Proposition XI.1.5] (but note that semimartingale local time is used there, while we shall only consider Markov local time—see Remark 2.2 later on). When $m \in (0,2)$, the process X accrues local time; i.e., with positive probability its Markov local time at zero is strictly positive. This is a consequence of Lemma 2.1.

Direct computations with the density of $X+Y$, or [9, Example 1], yield that

$$s(X+Y) \text{ is a strict local martingale in the filtration } \mathcal{G}_{\cdot}. \tag{2.3}$$

Similarly, the process $X_{\cdot\wedge\rho}^{1-m/2}$ is a martingale. This yields that the $\mathcal{F}_{\cdot}$ stopping time

$$\rho_\kappa := \inf\left\{t \geq 0 \;\middle|\; X_t \leq \frac{1}{\kappa}\right\}, \tag{2.4}$$

where $\kappa > 1$, has unbounded support. An alternative justification is provided in [3, Corollary 1.2].

2.2 Markov local time

The next result discusses properties of local time of X at zero.

Lemma 2.1. *Assume that* $0 < m < 2$. *Then the process* Λ *defined via*

$$\Lambda_t := \lim_{\varepsilon \downarrow 0} m\varepsilon^{-m/2} \int_0^t \mathbf{1}_{\{X_u<\varepsilon\}} \mathrm{d}u, \qquad t \geq 0.$$

is a nondecreasing continuous additive functional. Furthermore, we have

$$\frac{1}{1-m/2} X^{1-m/2} = \frac{1}{1-m/2} + 2\int_0^{\cdot} X_u^{(1-m)/2} \mathbf{1}_{\{X_u>0\}} \mathrm{d}B_u^X + \Lambda. \tag{2.5}$$

Remark 2.2. The process Λ of Lemma 2.1 is sometimes called “Markov” local time, in contrast to “semimartingale” local time, which only exists for semimartingales. For Markov semimartingales, these two local times may differ; however, as (2.5) shows, here the Markov local time Λ of X at zero is also the semimartingale local time of the process $X^{1-m/2}/(2-m)$ at zero. We refer to [7] and [5] for a deeper study of Bessel local time.

Proof of Lemma 2.1. We refer to [2, Section II.2 and Appendix 1.23], where properties of Λ are discussed, and further references are given.

The process $V := X^{1-m/2}/(2-m)$ is a diffusion in natural scale; thus a semimartingale; see, for example, [1, Lemma 5.22]. The Tanaka formula then yields

$$
\begin{aligned}
V &= \max(V,0) = V_0 + \int_0^{\cdot} \mathbf{1}_{\{V_u>0\}} \mathrm{d}V_u + \frac{1}{2} l^0 = V_0 + \int_0^{\cdot} \mathbf{1}_{\{X_u>0\}} X_u^{(1-m)/2} \mathrm{d}B_u^X + \frac{1}{2} l^0 \\
&= V_0 + \int_0^{\cdot} \mathbf{1}_{\{V_u>0\}} \left((2-m)V_u\right)^{(1-m)/(2-m)} \mathrm{d}B_u^X + \frac{1}{2} l^0. \qquad (2.6)
\end{aligned}
$$

Here, $(l^v)_{v\geq 0}$ denotes the semimartingale local time of V, continuous in time and right-continuous in the spatial variable $v \geq 0$, satisfying the *occupations time formula*

$$
\int_0^t g(V_u) \mathrm{d}[V,V]_u = \int_0^\infty g(v) l_t^v \mathrm{d}v, \qquad t \geq 0,
$$

for all Borel–measurable functions $g : [0,\infty) \to [0,\infty)$; see [11, Section VI.1]. Hence, we also have

$$
\int_0^t g(V_u) \mathrm{d}u = \int_0^t g(V_u) \mathbf{1}_{\{V_u>0\}} \mathrm{d}u = \int_0^\infty g(v) l_t^v \left((2-m)v\right)^{(2m-2)/(2-m)} \mathrm{d}v \qquad (2.7)
$$

for all Borel–measurable functions $g : [0,\infty) \to [0,\infty)$ Here, the first equality follows from the fact that X, and hence V, are Lebesgue–almost everywhere strictly positive, by (2.2). Now, the continuity properties of $(l^v)_{v\geq 0}$ and (2.7) yield

$$
\begin{aligned}
l_t^0 &= \lim_{\varepsilon \downarrow 0} \frac{1}{\int_0^\varepsilon \left((2-m)v\right)^{(2m-2)/(2-m)} \mathrm{d}v} \int_0^\varepsilon l_t^v \left((2-m)v\right)^{(2m-2)/(2-m)} \mathrm{d}v \\
&= \lim_{\varepsilon \downarrow 0} m \left((2-m)\varepsilon\right)^{m/(m-2)} \int_0^t \mathbf{1}_{\{V_u<\varepsilon\}} \mathrm{d}u \\
&= \lim_{\varepsilon \downarrow 0} m \left((2-m)\varepsilon\right)^{m/(m-2)} \int_0^t \mathbf{1}_{\{X_u<((2-m)\varepsilon)^{2/(2-m)}\}} \mathrm{d}u = \lim_{\varepsilon \downarrow 0} m\varepsilon^{-m/2} \int_0^t \mathbf{1}_{\{X_u<\varepsilon\}} \mathrm{d}u \\
&= \Lambda_t, \qquad t \geq 0,
\end{aligned}
$$

where the last equality follows from the definition of Λ. Then, (2.5) follows from (2.6) and the above equality. □

3 A Mostly Analytic Proof of Theorem 1.2

3.1 Three technical lemmas

Before we embark on proving Theorem 1.2, we shall provide some auxiliary analytic results.

Lemma 3.1. *Assume that $m \in [0,2)$ and recall the function f from (1.3). Then, with*

$$
\psi(x) := \frac{1}{\Gamma\left((n-m)/2\right)} \times \begin{cases} \int_0^\infty \left(1 - \left(w/\left(1+w\right)\right)^{n/2-1}\right) w^{-m/2} \mathrm{e}^{-xw} \mathrm{d}w, & \text{if } n > 2; \\ \int_0^\infty \log\left(1 + 1/w\right) w^{-m/2} \mathrm{e}^{-xw} \mathrm{d}w, & \text{if } n = 2, \end{cases} \qquad (3.1)
$$

for all $x \in (0,\infty)$, it holds that $\psi \in C^\infty((0,\infty))$, that

$$
f(t,x) = f(t,0) - \frac{x^{1-m/2}}{(2t)^{(n-m)/2}} \psi\left(\frac{x}{2t}\right), \qquad t > 0, \quad x > 0, \qquad (3.2)
$$

and that

$$
\lim_{x \downarrow 0} \frac{x^{1-m/2}}{(2t)^{(n-m)/2}} \psi\left(\frac{x}{2t}\right) = 0, \qquad t > 0. \qquad (3.3)
$$

Proof. Let us only consider the case $n > 2$; the case $n = 2$ follows in the same manner. Since

$$f(t, 2tx) = \frac{1}{\Gamma\left((n-m)/2\right)} (2t)^{1-n/2} \int_0^\infty \left(\frac{w}{x+w}\right)^{n/2-1} w^{-m/2} \mathrm{e}^{-w} \mathrm{d}w,$$

for all $t > 0$ and $x > 0$, we have

$$\begin{aligned}
(2t)^{n/2-1} \frac{f(t,0) - f(t,2tx)}{x^{1-m/2}} &= \frac{1}{\Gamma\left((n-m)/2\right)} \int_0^\infty \left(1 - \left(\frac{w}{x+w}\right)^{n/2-1}\right) \left(\frac{w}{x}\right)^{-m/2} e^{-w} \frac{\mathrm{d}w}{x} \\
&= \frac{1}{\Gamma\left((n-m)/2\right)} \int_0^\infty \left(1 - \left(\frac{w}{1+w}\right)^{n/2-1}\right) w^{-m/2} e^{-xw} \mathrm{d}w \\
&= \psi(x).
\end{aligned}$$

Therefore, substituting x for $2tx$, we obtain (3.2). Finally, (3.3) follows from the continuity of f as seen easily in (1.3). ☐

Lemma 3.2. *Assume that* $m \in [0, 2)$*, and recall the function* ψ *from* (3.1)*. Define the function*

$$(0, \infty) \ni x \mapsto p(x) := -x^{1-m/2} \psi'(x).$$

Then, p *is nonnegative and decreasing with* $0 < p(0+) < \infty$*. As a consequence,* $\sup_{x>0} p(x) < \infty$*.*

Proof. We just consider the case $n > 2$; the case $n = 2$ follows in the same manner with the appropriate modifications. To simplify notation we shall consider the function $p_0 := \Gamma((n-m)/2)p$. Simple algebra and a change of variables gives

$$\begin{aligned}
p_0(x) &= \int_0^\infty \frac{1}{x} \left(1 - \left(\frac{w}{1+w}\right)^{n/2-1}\right) (xw)^{1-m/2} e^{-xw} \mathrm{d}(xw) \\
&= \frac{1}{x} \int_0^\infty \left(1 - \left(\frac{v}{x+v}\right)^{n/2-1}\right) v^{1-m/2} e^{-v} \mathrm{d}v \\
&= \frac{1}{x} (L(0) - L(x)), \qquad x > 0,
\end{aligned}$$

where

$$L(x) = \int_0^\infty \left(\frac{v}{x+v}\right)^{n/2-1} v^{1-m/2} e^{-v} \mathrm{d}v, \qquad x > 0.$$

Hence we get

$$p_0(x) = -\int_0^1 L'(tx) \mathrm{d}t = \left(\frac{n}{2} - 1\right) \int_0^\infty \int_0^1 \frac{1}{(tx+v)^{n/2}} v^{(n-m)/2} e^{-v} \mathrm{d}t \mathrm{d}v, \qquad x > 0.$$

Thus, p_0 (and hence p) is nonnegative and decreasing with

$$p_0(0+) = \left(\frac{n}{2} - 1\right) \int_0^\infty v^{-m/2} e^{-v} \mathrm{d}v = \left(\frac{n}{2} - 1\right) \Gamma\left(1 - \frac{m}{2}\right).$$

This concludes the proof. ☐

Lemma 3.3. *Assume that* $m \in (0, 2)$*. Then, we have*

$$\psi(0) = \begin{cases} \frac{\Gamma(m/2)}{(1-m/2)\Gamma(n/2-1)}, & \text{if } n > 2; \\ \frac{\Gamma(m/2)}{1-m/2}, & \text{if } n = 2. \end{cases}$$

Proof. Again, we only treat the case $n > 2$, as the case $n = 2$ can be argued in the same way. Straightforward computations yield

$$\begin{aligned}\int_0^\infty \left(1 - \left(\frac{w}{1+w}\right)^{n/2-1}\right) w^{-m/2}\mathrm{d}w &= \left(\frac{n}{2}-1\right)\int_0^\infty \int_w^\infty \left(\frac{v^{n/2-2}}{(1+v)^{n/2}}\right)\mathrm{d}v w^{-m/2}\mathrm{d}w \\ &= \frac{n/2-1}{1-m/2}\int_0^\infty \frac{v^{(n-m)/2-1}}{(1+v)^{n/2}}\mathrm{d}v \\ &= \frac{n/2-1}{1-m/2}\int_0^1 w^{(n-m)/2-1}(1-w)^{m/2-1}\mathrm{d}w \\ &= \frac{n/2-1}{1-m/2}\frac{\Gamma\left((n-m)/2\right)\Gamma\left(m/2\right)}{\Gamma\left(n/2\right)} \\ &= \frac{1}{1-m/2}\frac{\Gamma\left((n-m)/2\right)\Gamma\left(m/2\right)}{\Gamma\left(n/2-1\right)},\end{aligned}$$

where we used the substitution $w = v/(1+v)$ in the third equality and the identity

$$\int_0^1 w^{a-1}(1-w)^{b-1}\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \qquad a, b > 0$$

in the fourth equality, which connects the Beta and Gamma functions. In the last equality of the long display, we have used the identity $\Gamma(k) = (k-1)\Gamma(k-1)$, which holds for all $k > 1$. □

3.2 Proof of Theorem 1.2

We proceed in five steps.

• Step 1: Using the density provided in (2.1), we obtain

$$f(t,x) = \mathsf{E}[s(x+Y_t)], \qquad t > 0, \quad x \geq 0,$$

where the function f is given in (1.3). Note that the process $f(\cdot, X_\cdot)$ is $\mathcal{F}_\cdot$–optional. Since we have already established the existence of the $\mathcal{F}_\cdot$-optimal projection Z of $s(X+Y)$ in Remark 1.1, it immediately follows that $Z_t = f(t, X_t)$ holds for all $t \geq 0$.

• Step 2: Consider first the case $n > 2$, fix some $\kappa > 1$, and recall the stopping times from (2.4). Then $s(X^{\rho_\kappa} + Y^{\rho_\kappa})$ is bounded, hence a martingale under $\mathcal{G}_\cdot$. Its $\mathcal{F}_\cdot$–optional projection, which is Z^{ρ_κ}, will also be a martingale. By Itô's formula and the fact that the derivatives of f are continuous and the product Lebesgue$\otimes$P measure of $\{(t,\omega) : (t, X_t(\omega)) \in U\}$ is strictly positive whenever U is a nonempty open subset of $(0,\infty)^2$ due to the unbounded support of ρ of (1.1), the partial differential equation in (1.7) holds for all $(t,x) \in (0,\infty)^2$.

Let us now consider the case $n = 2$ and fix again some $\kappa > 1$. In this case, Itô's formula yields

$$s(X+Y) = 2\int_0^\cdot \frac{1}{\sqrt{X_u + Y_u}}\mathrm{d}W_u$$

for some Brownian motion W. Hence, $s(X^{\rho_\kappa} + Y^{\rho_\kappa})$ is a martingale under $\mathcal{G}_\cdot$. Now we may conclude as in the case $n > 2$ that the partial differential equation in (1.7) holds.

• Step 3: When $m \geq 2$, then $\lim_{\kappa\uparrow\infty} \rho_\kappa = \infty$ holds for the stopping times of (2.4), thanks to the facts in §2.1. Hence, Z is indeed a local martingale satisfying (1.4) by Itô's formula and (1.7). It is, moreover, a strict local martingale since $s(X+Y)$ is not a martingale under $\mathcal{G}_\cdot$, as noted in (2.3).

• Step 4: We now focus on the case $0 < m < 2$ and argue the finite-variation term appearing in the Doob-Meyer decomposition of Z in (1.5). To make headway, Lemma 3.1 yields

$$Z_t = f(t, X_t) = f(t,0) - \frac{X_t^{1-m/2}}{(2t)^{(n-m)/2}} \psi\left(\frac{X_t}{2t}\right), \qquad t > 0.$$

where the function ψ is given in (3.1). Unfortunately, since $\psi'(0) = -\infty$ by Lemma 3.2, we cannot use the product rule directly. Instead, we shall approximate the function ψ. For $\varepsilon > 0$, define the function $\psi_\varepsilon : [0,\infty) \to \mathbb{R}$ by $\psi_\varepsilon(x) = \psi(x)$ for all $x > \varepsilon$ and by

$$\psi_\varepsilon(x) = \psi(\varepsilon) + \psi'(\varepsilon)(x - \varepsilon), \qquad x \in [0, \epsilon].$$

Since ψ is nonnegative, decreasing and convex, the same properties transfer to ψ_ε; furthermore, $\psi_\varepsilon \leq \psi$.

Next, fix some $t_0 > 0$. We shall first derive the dynamics of Z for $t \geq t_0$ via approximation, and then send t_0 to zero. Given that ψ_ε is convex and continuously differentiable on $[0,\infty)$, twice continuously differentiable except at $\varepsilon > 0$, and $\mathsf{E}\left[\int_0^\infty \mathbf{1}_{\{X_t = 2\varepsilon t\}} \mathrm{d}t\right] = 0$ holds, it follows that $(\psi_\varepsilon(X_t/(2t)))_{t \geq t_0}$ is a semimartingale satisfying

$$\psi_\varepsilon\left(\frac{X_t}{2t}\right) = \psi_\varepsilon\left(\frac{X_{t_0}}{2t_0}\right) + \int_{t_0}^t \mathbf{1}_{\{X_u > 2u\varepsilon\}} \mathrm{d}\psi\left(\frac{X_u}{2u}\right) + \psi'(\epsilon) \int_{t_0}^t \mathbf{1}_{\{X_u \leq 2u\varepsilon\}} \mathrm{d}\left(\frac{X_u}{2u}\right), \qquad t \geq t_0. \tag{3.4}$$

Define now the process

$$Z_t^\varepsilon := f(t,0) - \frac{X_t^{1-m/2}}{(2t)^{(n-m)/2}} \psi_\varepsilon\left(\frac{X_t}{2t}\right), \qquad t > 0. \tag{3.5}$$

An application of (3.2) and integration-by-parts, in conjunction with (3.4) and Tanaka's formula (see (2.5)), and recalling the partial differential equation (1.7) yield

$$\begin{aligned}
Z^\varepsilon - Z^\varepsilon_{t_0} &= 2 \int_{t_0}^{\cdot} f'_x(u, X_u) \sqrt{X_u} \mathbf{1}_{\{X_u > 2u\varepsilon\}} \mathrm{d}B_u^X \\
&\quad + \int_{t_0}^{\cdot} f'_t(u,0) \mathbf{1}_{\{X_u \leq 2u\varepsilon\}} \mathrm{d}u + \frac{n-m}{2} \int_{t_0}^{\cdot} \frac{2X_u^{1-m/2}}{(2u)^{(n-m)/2+1}} \psi_\varepsilon\left(\frac{X_u}{2u}\right) \mathbf{1}_{\{X_u \leq 2u\varepsilon\}} \mathrm{d}u \\
&\quad - (2-m) \int_{t_0}^{\cdot} \frac{\psi_\varepsilon\left(X_u/(2u)\right)}{(2u)^{(n-m)/2}} X_u^{(1-m)/2} \mathbf{1}_{\{0 < X_u \leq 2u\varepsilon\}} \mathrm{d}B_u^X - \left(1 - \frac{m}{2}\right) \psi_\varepsilon(0) \int_{t_0}^{\cdot} \frac{\mathrm{d}\Lambda_u}{(2u)^{(n-m)/2}} \\
&\quad - \psi'(\epsilon) \int_{t_0}^{\cdot} \frac{X_u^{1-m/2}}{(2u)^{(n-m)/2}} \mathbf{1}_{\{X_u \leq 2u\varepsilon\}} \mathrm{d}\left(\frac{X_u}{2u}\right) \\
&\quad - (2-m)\psi'(\epsilon) \int_{t_0}^{\cdot} \frac{2X_u^{1-m/2}}{(2u)^{(n-m)/2+1}} \mathbf{1}_{\{X_u \leq 2u\varepsilon\}} \mathrm{d}u.
\end{aligned}$$

To derive this long display, two cases are considered. Whenever $X_u > 2u\varepsilon$, then Z^ε has the dynamics of a local martingale, provided in the first line of the long display. For the second case, namely when $X_u \leq 2u\varepsilon$, we break up the second term on the right side of (3.5) in three components and apply the Itô product rule. The second line of the long display provides the contribution of the first term in (3.5) and of the component involving the power of t. The third line corresponds to the contribution of the power of X, after using the Tanaka formula in (2.5). The fourth line provides the contribution of $(\psi_\varepsilon(X_t/(2t)))_{t \geq t_0}$, worked out in (3.4). Finally, the last line yields the cross-product dynamics.

We now let ε go to zero. Then Z_t^ε tends to Z_t, for each $t > 0$. Let us next consider the right side of the long display. Using (2.2) and the bound $\psi_\varepsilon \leq \psi$, the dominated convergence theorem yields that the terms in the second line converge to zero. By a similar argument and Itô's isometry, so does the first term of the third line. For the fourth line, we bound the integrand

$$\left|\psi'(\epsilon)\frac{X_u^{1-m/2}}{(2u)^{(n-m)/2}}\mathbf{1}_{\{X_u\leq 2u\varepsilon\}}\right| \leq \frac{-\varepsilon^{1-m/2}\psi'(\varepsilon)}{(2t_0)^{(n-m)/2-1}}\mathbf{1}_{\{X_u\leq 2u\varepsilon\}} \leq \frac{p(0+)}{(2t_0)^{(n-m)/2-1}}\mathbf{1}_{\{X_u\leq 2u\varepsilon\}}$$

in the notation of Lemma 3.2. Hence, the term in the fourth line also converges to zero as ε tends to zero. By exactly the same arguments, so does the term in the last line of the long display.

We are left with two terms. Consider the integral in the first line. Lemma 3.1 yields that

$$f_x'(t,x) = -\left(1-\frac{m}{2}\right)\frac{x^{-m/2}}{(2t)^{(n-m)/2}}\psi\left(\frac{x}{2t}\right) - \frac{x^{1-m/2}}{(2t)^{(n-m)/2}}\psi'\left(\frac{x}{2t}\right), \qquad t>0, \quad x>0,$$

so that $x(f_x'(t,x))^2$ behaves like kx^{1-m}/t^{n-m} when $x\sim 0$, where $k>0$ is an appropriate constant. However, $\int_0^\cdot X_u^{1-m}\mathrm{d}u$ is a finite process, because it is (proportional to) the quadratic variation of the local martingale part in the dynamics of $X^{1-m/2}$. Therefore, it follows that the integral in the first line converges to

$$2\int_{t_0}^\cdot f_x'(u,X_u)\sqrt{X_u}\mathbf{1}_{\{X_u>0\}}\mathrm{d}B_u^X.$$

The only remaining term, namely the second term in the third line, converges to

$$-\left(1-\frac{m}{2}\right)\psi(0)\int_{t_0}^\cdot \frac{1}{(2u)^{(n-m)/2}}\mathrm{d}\Lambda_u.$$

To summarize, we have

$$Z - Z_{t_0} = 2\int_{t_0}^\cdot f_x'(u,X_u)\sqrt{X_u}\mathbf{1}_{\{X_u>0\}}\mathrm{d}B_u^X - \left(1-\frac{m}{2}\right)\psi(0)\int_{t_0}^\cdot \frac{1}{(2u)^{(n-m)/2}}\mathrm{d}\Lambda_u.$$

We can now sent t_0 to zero, noting that none of the integrals will explode because X is away from zero on the stochastic interval $[0,\rho_2]$, where ρ_2 is given as in (2.4) with $\kappa=2$. It then follows that

$$Z = 1 + 2\int_0^\cdot f_x'(u,X_u)\sqrt{X_u}\mathbf{1}_{\{X_u>0\}}\mathrm{d}B_u^X - \left(1-\frac{m}{2}\right)\psi(0)\int_0^\cdot \frac{1}{(2u)^{(n-m)/2}}\mathrm{d}\Lambda_u.$$

In conjunction with Lemma 3.3, this then yields (1.5).

• Step 5: Finally, for the case $m=0$ basic computations with (1.3) yield (1.6). Indeed, if $n>2$ we have

$$f(t,x) = \frac{1}{\Gamma(n/2)}\int_0^\infty (x+2tw)^{1-n/2}w^{n/2-1}\mathrm{e}^{-w}\mathrm{d}w, \qquad t>0, \quad x\geq 0.$$

This gives directly

$$f(t,0) = \frac{(2t)^{1-n/2}}{\Gamma(n/2)}\int_0^\infty w^{1-n/2}w^{n/2-1}\mathrm{e}^{-w}\mathrm{d}w = \frac{(2t)^{1-n/2}}{\Gamma(n/2)}, \qquad t>0.$$

One then concludes by observing that X gets absorbed when hitting zero; hence

$$f(t,X_t) = f(t\wedge\rho, X_{t\wedge\rho}) + \mathbf{1}_{\{\rho<t\}}(f(t,0)-f(\rho,0)).$$

The case $n=2$ is argued again in exactly the same way. □

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