GENERALISED LYAPUNOV FUNCTIONS AND FUNCTIONALLY GENERATED TRADING STRATEGIES

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Abstract. This paper investigates the dependence of functional portfolio generation, introduced by Fernholz (1999), on an extra finite variation process. The framework of Karatzas and Ruf (2017) is used to formulate conditions on trading strategies to be strong arbitrage relative to the market over sufficiently large time horizons. A mollification argument and Komlós theorem yield a general class of potential arbitrage strategies. These theoretical results are complemented by several empirical examples using data from the S&P 500 stocks.

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1. Introduction

E.R. Fernholz established Stochastic Portfolio Theory (SPT) to provide a theoretical tool for applications in equity markets, and for analysing portfolios with controlled behaviour; see Fernholz (1999) and Fernholz and Karatzas (2009), for example. SPT studies so called functionally generated portfolios. The value of a functionally generated portfolio relative to the total market capitalization is merely a function, known as the so called *master formula*, of the market weights. This formula does not involve stochastic integration or drifts, which makes the analysis very easy as the need for estimation is reduced.

One very interesting topic following up this construction is the study of relative arbitrage opportunities between functionally generated portfolios and the market portfolio. Fernholz (2002, 1999, 2001) states conditions for such relative arbitrage to exist over sufficiently large time horizons. To implement this relative arbitrage, trading strategies generated by suitable portfolio generating functions are required. Karatzas and Ruf (2017) interpret portfolio generating functions as Lyapunov functions. More precisely, the supermartingale property of the corresponding wealth processes after an appropriate change of measures is utilised to study the performance of functionally generated trading strategies. Relative arbitrage over arbitrary time horizons under appropriate conditions is also studied by Fernholz et al. (2018).

One offspring of portfolio generating functions is a generalised portfolio generating function, which depends on an additional argument with continuous path and finite variation. This is inspired by the fact that in practice, people tend to take historical data, such as past performance of stocks, or statistical estimates, into consideration when constructing portfolios. Besides, this generalisation provides additional flexibility in choosing portfolio generating functions. Section 3.2 of Fernholz (2002) formulates the concept of time-dependent generating functions, and presents the master formula under this situation. In the same framework, Strong (2014) shows an extension of the master formula to portfolios generated by functions that also depend on the current state of some continuous path process of finite variation. Also based on Fernholz's structure, Schied et al. (2018) provide a pathwise version of the relevant master formula. They also analyze examples where the additional process is chosen to be the moving average of the market weights. In a recent paper, Karatzas and Kim (2018) generalise the methodology developed by Karatzas and Ruf (2017) in a pathwise, probability-free setting. They also generalise portfolio generating functions with path-dependent functionals.

All the above mentioned papers (Fernholz (2002), Strong (2014), Schied et al. (2018), and Karatzas and Kim (2018)) make assumptions on the smoothness of the portfolio generating function with respect to both the finite variation process and the market weights. In this paper, we weaken these assumptions such that the choice for the portfolio generating function is less restricted. To this end, we use a mollification argument and the Komlós theorem. Then we study several examples empirically, using data from the S&P 500 index.[1]

An outline of the paper is as follows. Section 2 specifies the market model and recalls the definitions of trading strategies and relative arbitrage. Section 3 first gives the definitions of regular functions and Lyapunov functions, and then presents sufficient conditions for a function to be regular and Lyapunov, respectively. The appendix presents the proofs of these results. Section 4 defines additive and multiplicative generation, and the corresponding trading strategies and wealth processes. Section 4 also gives conditions for arbitrage relative to the market portfolio to exist. Section 5 describes the data involved and the processing method to implement the empirical analysis. Section 6 contains several examples of portfolio generating functions and discusses empirical results. Section 7 concludes.

[1] As the constituent list of the stocks in the S&P 500 index changes over time, we avoid a survivorship bias by not restricting the analysis to the current stocks in the S&P 500 index. Instead, we reconstruct the historical constituent list of the S&P 500 index and adjust the portfolios appropriately when the constituent list changes.

2. Model setup

We fix a filtered probability space $(\Omega, \mathcal{F}(\infty), \mathcal{F}(\cdot), \mathsf{P})$, where $\mathcal{F}(\cdot)$ is right-continuous and $\mathcal{F}(0) = \{\emptyset, \Omega\}$, and write

$$\Delta^d = \left\{ (x_1, \cdots, x_d)' \in [0,1]^d : \sum_{i=1}^d x_i = 1 \right\} \quad \text{and} \quad \Delta^d_+ = \Delta^d \cap (0,1)^d.$$

We consider an equity market with $d \geq 2$ companies, where each company has always one share of stock outstanding. We denote the market weights process by $\mu(\cdot) = (\mu_1(\cdot), \cdots, \mu_d(\cdot))'$. Here, $\mu_i(\cdot)$ is the market weight process of company i computed by dividing the capitalization of company i by the total capitalization of all d companies in the market, for all $i \in \{1, \cdots, d\}$. We assume that $\mu(\cdot)$ is Δ^d-valued with $\mu(0) \in \Delta^d_+$, and that $\mu_i(\cdot)$ is a continuous, non-negative semimartingale, for all $i \in \{1, \cdots, d\}$.

To define a trading strategy for $\mu(\cdot)$, let us consider a process $\vartheta(\cdot) = (\vartheta_1(\cdot), \cdots, \vartheta_d(\cdot))'$ in $\mathbb{R}^d$, which is predictable and integrable with respect to $\mu(\cdot)$. We denote the collection of all such processes by $\mathcal{L}(\mu)$.

For such a process $\vartheta(\cdot) \in \mathcal{L}(\mu)$, we interpret $\vartheta_i(t)$ as the number of shares in the stock of company i held at time $t \geq 0$, for all $i \in \{1, \cdots, d\}$. Then

$$V^\vartheta(\cdot) = \sum_{i=1}^d \vartheta_i(\cdot)\mu_i(\cdot)$$

can be interpreted as the wealth process corresponding to $\vartheta(\cdot)$.

Definition 1. (Trading strategies). A process $\varphi(\cdot) \in \mathcal{L}(\mu)$ is called a *trading strategy* if

$$V^\varphi(\cdot) - V^\varphi(0) = \int_0^\cdot \sum_{i=1}^d \varphi_i(t) \mathrm{d}\mu_i(t). \qquad \square$$

Remark 2. To convert a predictable process $\vartheta(\cdot) \in \mathcal{L}(\mu)$ into a trading strategy $\varphi(\cdot)$, we adapt the measure of the "defect of self-financibility" of $\vartheta(\cdot)$, introduced in Section 2 in Karatzas and Ruf (2017) and defined as

$$Q^\vartheta(\cdot) = V^\vartheta(\cdot) - V^\vartheta(0) - \int_0^\cdot \sum_{i=1}^d \vartheta_i(t) \mathrm{d}\mu_i(t). \tag{2.1}$$

As a result, the process $\varphi(\cdot)$ with components

$$\varphi_i(\cdot) = \vartheta_i(\cdot) - Q^\vartheta(\cdot) + C, \quad i \in \{1, \cdots, d\}, \tag{2.2}$$

where C can be any real constant, is a trading strategy for $\mu(\cdot)$. $\square$

Below, we shall analyze the performance of certain long-only portfolios, that is, portfolios for which the trading strategies $\varphi(\cdot)$ of Definition 1 are nonnegative at any time. Especially, we focus on studying the conditions for the existence of so called relative arbitrage.

Definition 3. (Arbitrage relative to the market). A trading strategy $\varphi(\cdot)$ is said to be *relative arbitrage* with respect to the market over a given time horizon $[0, T]$, for $T \geq 0$, if

$$V^\varphi(\cdot) \geq 0 \quad \text{and} \quad V^\varphi(0) = 1,$$

along with

$$\mathsf{P}\left[V^\varphi(T) \geq 1\right] = 1 \quad \text{and} \quad \mathsf{P}\left[V^\varphi(T) > 1\right] > 0. \tag{2.3}$$

If $\mathsf{P}[V^\varphi(T) > 1] = 1$ holds, we say that the relative arbitrage is strong over $[0, T]$. $\square$

Remark 4. The trading strategy that invests in each asset in proportion to its relative capitalization at all times has constant components 1 throughout $[0,T]$. This strategy is implemented in the so-called market portfolio. Hence, Definition 3 makes sense due to the fact that the wealth process of the market portfolio at any time is given by

$$V^{(1,\cdots,1)}(\cdot)=\sum_{i=1}^{d}\mu_i(\cdot)=1.$$

Then relative arbitrage exists over a given time horizon $[0,T]$ when a non-negative wealth process $V^{\varphi}(\cdot)$ has the same initial wealth as the market portfolio, the probability for $V^{\varphi}(T)$ to be greater than the wealth of the market portfolio is strictly positive, and $V^{\varphi}(T)$ is not lower than the wealth of the market portfolio. □

In the following sections, we study portfolio generating functions that depend on some $\mathbb{R}^m$-valued continuous process of finite variation on $[0,T]$, for $T\geq 0$ and some $m\in\mathbb{N}$. We use $\Lambda(\cdot)$ to denote such a process. This process allows for more flexibility in selecting portfolio generating functions. To this end, let $\mathcal{W}$ be some open subset of $\mathbb{R}^m\times\mathbb{R}^d$ such that

$$\mathsf{P}[(\Lambda(t),\mu(t))\in\mathcal{W},\ \forall\, t\geq 0]=1. \tag{2.4}$$

The following notations are introduced for the ranked market weights, which are studied in Theorem 11 and Example 12. For a vector $x=(x_1,\cdots,x_d)'\in\Delta^d$, denote its corresponding ranked vector as $\boldsymbol{x}=(x_{(1)},\cdots,x_{(d)})'$, where

$$\max_{i\in\{1,\cdots,d\}}x_i=x_{(1)}\geq x_{(2)}\geq\cdots\geq x_{(d-1)}\geq x_{(d)}=\min_{i\in\{1,\cdots,d\}}x_i$$

are the components of x in descending order. Denote

$$\mathbb{W}^d=\left\{\left(x_{(1)},\cdots,x_{(d)}\right)'\in\Delta^d:1\geq x_{(1)}\geq x_{(2)}\geq\cdots\geq x_{(d-1)}\geq x_{(d)}\geq 0\right\};$$

then the rank operator $\mathfrak{R}:\Delta^d\to\mathbb{W}^d$ is a mapping such that $\mathfrak{R}(x)=\boldsymbol{x}$. Moreover, denote $\mathbb{W}^d_+=\mathbb{W}^d\cap(0,1)^d$.

The ranked market weights process $\boldsymbol{\mu}(\cdot)$ is given by

$$\boldsymbol{\mu}(\cdot)=\mathfrak{R}(\mu(\cdot))=(\mu_{(1)}(\cdot),\cdots,\mu_{(d)}(\cdot))', \tag{2.5}$$

which is itself a continuous, Δ^d-valued semimartingale whenever $\mu(\cdot)$ is a continuous, Δ^d-valued semimartingale (see Theorem 2.2 in Banner and Ghomrasni (2008)). At last, let $\boldsymbol{W}$ be some open subset of $\mathbb{R}^m\times\mathbb{R}^d$ such that

$$\mathsf{P}[(\Lambda(t),\boldsymbol{\mu}(t))\in\boldsymbol{W},\ \forall\, t\geq 0]=1. \tag{2.6}$$

To conclude this section, we introduce several notions that will be used in the following. For a continuous function F, write $F\in C^\infty$ if F is infinitely differentiable. If $F=F(\lambda,x)$, write $F\in C^{0,1}$ if F is differentiable with respect to the second argument and $\partial F/\partial x$ is jointly continuous; write $F\in C^{1,2}$ if F is once differentiable with respect to the first argument, twice differentiable with respect to the second arguments, and $\partial F/\partial\lambda$ and $\partial^2F/\partial x^2$ are both jointly continuous. In addition, write $\|z\|_2=\left(\sum_{i=1}^n z_i^2\right)^{1/2}$ to denote the L^2 norm of $z=(z_1,\cdots,z_n)'\in\mathbb{R}^n$.

3. Generalised regular and Lyapunov functions

In this section, we consider two classes of portfolio generating functions, regular and Lyapunov functions, which are introduced in Karatzas and Ruf (2017). We generalise these notions here to allow for the additional process $\Lambda(\cdot)$. To this end, recall the open set $\mathcal{W}$, in which $(\Lambda(\cdot),\mu(\cdot))$ take values, from (2.4).

Definition 5. (Generalised regular function). A continuous function $G:\mathcal{W}\to\mathbb{R}$ is said to be *generalised regular* for $\Lambda(\cdot)$ and $\mu(\cdot)$ if

(1) there exists a measurable function $DG = (D_1G, \cdots, D_dG)' : \mathcal{W} \to \mathbb{R}^d$ such that the process $\boldsymbol{\vartheta}(\cdot) = (\boldsymbol{\vartheta}_1(\cdot), \cdots, \boldsymbol{\vartheta}_d(\cdot))'$ with components

$$\boldsymbol{\vartheta}_i(\cdot) = D_iG(\Lambda(\cdot), \mu(\cdot)), \quad i \in \{1, \cdots, d\}, \tag{3.1}$$

is in $\mathcal{L}(\mu)$; and

(2) the continuous, adapted process

$$\Gamma^G(\cdot) = G(\Lambda(0), \mu(0)) - G(\Lambda(\cdot), \mu(\cdot)) + \int_0^{\cdot} \sum_{i=1}^{d} \boldsymbol{\vartheta}_i(t) \mathrm{d}\mu_i(t) \tag{3.2}$$

is of finite variation on the interval $[0, T]$, for all $T \geq 0$.

□

Definition 6. (Generalised Lyapunov function). A generalised regular function $G : \mathcal{W} \to \mathbb{R}$ is said to be a *generalised Lyapunov function* for $\Lambda(\cdot)$ and $\mu(\cdot)$ if, for some function DG as in Definition 5, the finite variation process $\Gamma^G(\cdot)$ of (3.2) is non-decreasing. □

A Lyapunov function turns the semimartingale μ together with the finite variation process Λ into a supermartingale under a related measure; also see Remark 3.4 in Karatzas and Ruf (2017) for further explanations. In the following, we shall omit the terminology "generalised" for simplicity.

In the next example, we discuss sufficient conditions for a smooth function to be regular or Lyapunov.

Example 7. Consider a $C^{1,2}$ function $G : \mathcal{W} \to \mathbb{R}$. Setting $\boldsymbol{\vartheta}_i(\cdot) = \partial G / \partial x_i(\Lambda(\cdot), \mu(\cdot))$ and applying Itô's formula yield that G is regular for $\Lambda(\cdot)$ and $\mu(\cdot)$. Indeed, we get the finite variation process

$$\Gamma^G(\cdot) = -\int_0^{\cdot} \sum_{v=1}^{m} \frac{\partial G}{\partial \lambda_v}(\Lambda(t), \mu(t)) \mathrm{d}\Lambda_v(t) - \frac{1}{2} \sum_{i,j=1}^{d} \int_0^{\cdot} \frac{\partial^2 G}{\partial x_i \partial x_j}(\Lambda(t), \mu(t)) \mathrm{d}\left[\mu_i, \mu_j\right](t). \tag{3.3}$$

Moreover, if the process $\Gamma^G(\cdot)$ is non-decreasing, then G is not only a regular function, but also a Lyapunov function for $\Lambda(\cdot)$ and $\mu(\cdot)$. For instance, this holds if G is non-decreasing in every dimension with respect to the first argument and $\Lambda(\cdot)$ is decreasing in every dimension, and G is concave with respect to the second argument. □

Below we give sufficient conditions for a function G to be regular (Lyapunov). To this end, recall the open set $\mathcal{W}$ from (2.4).

Theorem 8. *For a continuous function $G : \mathcal{W} \to \mathbb{R}$, consider the following conditions.*

(ai) On any compact set $\overline{\mathcal{V}} \subset \mathcal{W}$, there exists a constant $L = L(\overline{\mathcal{V}}) \geq 0$ such that, for all $(\lambda_1, x), (\lambda_2, x) \in \overline{\mathcal{V}}$,

$$|G(\lambda_1, x) - G(\lambda_2, x)| \leq L \|\lambda_1 - \lambda_2\|_2.$$

(aii) $G(\cdot, x)$ is non-increasing, for fixed x, and $\Lambda(\cdot)$ is non-decreasing in every dimension.

(bi) G is differentiable in the second argument and $\partial G / \partial x$ is jointly continuous. Moreover, on any compact set $\overline{\mathcal{V}} \subset \mathcal{W}$, there exists a constant $L = L(\overline{\mathcal{V}}) \geq 0$ such that, for all $(\lambda, x_1), (\lambda, x_2) \in \overline{\mathcal{V}}$,

$$\left\| \frac{\partial G}{\partial x}(\lambda, x_1) - \frac{\partial G}{\partial x}(\lambda, x_2) \right\|_2 \leq L \|x_1 - x_2\|_2.$$

(bii) $G(\lambda, \cdot)$ is concave, for fixed λ.

If one of the conditions (ai) or (aii) holds and one of the conditions (bi) or (bii) holds, G is a regular function for $\Lambda(\cdot)$ and $\mu(\cdot)$. Moreover, in the case that (aii) and (bii) hold, G is Lyapunov.

The proof of Theorem 8 is given in the appendix. A generalised version of Itô's formula studied in Krylov (2009) is related but can only be applied in a Markovian setting.

Theorem 8 can be applied to functions not in $C^{1,2}$, such as in Example 12. Another choice of a non-$C^{1,2}$ function G is the Gini function; see Example 6.1 in Karatzas and Ruf (2017) for details.

Remark 9. Consider the special case where $\Lambda(\cdot)$ is set to be a constant λ. Then Theorem 8 generalises Theorem 3.7(i) and (ii) in Karatzas and Ruf (2017). If $\Lambda(\cdot)$ is non-constant, in contrast to Theorem 3.7 in Karatzas and Ruf (2017), even if G can be extended to a continuous function concave in the second argument, G may not be Lyapunov. A counterexample is given in Example 10. Therefore, for the generalised case, Theorem 3.7 in Karatzas and Ruf (2017) cannot be applied, and instead we have to use modified conditions such as given by Theorem 8. □

Example 10. Assume that $\mu(\cdot) \in \Delta^d$ with $[\mu_1, \mu_1](t) > 0$, for all $t > 0$, and that

$$\Lambda(\cdot) = \gamma \sum_{i=1}^{d} [\mu_i, \mu_i](\cdot),$$

where γ is a constant.

Define the concave quadratic function

$$G(\lambda, x) = \lambda - \sum_{i=1}^{d} x_i^2, \quad \lambda \in \mathbb{R},\ x \in \Delta^d.$$

Then from (3.3) we have

$$\Gamma^G(\cdot) = -\int_0^{\cdot} \mathrm{d}\Lambda(t) + \sum_{i=1}^{d} \int_0^{\cdot} \mathrm{d}\,[\mu_i, \mu_i](t) = (1-\gamma) \sum_{i=1}^{d} [\mu_i, \mu_i](\cdot).$$

Observe that $\Gamma^G(\cdot)$ is decreasing for $\gamma > 1$; hence G is not a Lyapunov function for $\Lambda(\cdot)$ and $\mu(\cdot)$, although it is concave in its second argument.

Define now $\overline{G}(\lambda, x) = -G(\lambda, x)$. Then we have $\Gamma^{\overline{G}}(\cdot) = -\Gamma^G(\cdot)$. Therefore, if $\gamma > 1$ holds, $\Gamma^{\overline{G}}(\cdot)$ is increasing; hence $\overline{G}$ is Lyapunov although convex in its second argument. □

Recall the ranked market weights process $\boldsymbol{\mu}(\cdot)$ defined in (2.5) and the open set $\boldsymbol{W}$ from (2.6).

Theorem 11. *If a function $\boldsymbol{G} : \boldsymbol{W} \to \mathbb{R}$ is regular for $\Lambda(\cdot)$ and $\boldsymbol{\mu}(\cdot) = \mathfrak{R}(\mu(\cdot))$, then the composition $G = \boldsymbol{G} \circ \mathfrak{R}$ is regular for $\Lambda(\cdot)$ and $\mu(\cdot)$.*

To prove Theorem 2, we can apply the same techniques used in the proof of Theorem 3.8 in Karatzas and Ruf (2017), but now with the generalised form of the function $\boldsymbol{G}$; see the appendix for details.

The following example concerns a function G which is not in $C^{1,2}$.

Example 12. Assume that $\mu(\cdot) \in \Delta^d_+$ and consider the $C^{1,2}$ function

$$\boldsymbol{G}(\lambda, \boldsymbol{x}) = -\lambda \sum_{l=1}^{d_1} x_{(l)} \log x_{(l)} + 1 - \sum_{l=d_1+1}^{d_2} x_{(l)}^2, \quad \lambda \in \mathbb{R},\ \boldsymbol{x} \in \mathbb{W}^d_+,$$

where d_1 and d_2 are positive integers with $d_1 < d_2 \leq d$. According to Example 7, $\boldsymbol{G}$ is regular for $\Lambda(\cdot)$ and $\boldsymbol{\mu}(\cdot)$. In particular, the corresponding measurable function $D\boldsymbol{G}$ as in Definition 5 can be chosen with components

$$D_l \boldsymbol{G}(\lambda, \boldsymbol{x}) = \begin{cases} -\lambda \log x_{(l)} - \lambda, & \text{if } l \in \{1, \cdots, d_1\} \\ -2x_{(l)}, & \text{if } l \in \{d_1+1, \cdots, d_2\}. \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

In this case, Itô's lemma yields

$$\begin{aligned}\boldsymbol{G}(\Lambda(\cdot),\boldsymbol{\mu}(\cdot)) =&\boldsymbol{G}(\Lambda(0),\boldsymbol{\mu}(0)) + \int_0^{\cdot}\sum_{l=1}^{d} D_l\boldsymbol{G}(\Lambda(t),\boldsymbol{\mu}(t))\mathrm{d}\mu_{(l)}(t) - \Gamma^{\boldsymbol{G}}(\cdot)\\ &- \int_0^{\cdot}\sum_{l=1}^{d_1}\mu_{(l)}(t)\log\mu_{(l)}(t)\mathrm{d}\Lambda(t)\end{aligned} \tag{3.5}$$

with $D_l\boldsymbol{G}$ given in (3.4) and

$$\Gamma^{\boldsymbol{G}}(\cdot) = \frac{1}{2}\int_0^{\cdot}\sum_{l=1}^{d_1}\frac{\Lambda(t)}{\mu_{(l)}(t)}\mathrm{d}\left[\mu_{(l)},\mu_{(l)}\right](t) + \int_0^{\cdot}\sum_{l=d_1+1}^{d_2}\mathrm{d}\left[\mu_{(l)},\mu_{(l)}\right](t). \tag{3.6}$$

Denote the number of components of $x = (x_1,\cdots,x_d)' \in \Delta^d$ that coalesce at a given rank $l \in \{1,\cdots,d\}$ by

$$N_l(x) = \sum_{i=1}^{d}\mathbf{1}_{x_i = x_{(l)}}.$$

Then by Theorem 2.3 in Banner and Ghomrasni (2008), the ranked market weights process $\boldsymbol{\mu}(\cdot)$ has components

$$\begin{aligned}\mu_{(l)}(\cdot) =&\mu_{(l)}(0) + \int_0^{\cdot}\sum_{i=1}^{d}\frac{\mathbf{1}_{\{\mu_i(t)=\mu_{(l)}(t)\}}}{N_l(\mu(t))}\mathrm{d}\mu_i(t) + \sum_{k=l+1}^{d}\int_0^{\cdot}\frac{\mathrm{d}\boldsymbol{\Lambda}^{(l,k)}(t)}{N_l(\mu(t))}\\ &-\sum_{k=1}^{l-1}\int_0^{\cdot}\frac{\mathrm{d}\boldsymbol{\Lambda}^{(k,l)}(t)}{N_l(\mu(t))}, \quad l \in \{1,\cdots,d\},\end{aligned} \tag{3.7}$$

where $\boldsymbol{\Lambda}^{(i,j)}(\cdot)$ with $1 \le i < j \le d$ is the local time process (refer to Section 6, Chapter 3 in Karatzas and Shreve (2012) for details) of the continuous semimartingale $\mu_{(i)}(\cdot) - \mu_{(j)}(\cdot) \ge 0$ at the origin.

By Theorem 11, the function

$$G(\lambda,x) = \boldsymbol{G}(\lambda,\mathfrak{R}(x)) = -\lambda\sum_{l=1}^{d_1}\sum_{i=1}^{d}\frac{\mathbf{1}_{x_i=x_{(l)}}}{N_l(x)}x_i\log x_i + 1 - \sum_{l=d_1+1}^{d_2}\sum_{i=1}^{d}\frac{\mathbf{1}_{x_i=x_{(l)}}}{N_l(x)}x_i^2$$

is regular for $\Lambda(\cdot)$ and $\mu(\cdot)$, since $\boldsymbol{G}$ is regular for $\Lambda(\cdot)$ and $\boldsymbol{\mu}(\cdot)$.

Now, let us assume that $\Lambda(\cdot)$ is of the form

$$\Lambda(\cdot) = \overline{\xi}\wedge\left(\underline{\xi}\vee\Lambda'(\cdot)\right),$$

where $\overline{\xi}$ and $\underline{\xi}$ are two positive constants with $\underline{\xi} < \overline{\xi}$, and the process $\Lambda'(\cdot)$ is of finite variation. Let $\mathcal{G}(\lambda',x) = G(\overline{\xi}\wedge(\underline{\xi}\vee\lambda'),x)$, for all $\lambda' \in \mathbb{R}$ and $x \in \Delta_+^d$. Then with $D_l\boldsymbol{G}$ and $\Gamma^{\boldsymbol{G}}(\cdot)$ given in (3.4) and (3.6), respectively, inserting (3.7) into (3.5) yields

$$\mathcal{G}(\Lambda'(\cdot),\mu(\cdot)) = \mathcal{G}(\Lambda'(0),\mu(0)) + \int_0^{\cdot}\sum_{i=1}^{d}D_i\mathcal{G}(\Lambda'(t),\mu(t))\mathrm{d}\mu_i(t) - \Gamma^{\mathcal{G}}(\cdot),$$

where

$$D_i\mathcal{G}(\lambda',x) = \sum_{l=1}^{d}\frac{\mathbf{1}_{x_i=x_{(l)}}}{N_l(x)}D_l\boldsymbol{G}(\overline{\xi}\wedge(\underline{\xi}\vee\lambda'),\mathfrak{R}(x)), \quad i \in \{1,\cdots,d\},$$

and

$$\begin{aligned}\Gamma^{\mathcal{G}}(\cdot) =&\Gamma^{\boldsymbol{G}}(\cdot) + \int_0^{\cdot} \sum_{l=1}^{d_1} \mu_{(l)}(t) \log \mu_{(l)}(t) \mathbf{1}_{\{\underline{\xi} \le \Lambda'(t) \le \bar{\xi}\}} \mathrm{d}\Lambda'(t) \\ &- \sum_{l=1}^{d-1} \sum_{k=l+1}^{d} \int_0^{\cdot} \frac{D_l \boldsymbol{G}(\Lambda(t), \mathfrak{R}(\mu(t)))}{N_l(\mu(t))} \mathrm{d}\boldsymbol{\Lambda}^{(l,k)}(t) \\ &+ \sum_{l=2}^{d} \sum_{k=1}^{l-1} \int_0^{\cdot} \frac{D_l \boldsymbol{G}(\Lambda(t), \mathfrak{R}(\mu(t)))}{N_l(\mu(t))} \mathrm{d}\boldsymbol{\Lambda}^{(k,l)}(t).\end{aligned}$$

Observe that $\mathcal{G}$ is regular for $\Lambda'(\cdot)$ and $\mu(\cdot)$, yet it is not in $C^{1,2}$. □

4. Functional generation and relative arbitrage

In Karatzas and Ruf (2017), two types of functional generation, additive and multiplicative generation, are constructed to study the properties of relative values of functionally generated portfolios. In this section, we first discuss the generalised versions of these functional generations and corresponding properties. Then we consider sufficient conditions for strong arbitrage relative to the market to exist.

4.1. Additive generation.

Recall the open set $\mathcal{W}$ from (2.4).

Definition 13. (Additive generation). For a function $G : \mathcal{W} \to \mathbb{R}$, regular for $\Lambda(\cdot)$ and $\mu(\cdot)$, and the process $\boldsymbol{\vartheta}(\cdot)$ given in (3.1), the trading strategy $\boldsymbol{\varphi}(\cdot)$ with components

$$\boldsymbol{\varphi}_i(\cdot) = \boldsymbol{\vartheta}_i(\cdot) - Q^{\boldsymbol{\vartheta}}(\cdot) + C, \quad i \in \{1, \cdots, d\},$$

in the manner of (2.2) and (2.1), and with the real constant

$$C = G(\Lambda(0), \mu(0)) - \sum_{j=1}^{d} \mu_j(0) D_j G(\Lambda(0), \mu(0)), \tag{4.1}$$

is said to be *additively generated* by the regular function G. □

Proposition 14. *The trading strategy $\boldsymbol{\varphi}(\cdot)$, generated additively by a regular function $G : \mathcal{W} \to \mathbb{R}$, has components*

$$\boldsymbol{\varphi}_i(\cdot) =D_i G(\Lambda(\cdot), \mu(\cdot)) + \Gamma^G(\cdot) + G(\Lambda(\cdot), \mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G(\Lambda(\cdot), \mu(\cdot)), \tag{4.2}$$

for all $i \in \{1, \cdots, d\}$. Moreover, the wealth process of $\boldsymbol{\varphi}(\cdot)$ is given by

$$V^{\boldsymbol{\varphi}}(\cdot) = G(\Lambda(\cdot), \mu(\cdot)) + \Gamma^G(\cdot). \tag{4.3}$$

Proof. The proposition is proven exactly like Proposition 4.3 in Karatzas and Ruf (2017). □

4.2. Multiplicative generation.

Definition 15. (Multiplicative generation). For a function $G : \mathcal{W} \to (0, \infty)$, regular for $\Lambda(\cdot)$ and $\mu(\cdot)$, let the process $\boldsymbol{\vartheta}(\cdot)$ be given in (3.1) and assume that $1/G(\Lambda(\cdot), \mu(\cdot))$ is locally bounded. Consider the process $\widetilde{\boldsymbol{\vartheta}}(\cdot) \in \mathcal{L}(\mu)$ with components

$$\widetilde{\boldsymbol{\vartheta}}_i(\cdot) = \boldsymbol{\vartheta}_i(\cdot) \times \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G(\Lambda(t), \mu(t))} \right), \quad i \in \{1, \cdots, d\}. \tag{4.4}$$

Then the trading strategy $\boldsymbol{\psi}(\cdot)$ with components

$$\boldsymbol{\psi}_i(\cdot) = \widetilde{\boldsymbol{\vartheta}}_i(\cdot) - Q^{\widetilde{\boldsymbol{\vartheta}}}(\cdot) + C, \quad i \in \{1, \cdots, d\},$$

in the manner of (2.2) and (2.1), and with C given in (4.1), is said to be *multiplicatively generated* by the regular function G. □

Proposition 16. *The trading strategy $\boldsymbol{\psi}(\cdot)$, generated multiplicatively by a regular function $G : \mathcal{W} \to (0, \infty)$ with $1/G(\Lambda(\cdot), \mu(\cdot))$ locally bounded, has components*

$$\boldsymbol{\psi}_i(\cdot) = V^{\boldsymbol{\psi}}(\cdot) \left(1 + \frac{D_i G(\Lambda(\cdot), \mu(\cdot)) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G(\Lambda(\cdot), \mu(\cdot))}{G(\Lambda(\cdot), \mu(\cdot))} \right), \tag{4.5}$$

for all $i \in \{1, \cdots, d\}$, where the wealth process of $\boldsymbol{\psi}(\cdot)$ is given by

$$V^{\boldsymbol{\psi}}(\cdot) = G(\Lambda(\cdot), \mu(\cdot)) \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G(\Lambda(t), \mu(t))} \right) > 0. \tag{4.6}$$

Proof. The same argument as in Proposition 4.8 in Karatzas and Ruf (2017) applies. □

4.3. **Sufficient conditions for arbitrage relative to the market.** In Karatzas and Ruf (2017), Theorems 5.1 and 5.2 give sufficient conditions for strong arbitrage relative to the market to exist for both additively and multiplicatively generated portfolios, respectively. These results still hold for a regular / Lyapunov function $G : \mathcal{W} \to [0, \infty)$ under specific conditions.

To be consistent with the conditions of arbitrage relative to the market in (2.3), we normalise $G(\Lambda(0), \mu(0)) = 1$ such that both of the wealth processes in (4.3) and (4.6) have initial values 1. This normalisation is guaranteed by replacing G with $G+1$ when $G(\Lambda(0), \mu(0)) = 0$, or with $G/G(\Lambda(0), \mu(0))$ when $G(\Lambda(0), \mu(0)) > 0$.

Theorem 17. *Fix a function $G : \mathcal{W} \to [0, \infty)$, Lyapunov for $\Lambda(\cdot)$ and $\mu(\cdot)$, with $G(\Lambda(0), \mu(0)) = 1$. For some real number $T_* > 0$, suppose that*

$$\mathsf{P}\left[\Gamma^G(T_*) > 1\right] = 1.$$

Then the additively generated trading strategy $\boldsymbol{\varphi}(\cdot)$ of Definition 13 is strong arbitrage relative to the market over every time horizon $[0, T]$ with $T \geq T_*$.

Proof. Use the same reasoning as in the proof of Theorem 5.1 in Karatzas and Ruf (2017). □

Theorem 18. *Assume that $|\Lambda(\cdot)|$ is uniformly bounded. Fix a function $G : \mathcal{W} \to [0, \infty)$, regular for $\Lambda(\cdot)$ and $\mu(\cdot)$, with $G(\Lambda(0), \mu(0)) = 1$. For some real numbers $T_* > 0$, suppose that we can find an $\varepsilon = \varepsilon(T_*) > 0$ such that*

$$\mathsf{P}\left[\Gamma^G(T_*) > 1 + \varepsilon\right] = 1.$$

Then there exists a constant $c = c(T_*, \varepsilon) > 0$ such that the trading strategy $\boldsymbol{\psi}^{(c)}(\cdot)$, generated multiplicatively by the regular function

$$G^{(c)} = \frac{G + c}{1 + c}$$

as in Definition 15, is strong arbitrage relative to the market over the time horizon $[0, T_*]$. Moreover, if G is a Lyapunov function for $\Lambda(\cdot)$ and $\mu(\cdot)$, then $\boldsymbol{\psi}^{(c)}(\cdot)$ is also a strong relative arbitrage over every time horizon $[0, T]$ with $T \geq T_*$.

Proof. See the proof of Theorem 5.2 in Karatzas and Ruf (2017). Note that $G(\Lambda(\cdot), \mu(\cdot))$ is uniformly bounded thanks to the assumptions. □

5. Data source and processing

We start this section by describing the data used in the next section, where several trading strategies are implemented. Then we discuss the method to process the data.

5.1. **Data source and description.** We shall consider a market consisting of all stocks in the S&P 500 index. We are interested in the beginning of day and the end of day market weights of each of these stocks. To calculate these market weights accurately (according to the method in Subsection 5.2), we make use of two time series: the daily market values (market capitalizations, which exclude all the dividend payments) and the daily return indexes (used to consider the effect of reinvestment of dividend payments) of the corresponding component stocks in the S&P 500 index. Both of these time series are available at the end of each trading day.

The data of the market values and return indexes is downloaded from DataStream[2]. The first day, for which the data is available on DataStream, is September 29th, 1989. Since then there are in total 1140 constituents that have belonged to the S&P 500 index. A list of stocks in the S&P 500 index is also attainable on DataStream. In particular, for each month, we derive the list of constituents of the index at the last day of this month. For a constituent delisted from the index in that month, we keep it in our portfolio provided that the constituent still remains in the market till the end of that month. However, we get rid of it from our portfolio on the same day when the constituent does no longer exist in the market, usually due to mergers and acquisitions, bankruptcies, etc. For a constituent newly added to the index in that month, we put it into our portfolio from the first day of the following month.

5.2. **Data processing.** Theoretically, trading strategies vary continuously in time, while in the empirical analysis a daily trading frequency is used. The following procedure illustrates how we examine the gains and losses in our portfolio relative to the market portfolio.

We discretise the time horizon as $0 = t_0 < t_1 < \cdots < t_{N-1} = T$, where N is the total number of trading days.

- The transaction on day t_l, for all $l \in \{1, \cdots, N-1\}$, is made at the beginning of day $(\underline{t}_l)$, taking the beginning of day t_l market weights $\mu(\underline{t}_l)$ as inputs. These market weights $\mu(\underline{t}_l)$ are computed by

$$\mu_i(\underline{t}_l) = \frac{\mathrm{MV}_i(\underline{t}_l)}{\Sigma(\underline{t}_l)}, \quad i \in \{1, \cdots, d\},$$

  where $\mathrm{MV}_i(\underline{t}_l)$ is the market value of stock i at the beginning of day t_l, which is assumed to be equal to the market value attainable at the end of the last trading day t_{l-1}, and $\Sigma(\underline{t}_l) = \sum_{j=1}^{d} \mathrm{MV}_j(\underline{t}_l)$ denotes the total market capitalization at the beginning of day t_l.
- The theoretical (non-self-financing) trading strategy throughout t_l, denoted by $\theta(t_l)$, is computed based on either (3.1) or (4.4), taking $\mu(\underline{t}_l)$ as inputs. Denote the implemented (self-financing) trading strategy corresponding to $\theta(t_l)$ by $\phi(t_l)$. Then $V^{\phi}(\underline{t}_l)$, the beginning of day t_l wealth of the portfolio corresponding to $\phi(t_l)$, is given by

$$V^{\phi}(\underline{t}_1) = \frac{V^{\phi}(\bar{t}_{l-1})\Sigma(\bar{t}_{l-1})}{\Sigma(\underline{t}_1)}. \tag{5.1}$$

  This is based on the assumption that the real portfolio wealth does not change overnight. In (5.1), $V^{\phi}(\bar{t}_{l-1})$ and $\Sigma(\bar{t}_{l-1})$ are the end of day t_{l-1} portfolio wealth and total market capitalization, respectively, computed at $\bar{t}_{l-1}$ (thus already known at $\underline{t}_l$).
- To derive the implemented (self-financing) trading strategy $\phi(t_l)$ corresponding to $\theta(t_l)$, we compute the number

$$C(t_l) = \sum_{j=1}^{d} \theta_j(t_l)\mu_j(\underline{t}_l) - V^{\phi}(\underline{t}_l). \tag{5.2}$$

[2]DataStream, operated by Thomson Reuters, is a financial time series database; see `https://financial.thomsonreuters.com/en/products/data-analytics/economic-data.html`.

Then $\phi(t_l)$ is derived by

$$\phi_i(t_l) = \theta_i(t_l) - C(t_l), \quad i \in \{1, \cdots, d\}. \tag{5.3}$$

This guarantees $V^\phi(\underline{t}_l) = \sum_{i=1}^d \phi_i(t_l)\mu_i(\underline{t}_l)$.

- At the end of day t_l, the return indexes of the stocks for t_l are available, and the total returns $\mathrm{TR}(t_l)$ are computed through dividing the return indexes of t_l with the return indexes of t_{l-1}. Then the end of day t_l implied market values $\mathrm{MV}(\bar{t}_l)$, which take the dividend payments into consideration, are given by

$$\mathrm{MV}_i(\bar{t}_l) = \mathrm{MV}_i(\underline{t}_l)\mathrm{TR}_i(t_l), \quad i \in \{1, \cdots, d\}.$$

  The end of day t_l modified total market capitalization $\Sigma(\bar{t}_l)$ and market weights $\mu(\bar{t}_l)$ are calculated similarly as $\Sigma(\underline{t}_l)$ and $\mu(\underline{t}_l)$, with $\mathrm{MV}(\underline{t}_l)$ replaced by $\mathrm{MV}(\bar{t}_l)$.
- The end of day t_l portfolio wealth is then computed by

$$V^\phi(\bar{t}_l) = \sum_{j=1}^d \phi_j(t_l)\mu_j(\bar{t}_l).$$

  Note that we have

$$V^\phi(\bar{t}_l) = V^\phi(\underline{t}_l) + \sum_{j=1}^d \theta_j(t_l)\left(\mu_j(\bar{t}_l) - \mu_j(\underline{t}_l)\right). \tag{5.4}$$

In particular, at the beginning of day t_0, all of the above steps are still applied, except that we have $V^\phi(\underline{t}_0) = 1$ instead of (5.1) due to Definition 3.

6. Examples and empirical results

In this section, several examples of portfolio generating functions are empirically studied.

Example 19. Define the generalised *entropy function*

$$G(\lambda, x) = \lambda \sum_{i=1}^d x_i \log\left(\frac{1}{x_i}\right), \quad \lambda \in \mathbb{R}_+, \ x \in \Delta_+^d,$$

with values in $(0, \lambda \log d)$, for fixed $\lambda > 0$. Suppose that $\mu(\cdot)$ takes values in Δ_+^d and that $\Lambda(\cdot)$ is $(0, \infty)$-valued.

From (3.3) we have

$$\Gamma^G(\cdot) = \sum_{i=1}^d \int_0^\cdot \mu_i(t)\log\mu_i(t)\mathrm{d}\Lambda(t) + \frac{1}{2}\sum_{i=1}^d \int_0^\cdot \Lambda(t)\frac{\mathrm{d}\left[\mu_i, \mu_i\right](t)}{\mu_i(t)}. \tag{6.1}$$

Then G is a Lyapunov function for $\Lambda(\cdot)$ and $\mu(\cdot)$ provided that $\Gamma^G(\cdot)$ is non-decreasing. One sufficient condition for this to hold is that $\Lambda(\cdot)$ is non-increasing.

From (4.2), the trading strategy $\boldsymbol{\varphi}(\cdot)$, generated additively by G, has components

$$\boldsymbol{\varphi}_i(\cdot) = \Gamma^G(\cdot) - \Lambda(\cdot)\log\mu_i(\cdot), \quad i \in \{1, \cdots, d\}. \tag{6.2}$$

Using (4.3), the corresponding wealth process $V^{\boldsymbol{\varphi}}(\cdot) = G(\Lambda(\cdot), \mu(\cdot)) + \Gamma^G(\cdot)$ is strictly positive if G is Lyapunov for $\Lambda(\cdot)$ and $\mu(\cdot)$.

For the multiplicative generation, G is required to be bounded away from zero. One sufficient condition for this to hold is that $\Lambda(\cdot)$ is bounded away from 0 and the market is diverse on $[0, \infty)$, i.e., there exists $\epsilon > 0$ such that $G(\Lambda(t), \mu(t)) \geq \Lambda(t)\epsilon$, for all $t \geq 0$ (see Proposition 2.3.2 in Fernholz (2002)). Then from (4.5), the trading strategy $\boldsymbol{\psi}(\cdot)$, generated multiplicatively by G, has components

$$\boldsymbol{\psi}_i(\cdot) = -\Lambda(\cdot)\log\mu_i(\cdot)\exp\left(\int_0^\cdot \frac{\mathrm{d}\Gamma^G(t)}{G(\Lambda(t), \mu(t))}\right), \quad i \in \{1, \cdots, d\}.$$

The corresponding wealth process $V^{\boldsymbol{\psi}}(\cdot)$ is given in (4.6).

Now, let us discuss sufficient conditions for the existence of arbitrage relative to the market. To this end, let $\Lambda(\cdot)$ be such that G is Lyapunov for $\Lambda(\cdot)$ and $\mu(\cdot)$, for example, let $\Lambda(\cdot)$ be non-increasing. Next consider

$$\mathcal{G} = \frac{G}{G(\Lambda(0), \mu(0))}, \tag{6.3}$$

together with the non-decreasing process

$$\Gamma^{\mathcal{G}}(\cdot) = \frac{\Gamma^{G}(\cdot)}{G(\Lambda(0), \mu(0))}. \tag{6.4}$$

Then from Theorem 17, if

$$\mathsf{P}\left[\Gamma^{\mathcal{G}}(T_*) > 1\right] = \mathsf{P}\left[\Gamma^{G}(T_*) > G(\Lambda(0), \mu(0))\right] = 1,$$

then the trading strategy $\boldsymbol{\varphi}(\cdot)/G(\Lambda(0), \mu(0))$, generated additively by $\mathcal{G}$, is strong relative arbitrage over every time horizon $[0, T]$ with $T \geq T_*$.

Similarly, from Theorem 18, if

$$\mathsf{P}\left[\Gamma^{\mathcal{G}}(T_*) > 1 + \varepsilon\right] = \mathsf{P}\left[\Gamma^{G}(T_*) > G(\Lambda(0), \mu(0))(1 + \varepsilon)\right] = 1,$$

then the trading strategy $\boldsymbol{\psi}^{(c)}(\cdot)$, generated multiplicatively by

$$G^{(c)} = \frac{G + c}{G(\Lambda(0), \mu(0)) + c}, \tag{6.5}$$

for some sufficiently large $c > 0$, is strong relative arbitrage over every time horizon $[0, T]$ with $T \geq T_*$.

To empirically examine the performance of the portfolio generated by G, we only restrict G to be regular for $\Lambda(\cdot)$ and $\mu(\cdot)$, although G is Lyapunov for some of the choices of $\Lambda(\cdot)$ in the following.

Recall that the wealth processes of portfolios generated either additively or multiplicatively are relative to the S&P 500 index. For a specific day t_n, we estimate

$$[\mu_i, \mu_i](t_n) \approx \sum_{l=1}^{n} \left(\mu_i(\bar{t}_l) - \mu_i(\underline{t}_l)\right)^2, \quad i \in \{1, \cdots, d\},$$

where $\underline{t}_l$ ($\bar{t}_l$) denotes the beginning (end) of the day t_l.

Figure 6.1 presents $\Gamma^{\mathcal{G}}(\cdot)$ given in (6.4) and the relative wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ and $V^{\boldsymbol{\psi}^{(0)}}(\cdot)$ (minus 1 to start from 0 as $\Gamma^{\mathcal{G}}(\cdot)$) of portfolios generated additively and multiplicatively by $\mathcal{G}$, respectively, with finite variation process $\Lambda(\cdot) = 1$. As we can observe from the figure, both $V^{\boldsymbol{\varphi}}(\cdot)$ and $V^{\boldsymbol{\psi}^{(0)}}(\cdot)$ have been continuously outperforming the market portfolio since the year 2000.

Next, we examine the effect that choosing some non-constant $\Lambda(\cdot)$ may have on the portfolio performance. Figures 6.2 and 6.3 display the relative wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ (in logarithmic scale) generated additively corresponding to two different groups of $\Lambda(\cdot)$. The first group of $\Lambda(\cdot)$ is increasing, which results in decreasing $\Gamma^{G}(\cdot)$ given by (6.1); the corresponding G is only regular but not Lyapunov for $\Lambda(\cdot)$ and $\mu(\cdot)$. The second group of $\Lambda(\cdot)$ is decreasing; the corresponding $\Gamma^{G}(\cdot)$ given by (6.1) is increasing and G is Lyapunov for $\Lambda(\cdot)$ and $\mu(\cdot)$.

More precisely, for all $l \in \{1, \cdots, N\}$, in Figure 6.2, the wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ corresponding to $\Lambda(t_l) = \exp(10^{-4} l)$ and $\Lambda(t_l) = \exp(-10^{-4} l)$ are plotted; in Figure 6.3, the wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ corresponding to

$$\Lambda(t_l) = \exp\left(100 \sum_{j=1}^{d} [\mu_j, \mu_j](t_l)\right) \quad \text{and} \quad \Lambda(t_l) = \exp\left(-100 \sum_{j=1}^{d} [\mu_j, \mu_j](t_l)\right)$$

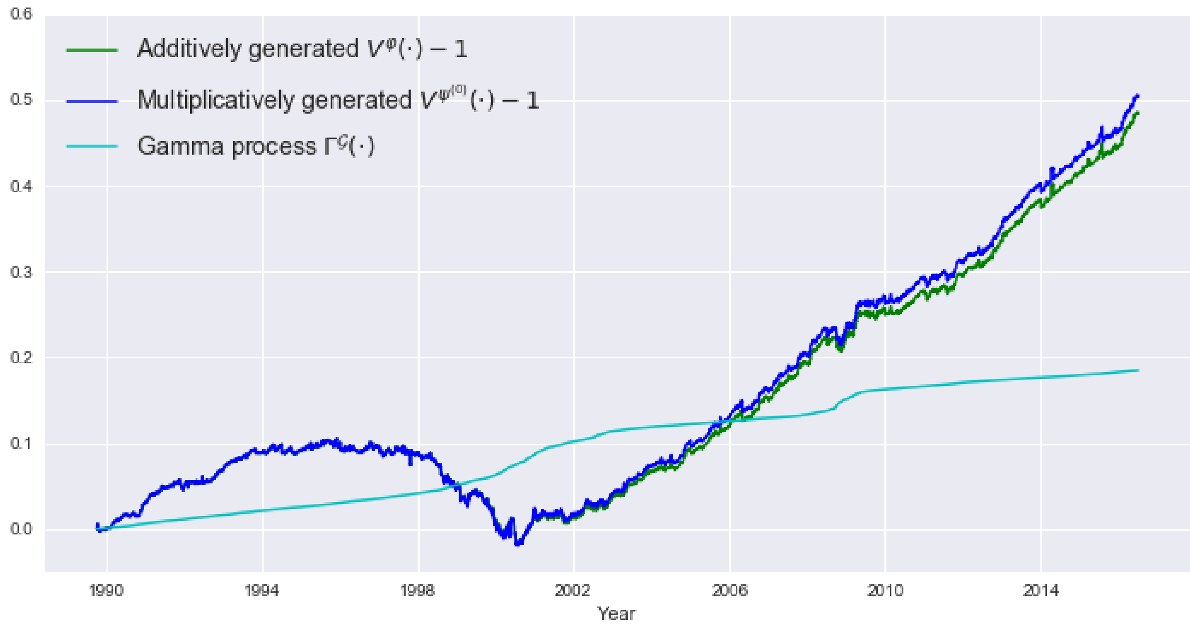

FIGURE 6.1. Gamma process $\Gamma^{\mathcal{G}}(\cdot)$ and relative wealth processes (minus 1) of both the additively and the multiplicatively generated portfolios with constant $\Lambda(\cdot) = 1$.

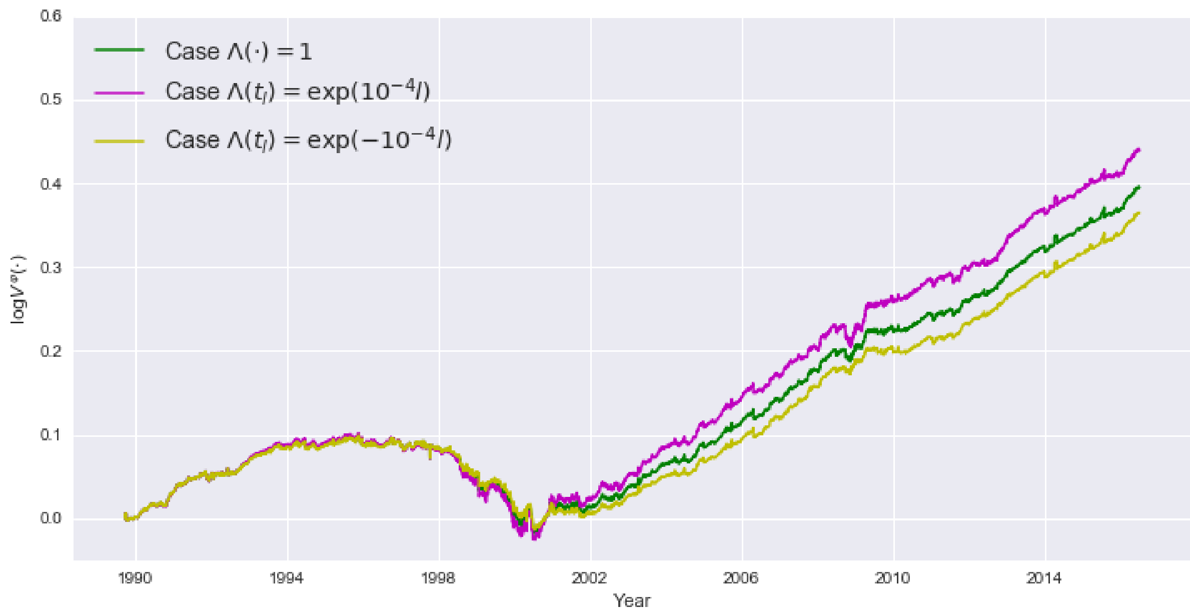

FIGURE 6.2. Relative wealth process $V^{\varphi}(\cdot)$ (in logarithmic scale) of additively generated portfolios with $\Lambda(\cdot)$ a deterministic exponential.

are plotted. The constants 10^{-4} and 100 are chosen such that, with these forms, the daily changes of both $\mathcal{G}(\Lambda(\cdot), \mu(\cdot))$ and $\Gamma^{\mathcal{G}}(\cdot)$ are roughly at the same level of magnitude. Hence, in (4.3), neither part on the right hand side dominates the other.

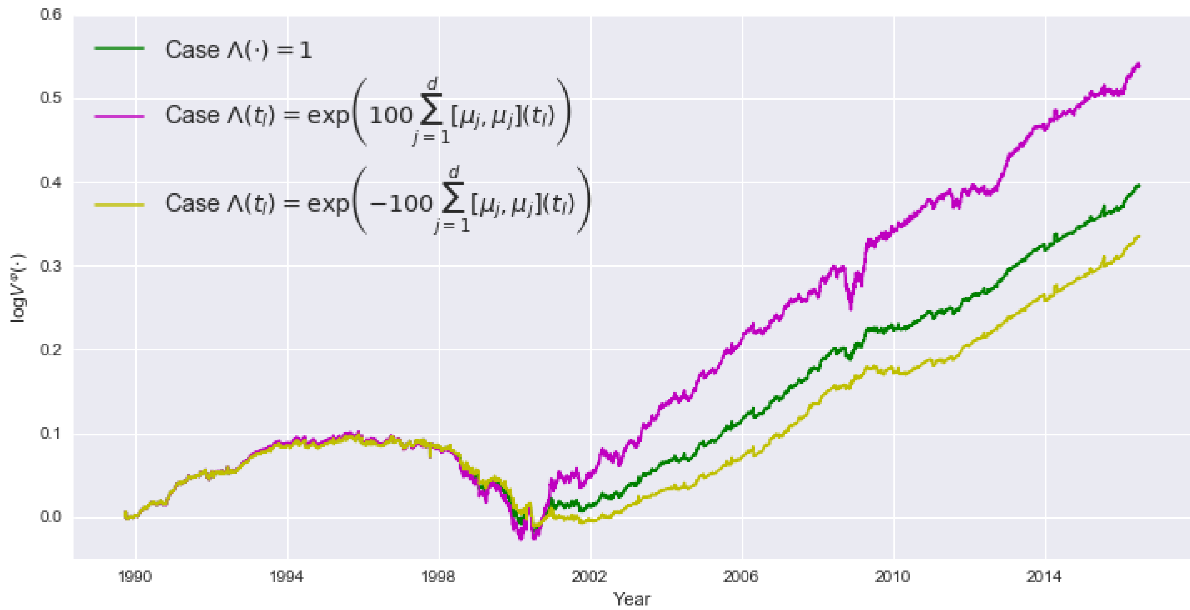

FIGURE 6.3. Relative wealth process $V^{\varphi}(\cdot)$ (in logarithmic scale) of additively generated portfolios with $\Lambda(\cdot)$ an exponential of the quadratic variation of $\mu(\cdot)$.

As we can observe from the figures, choosing $\Lambda(\cdot)$ increasing seems to lead to a better performance than choosing $\Lambda(\cdot)$ constant, which again seems to be better than choosing $\Lambda(\cdot)$ decreasing. We attribute the reason behind this observation to the state of market diversification as follows.

Observe that (5.4) yields

$$V^{\varphi}(\bar{t}_l) = V^{\varphi}(\underline{t}_l) + \frac{1}{G(\Lambda(0), \mu(\underline{0}))}\Lambda(t_l)D(t_l), \quad l \in \{0, \cdots, N\}, \tag{6.6}$$

where $D(t_l)$ is given by

$$D(t_l) = \sum_{j=1}^{d} -\log \mu_j(\underline{t}_l)\left(\mu_j(\bar{t}_l) - \mu_j(\underline{t}_l)\right). \tag{6.7}$$

The value $D(t_l)$ can be considered as an indicator of the direction of changes in market weights from the beginning to the end of date t_l. The value $D(t_l)$ will be positive (negative), if market weights are shifted from companies with large (small) beginning of day market weights to companies with small (large) beginning of day market weights throughout date t_l. We consider a simple example to better understand why this is the case.

Fix $d = 2$ and assume that $\mu_1(\underline{t}_l) > \mu_2(\underline{t}_l)$. Then

$$\begin{aligned} D(t_l) &= -\log \mu_1(\underline{t}_l)\left(\mu_1(\bar{t}_l) - \mu_1(\underline{t}_l)\right) - \log \mu_2(\underline{t}_l)\left(\mu_2(\bar{t}_l) - \mu_2(\underline{t}_l)\right) \\ &= \left(-\log \mu_1(\underline{t}_l) + \log \mu_2(\underline{t}_l)\right)\left(\mu_1(\bar{t}_l) - \mu_1(\underline{t}_l)\right) \end{aligned}$$

holds due to the fact that $(\mu_1(\bar{t}_l) - \mu_1(\underline{t}_l)) = -(\mu_2(\bar{t}_l) - \mu_2(\underline{t}_l))$. Hence, $D(t_l) > 0$ if and only if $\mu_1(\bar{t}_l) < \mu_1(\underline{t}_l)$, i.e., the market weight of the company with larger beginning of day market weight decreases, while the market weight of the company with smaller beginning of day market weight increases.

Hence, a positive $D(\cdot)$ indicates an enhancement in market diversification, while $D(\cdot)$ being negative actually implies a reduction in market diversification. Figure 6.4 plots the cumulative process $E(\cdot) = \sum_{t_l = t_1} D(t_l)$. The process $E(\cdot)$ is increasing (decreasing) whenever $D(\cdot)$ is positive (negative). From Figure 6.4 we can observe that after a slight increase from the year

1991 to the year 1995, $E(\cdot)$ keeps declining till the year 2000. Then $E(\cdot)$ rises up in the long run from the year 2000 until now

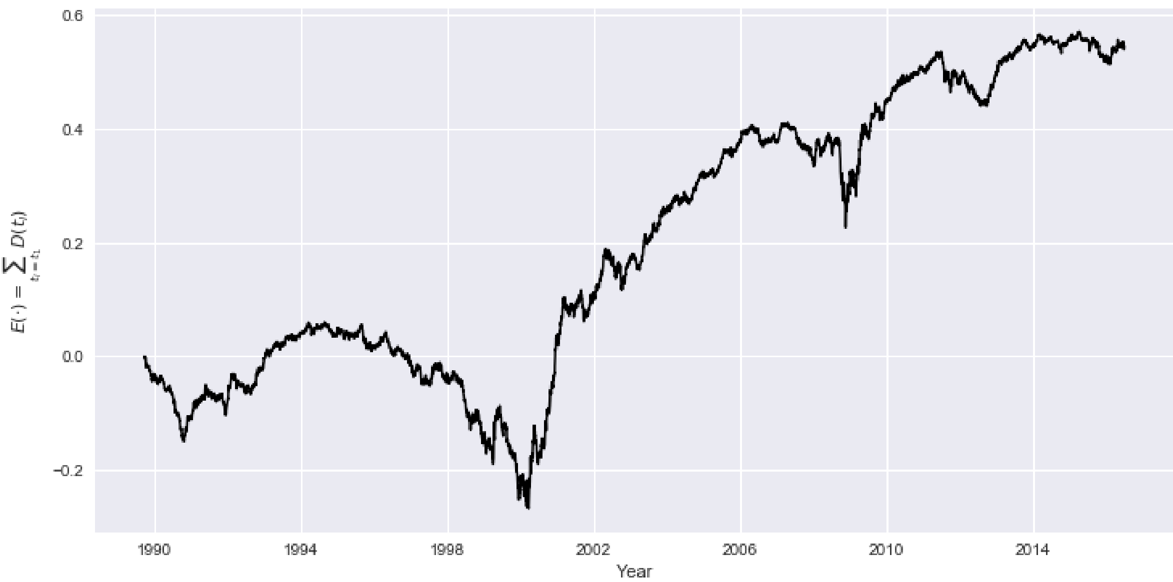

FIGURE 6.4. Integration process $E(\cdot)$ with components given by (6.7).

The behaviour of the process $E(\cdot)$ is in line with another measurement of the market diversification. More precisely, let us consider the process $\sum_{i=1}^{d}(\mu_i \wedge 0.002)(\cdot)$. Note that the value $0.002 = 1/500$, which is roughly the number of constituents in the portfolio. This process is a measure of the market diversification, as it goes up when the market weights of small companies become larger, i.e., the market diversification is strengthened. Figure 6.5 plots the process, which first grows from the year 1991 to the year 1995. Then from the year 1995 to 2000, the process declines fast. This indicates that during this period, the market diversification weakens. On the contrary, the market diversification strengthens afterwards until the year 2008, as the process goes up. Then the level of market diversification remains within a relatively small range.

As a result, according to (6.6), if the market presents a trend of increasing diversification, an increasing positive $\Lambda(\cdot)$ helps to reinforce this effect, and further assists in pulling up $V^{\boldsymbol{\varphi}}(\cdot)$, while a decreasing positive $\Lambda(\cdot)$ is counteractive. On the other hand, if the market presents a trend of decreasing diversification, then a decreasing positive $\Lambda(\cdot)$ helps to slow down the declining speed of $V^{\boldsymbol{\varphi}}(\cdot)$, while an increasing positive $\Lambda(\cdot)$ would make the speed even faster. This is confirmed in Figures 6.2 and 6.3, as from the year 1991 to the year 1995 and from the year 2000 till now, an increasing positive $\Lambda(\cdot)$ makes $V^{\boldsymbol{\varphi}}(\cdot)$ perform better, while from the year 1995 to the year 2000, $V^{\boldsymbol{\varphi}}(\cdot)$ corresponding to a decreasing positive $\Lambda(\cdot)$ is slightly larger.

Although an increasing positive $\Lambda(\cdot)$ has positive effect on the portfolio performance $V^{\boldsymbol{\varphi}}(\cdot)$ whenever the market diversification strengthens, we are not allowed to choose $\Lambda(\cdot)$ arbitrarily fast increasing. The reason is that the portfolio is required to be long-only in our framework, i.e., the trading strategy $\boldsymbol{\varphi}(\cdot)$ given by (6.2) must be nonnegative at any time. If $\Lambda(\cdot)$ is increasing fast enough, $\Gamma^{\mathcal{G}}(\cdot)$ will become negative and decrease fast, which may result in negative $\boldsymbol{\varphi}(\cdot)$ according to (6.2).

As for the multiplicative generation, the different choices of finite variation processes do not change the wealth processes significantly. Indeed, according to (6.1), an increasing $\Lambda(\cdot)$ may slow down the growth rate of $\Gamma(\cdot)$, or even turn $\Gamma(\cdot)$ into a decreasing one. When applying (5.3)

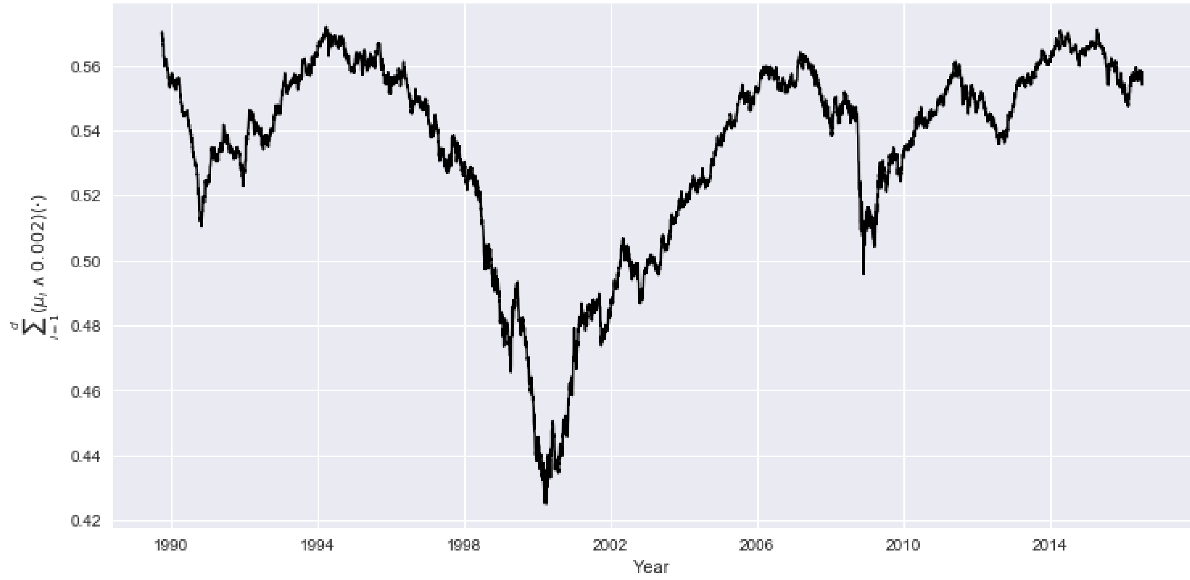

FIGURE 6.5. Process $\sum_{i=1}^{d}(\mu_i \wedge 0.002)(\cdot)$ as a measure of the market diversification degree in the S&P 500 market.

to $\widetilde{\boldsymbol{\vartheta}}(\cdot)$ from (4.4), we have

$$V^{\boldsymbol{\psi}^{(c)}}(\bar{t}_l) = \exp\left(\int_0^{t_l} \frac{\mathrm{d}\Gamma^G(t)}{G(\Lambda(t),\mu(t))+c}\right)\frac{\Lambda(t_l)}{G(\Lambda(0),\mu(\underline{0}))+c}D(t_l) + V^{\boldsymbol{\psi}^{(c)}}(\underline{t}_l),$$

for all $l \in \{0, \cdots, N\}$, with $D(\cdot)$ given in (6.7). In this example, according to the above equation, the positive effect in boosting $V^{\boldsymbol{\psi}^{(c)}}(\cdot)$ contributed by an increasing positive $\Lambda(\cdot)$ is counteracted more or less by the opposite impact the same $\Lambda(\cdot)$ has on the exponential part. A similar analysis also applies to a decreasing positive $\Lambda(\cdot)$. Therefore, under the above mentioned situation (market diversification increases in general), the different choices of a monotone $\Lambda(\cdot)$ do not influence $V^{\boldsymbol{\psi}^{(c)}}(\cdot)$ as much as they do on $V^{\boldsymbol{\varphi}}(\cdot)$.

Note that our process $D(\cdot)$ is related but not the same as the Bregman divergence

$$D_{B,G}\left[\mu(\bar{t}_l)|\mu(\underline{t}_l)\right] = \Lambda(t_l)D(t_l) - \left(G(\Lambda(t_l),\mu(\bar{t}_l)) - G(\Lambda(t_l),\mu(\underline{t}_l))\right),$$

defined in Definition 3.6 in Wong (2017). For its connection to optimal transport, we refer to Wong (2017).

To conclude this example, we compute several empirical indicators corresponding to the performance of above mentioned portfolios over the chosen time horizon. The S&P 500 market portfolio has an averaged yearly return of 9.87% and a Sharpe ratio[3] of 0.37. As for the functionally generated portfolios analyzed in this example, their averaged yearly returns are ranging from 11.12% to 12%, their Sharpe ratios lie between 0.45 and 0.49, and their excess returns with respect to the market portfolio vary from 1.25% to 2.13%. We refer to Banner et al. (2018) for a detailed empirical study to explain these excess returns. □

The following example is motivated by Schied et al. (2018).

[3]To compute the Sharpe ratios of the market portfolio and other functionally generated portfolios, the one-year U.S. Treasury yields are used. The data of these yields can be downloaded from `https://www.federalreserve.gov`.

Example 20. Consider the function

$$G(\lambda, x) = \left(\sum_{i=1}^{d} (\alpha x_i + (1-\alpha)\lambda_i)^p \right)^{\frac{1}{p}}, \quad \lambda \in \mathbb{R}_+^d, \ x \in \Delta_+^d,$$

with constants $\alpha, p \in (0, 1)$. Then G is concave.

For fixed constant $\delta > 0$, define the $\mathbb{R}_+^d$-valued moving average process $\Lambda(\cdot)$ by

$$\Lambda_i(\cdot) = \begin{cases} \frac{1}{\delta} \int_0^{\cdot} \mu_i(t) \mathrm{d}t + \frac{1}{\delta} \int_{\cdot-\delta}^{0} \mu_i(0) \mathrm{d}t & \text{on } [0, \delta) \\ \frac{1}{\delta} \int_{\cdot-\delta}^{\cdot} \mu_i(t) \mathrm{d}t & \text{on } [\delta, \infty) \end{cases},$$

for all $i \in \{1, \cdots, d\}$.

Write $\overline{\mu}(\cdot) = \alpha\mu(\cdot) + (1-\alpha)\Lambda(\cdot)$. Then by (3.3),

$$\begin{aligned} \Gamma^G(\cdot) = & -(1-\alpha) \sum_{i=1}^{d} \int_0^{\cdot} \left(\frac{G(\Lambda(t), \mu(t))}{\overline{\mu}_i(t)} \right)^{1-p} \mathrm{d}\Lambda_i(t) \\ & - \frac{\alpha^2(1-p)}{2} \sum_{i,j=1}^{d} \int_0^{\cdot} \left(\frac{G(\Lambda(t), \mu(t))}{\overline{\mu}_i(t)\overline{\mu}_j(t)} \right)^{1-p} \frac{1}{\sum_{v=1}^{d} (\overline{\mu}_v(t))^p} \mathrm{d}\left[\mu_i, \mu_j\right](t) \\ & + \frac{\alpha^2(1-p)}{2} \sum_{i=1}^{d} \int_0^{\cdot} \left(\frac{G(\Lambda(t), \mu(t))}{\overline{\mu}_i(t)} \right)^{1-p} \frac{1}{\overline{\mu}_i(t)} \mathrm{d}\left[\mu_i, \mu_i\right](t). \end{aligned}$$

Notice that G is not Lyapunov in general.

The trading strategies $\boldsymbol{\varphi}(\cdot)$ and $\boldsymbol{\psi}(\cdot)$, generated additively and multiplicatively by G, respectively, are given by

$$\boldsymbol{\varphi}_i(\cdot) = G(\Lambda(\cdot), \mu(\cdot)) \left(\frac{\alpha\left(\overline{\mu}_i(\cdot)\right)^p}{\overline{\mu}_i(\cdot) \sum_{v=1}^{d} (\overline{\mu}_v(\cdot))^p} - \sum_{j=1}^{d} \frac{\alpha\mu_j(\cdot)\left(\overline{\mu}_j(\cdot)\right)^p}{\overline{\mu}_j(\cdot) \sum_{v=1}^{d} (\overline{\mu}_v(\cdot))^p} + 1 \right) + \Gamma^G(\cdot)$$

and

$$\boldsymbol{\psi}_i(\cdot) = \left(\boldsymbol{\varphi}_i(\cdot) - \Gamma^G(\cdot)\right) \exp\left(\int_0^{\cdot} \frac{\mathrm{d}\Gamma^G(t)}{G(\Lambda(t), \mu(t))} \right),$$

for all $i \in \{1, \cdots, d\}$. The corresponding wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ and $V^{\boldsymbol{\psi}}(\cdot)$ can be derived from (4.3) and (4.6), respectively.

Consider the normalised regular function $\mathcal{G}$ given in (6.3) and the corresponding process $\Gamma^{\mathcal{G}}(\cdot)$ given in (6.4). By Theorem 18, if

$$\mathsf{P}\left[\Gamma^{\mathcal{G}}(T_*) > 1 + \varepsilon\right] = \mathsf{P}\left[\Gamma^G(T_*) > G(\Lambda(0), \mu(0))(1+\varepsilon)\right] = 1,$$

then the trading strategy $\boldsymbol{\psi}^{(c)}(\cdot)$, generated multiplicatively by $G^{(c)}$ given in (6.5) for some sufficiently large $c > 0$, is strong relative arbitrage over the time horizon $[0, T_*]$.

To simulate the relative performance of the portfolio, we use the parameters $\delta = 250$ days and $p = 0.8$. Figure 6.6 shows $\Gamma^{\mathcal{G}}(\cdot)$ and the wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ and $V^{\boldsymbol{\psi}^{(0)}}(\cdot)$ without the effect of the moving average part, i.e., $\alpha = 1$. In this case, $\mathcal{G}$ is Lyapunov. The relative performance of the portfolio is similar to that in Example 19, when the finite variation process is chosen to be constant. Figure 6.7 presents the case when $\alpha = 0.6$. It can be observed that $\Gamma^{\mathcal{G}}(\cdot)$ increases slower when the moving average part is considered. Compared with the case that the moving average part is not included, the wealth processes $V^{\boldsymbol{\varphi}}(\cdot)$ and $V^{\boldsymbol{\psi}^{(0)}}(\cdot)$ also take smaller values in the long run. This is due to the fact that when α decreases, the volatility of $\overline{\mu}(\cdot)$ decreases as well. In this case, we trade slower, and the gains and losses will also be relatively less.

For the four functionally generated portfolios examined in this example, their averaged yearly returns range from 11.21% to 11.47%, their Sharpe ratios lie between 0.45 and 0.47, and their excess returns with respect to the market portfolio vary from 1.34% to 1.6%. □

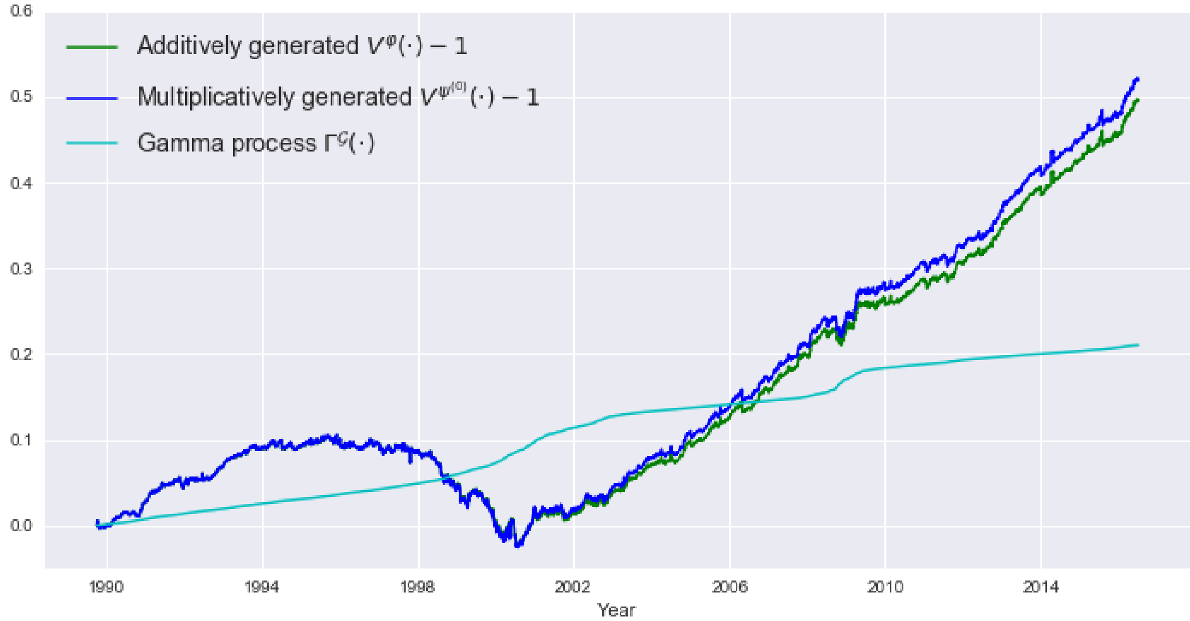

FIGURE 6.6. Gamma process $\Gamma^{\mathcal{G}}(\cdot)$ and relative wealth processes (minus 1) of both the additively and the multiplicatively generated portfolios with $\delta = 250$ days, $p = 0.8$, and $\alpha = 1$.

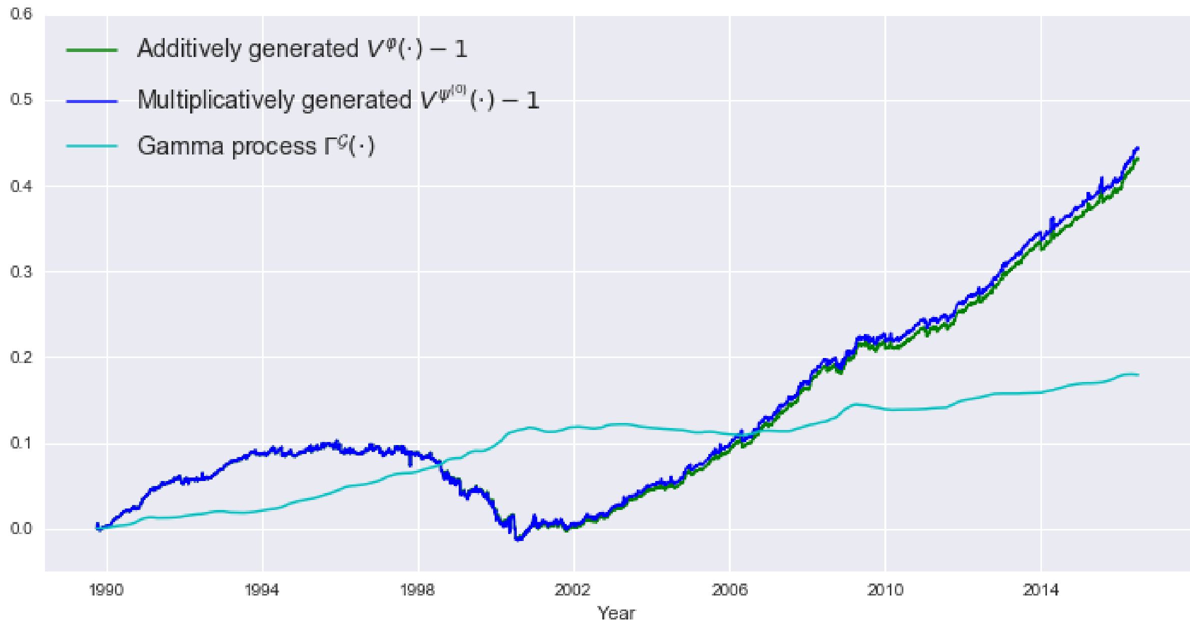

FIGURE 6.7. Gamma process $\Gamma^{\mathcal{G}}(\cdot)$ and relative wealth processes (minus 1) of both the additively and the multiplicatively generated portfolios with $\delta = 250$ days, $p = 0.8$, and $\alpha = 0.6$.

The above two examples illustrate that the choice of the finite variation process $\Lambda(\cdot)$ has an effect on the corresponding portfolio performance. The process $\Lambda(\cdot)$ can be chosen to magnify the impact of market diversification on the portfolio performance, to speed up or slow down the trading frequency, etc. In addition, an extra source of randomness, such as market sentiment indicators used in sentiment trading strategies, could be introduced when constructing $\Lambda(\cdot)$. We leave it to future research to develop a methodology to construct such processes $\Lambda(\cdot)$ systematically.

7. Conclusion

Karatzas and Ruf (2017) build a simple and intuitive structure by interpreting the portfolio generating functions G initiated by Fernholz (2002, 1999, 2001) as Lyapunov functions. They formulate conditions for the existence of strong arbitrage relative to the market over appropriate time horizons. The purpose of this paper is to investigate the dependence of the portfolio generating functions G on an extra $\mathbb{R}^m$-valued, progressive, continuous process $\Lambda(\cdot)$ of finite variation on $[0,T]$, for all $T \geq 0$.

The results of this paper are illuminated by several examples and shown to work on empirical data using stocks from the S&P 500 index. The effects that different choices of $\Lambda(\cdot)$ have on the portfolio wealths are analyzed. Provided that the market undergoes an explicit trend of either increasing or decreasing market diversification, certain choices of $\Lambda(\cdot)$ are better than others.

Appendix A. Proofs of Theorems 8 and 11

A.1. **Preliminaries.** Before providing the proof of Theorem 8, we discuss some technical details.

Recall the open set $\mathcal{W}$ from (2.4) and consider a continuous function $g : \mathcal{W} \to \mathbb{R}$. Define a function $\overline{g} : \mathbb{R}^{m+d} \to \mathbb{R}$ by

$$\overline{g}(z) = \begin{cases} g(z), & \text{if } z \in \mathcal{W} \\ 0, & \text{if } z \notin \mathcal{W} \end{cases}.$$

Next, let $(g_{n_1,n_2})_{n_1,n_2\in\mathbb{N}}$ be the family of functions $g_{n_1,n_2} : \mathcal{W} \to \mathbb{R}$ given by

$$g_{n_1,n_2}(\lambda, x) = \int_{\mathbb{R}^d} \eta_{n_2}(y) \int_{\mathbb{R}^m} \eta_{n_1}(u)\overline{g}(\lambda - u, x - y)\mathrm{d}u\mathrm{d}y, \tag{A.1}$$

for all $(\lambda, x) \in \mathcal{W}$, with $g_{n_1,n_2}(\lambda, x) = 0$ whenever the right hand side of (A.1) is not defined. Here in (A.1), for $z \in \mathbb{R}^l$ and $n \in \mathbb{N}$,

$$\eta_n(z) = \begin{cases} \beta n^l \exp\left(\frac{1}{n^2\|z\|_2^2 - 1}\right), & \text{if } \|z\|_2 < \frac{1}{n} \\ 0, & \text{if } \|z\|_2 \geq \frac{1}{n} \end{cases} \tag{A.2}$$

is used with the normalisation constant

$$\beta = \left(\int_{\mathbb{R}^l} \exp\left(\frac{1}{\|y\|_2^2 - 1}\right) \mathrm{d}y\right)^{-1},$$

independent of n.

Lemma 21. *Let $\overline{\mathcal{V}}$ denote any closed subset of $\mathcal{W}$. Consider a continuous function $g : \mathcal{W} \to \mathbb{R}$ and the mollification $(g_{n_1,n_2})_{n_1,n_2\in\mathbb{N}}$ of g defined as in* (A.1)*.*

(i) We have

$$\lim_{n_2\uparrow\infty} \lim_{n_1\uparrow\infty} g_{n_1,n_2} = g.$$

(ii) For $n_1, n_2 \in \mathbb{N}$ large enough, $g_{n_1,n_2} \in C^\infty(\overline{\mathcal{V}})$.

(iii) If there exists a constant $L = L(\overline{\mathcal{V}}) \geq 0$ such that, for all $(\lambda_1, x), (\lambda_2, x) \in \overline{\mathcal{V}}$,

$$|g(\lambda_1, x) - g(\lambda_2, x)| \leq L\|\lambda_1 - \lambda_2\|_2,$$

then, for $n_1, n_2 \in \mathbb{N}$ large enough and all $(\lambda, x) \in \overline{\mathcal{V}}$, we have

$$\left|\frac{\partial g_{n_1,n_2}}{\partial \lambda_v}(\lambda, x)\right| \leq L, \quad v \in \{1, \cdots, m\}.$$

(iv) If $g \in C^{0,1}$, then, for all $(\lambda, x) \in \mathcal{W}$, we have

$$\lim_{n_2 \uparrow \infty} \lim_{n_1 \uparrow \infty} \frac{\partial g_{n_1,n_2}}{\partial x_i}(\lambda, x) = \frac{\partial g}{\partial x_i}(\lambda, x), \quad i \in \{1, \cdots, d\}.$$

(v) If $g \in C^{0,1}$ and if there exists a constant $L = L(\overline{\mathcal{V}}) \geq 0$ such that, for all $(\lambda, x_1), (\lambda, x_2) \in \overline{\mathcal{V}}$,

$$\left\| \frac{\partial g}{\partial x}(\lambda, x_1) - \frac{\partial g}{\partial x}(\lambda, x_2) \right\|_2 \leq L \|x_1 - x_2\|_2,$$

then, for $n_1, n_2 \in \mathbb{N}$ large enough and all $(\lambda, x) \in \overline{\mathcal{V}}$, we have

$$\left| \frac{\partial^2 g_{n_1,n_2}}{\partial x_i \partial x_j}(\lambda, x) \right| \leq L, \quad i, j \in \{1, \cdots, d\}.$$

Proof. For (i) and (ii), see Theorem 6 in Appendix C in Evans (1998).

For (iii), observe that, for each $n_1, n_2 \in \mathbb{N}$ large enough and all $v \in \{1, \cdots, m\}$, (A.1) yields

$$\begin{aligned}
&\left| \frac{\partial g_{n_1,n_2}}{\partial \lambda_v}(\lambda, x) \right| = \left| \lim_{\delta \to 0} \frac{g_{n_1,n_2}(\lambda + \delta \mathbf{e}_v, x) - g_{n_1,n_2}(\lambda, x)}{\delta} \right| \\
&= \left| \lim_{\delta \to 0} \frac{1}{\delta} \int_{\mathbb{R}^d} \eta_{n_2}(y) \int_{\mathbb{R}^m} \eta_{n_1}(u) (\overline{g}(\lambda + \delta \mathbf{e}_v - u, x - y) - \overline{g}(\lambda - u, x - y)) \mathrm{d}u \mathrm{d}y \right| \\
&\leq \lim_{\delta \to 0} \frac{1}{\delta} \int_{\mathbb{R}^d} \eta_{n_2}(y) \int_{\mathbb{R}^m} \eta_{n_1}(u) \left| \overline{g}(\lambda + \delta \mathbf{e}_v - u, x - y) - \overline{g}(\lambda - u, x - y) \right| \mathrm{d}u \mathrm{d}y \\
&\leq \lim_{\delta \to 0} \frac{1}{\delta} \delta L \int_{\mathbb{R}^d} \eta_{n_2}(y) \int_{\mathbb{R}^m} \eta_{n_1}(u) \mathrm{d}u \mathrm{d}y = L,
\end{aligned}$$

for all $(\lambda, x) \in \overline{\mathcal{V}}$, where $\mathbf{e}_v$ is the unit vector in the v-th dimension.

For (iv), apply the dominated convergence theorem and (i) to $\frac{\partial g}{\partial x_i}$, for all $i \in \{1, \cdots, d\}$.

For (v), apply the dominated convergence theorem and a similar argument as in (iii). □

The following lemma is an extension of Lemma 2 in Bouleau (1981). For a continuous function $g : \mathcal{W} \to \mathbb{R}$, consider its corresponding mollification $(g_{n_1,n_2})_{n_1,n_2 \in \mathbb{N}}$ defined as in (A.1).

Lemma 22. *If a continuous function $g : \mathcal{W} \to \mathbb{R}$ is concave in its second argument, then*

$$\lim_{n_2 \uparrow \infty} \lim_{n_1 \uparrow \infty} \frac{\partial g_{n_1,n_2}}{\partial x_i} = f_i, \quad i \in \{1, \cdots, d\},$$

for some measurable function $f_i : \mathcal{W} \to \mathbb{R}$, bounded on any compact $\overline{\mathcal{V}} \subset \mathcal{W}$.

Proof. Fix $i \in \{1, \cdots, d\}$. With the notation in (A.2), we have

$$\eta_n(z) = n^l \eta_1(nz), \quad z \in \mathbb{R}^l, \; n \in \mathbb{N}.$$

For $(\lambda, x) \in \mathcal{W}$ and $n_2 \in \mathbb{N}$ large enough, the definition of g_{n_1,n_2} in (A.1), the dominated convergence theorem, and Lemma 21(i)&(ii) yield

$$\begin{aligned}
\lim_{n_1\uparrow\infty} \frac{\partial g_{n_1,n_2}}{\partial x_i}(\lambda, x) &= \lim_{n_1\uparrow\infty} \int_{\mathbb{R}^d} \frac{\partial \eta_{n_2}}{\partial x_i}(x-y) \int_{\mathbb{R}^m} \eta_{n_1}(u)\overline{g}(\lambda-u, y)\mathrm{d}u\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \frac{\partial \eta_{n_2}}{\partial x_i}(x-y) \lim_{n_1\uparrow\infty} \int_{\mathbb{R}^m} \eta_{n_1}(u)\overline{g}(\lambda-u, y)\mathrm{d}u\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \frac{\partial \eta_{n_2}}{\partial x_i}(x-y)\overline{g}(\lambda, y)\mathrm{d}y \\
&= -\int_{\mathbb{R}^d} \frac{\partial \eta_{n_2}}{\partial y_i}(y)\overline{g}(\lambda, x-y)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} n_2 \frac{\partial \eta_1}{\partial y_i}(y)\overline{g}\left(\lambda, x+\frac{y}{n_2}\right)\mathrm{d}y \\
&= \int_{\mathbb{R}^d} \frac{\partial \eta_1}{\partial y_i}(y) n_2\left(\overline{g}\left(\lambda, x+\frac{y}{n_2}\right) - \overline{g}\left(\lambda, x\right)\right)\mathrm{d}y.
\end{aligned}$$

Note that the last equality holds due to the fact that

$$\int_{\mathbb{R}^d} \frac{\partial \eta_1}{\partial y_i}(y)\mathrm{d}y = 0.$$

Next, for all $(\lambda, x) \in \mathcal{W}$ and $y \in \mathbb{R}^d$, define the one-sided directional partial derivative as

$$\nabla g(\lambda, x; y) = \lim_{n_2\uparrow\infty} \frac{g\left(\lambda, x+y/n_2\right) - g(\lambda, x)}{1/n_2}.$$

Such ∇g exists according to Theorem 23.1 in Rockafellar (1970). Since g is concave in the second argument, it is locally Lipschitz in its second argument on $\mathcal{W}$ (see Theorem 10.4 in Rockafellar (1970)). Hence, for each compact $\overline{\mathcal{V}} \subset \mathcal{W}$, there exists a constant $L = L(\overline{\mathcal{V}}) \geq 0$ such that $\nabla g(\lambda, x; y) \leq L$, for all $y \in \mathbb{R}^d$ and (λ, x) in the interior of $\overline{\mathcal{V}}$.

The statement now follows with

$$f_i(\lambda, x) = \int_{\mathbb{R}^d} \nabla \overline{g}(\lambda, x; y)\frac{\partial \eta_1}{\partial y_i}(y)\mathrm{d}y,$$

for all $(\lambda, x) \in \mathcal{W}$, by the dominated convergence theorem. □

Lemma 23. *Assume that $\mu(\cdot)$ has Doob-Meyer decomposition $\mu(\cdot) = \mu(0) + M(\cdot) + V(\cdot)$, where $M(\cdot)$ is a d-dimensional continuous local martingale and $V(\cdot)$ is a d-dimensional finite variation process with $M(0) = V(0) = 0$. Moreover, suppose that,*

(i) for some open $\mathcal{V} \subset \mathcal{W}$, we have $(\Lambda(\cdot), \mu(\cdot)) = (\Lambda(\cdot \wedge \tau), \mu(\cdot \wedge \tau))$, where

$$\tau = \inf\left\{t \geq 0;\ (\Lambda(t), \mu(t)) \notin \mathcal{V}\right\};$$

(ii) for some constant $\kappa \geq 0$, we have

$$\sum_{i=1}^d \left([M_i, M_i](\infty) + \int_0^\infty \mathrm{d}|V_i(t)| \right) + \sum_{v=1}^m \int_0^\infty \mathrm{d}|\Lambda_v(t)| \leq \kappa < \infty. \tag{A.3}$$

Let $(h_i)_{i\in\{1,\cdots,d\}}$ be a family of functions $h_i : \mathcal{V} \to \mathbb{R}$ and let $(h_i^{n_1,n_2})_{n_1,n_2\in\mathbb{N}, i\in\{1,\cdots,d\}}$ be a family of doubly indexed sequences of uniformly bounded functions $h_i^{n_1,n_2} : \mathcal{V} \to \mathbb{R}$. If

$$\lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty} h_i^{n_1,n_2} = h_i, \quad i \in \{1, \cdots, d\},$$

then there exist two random subsequences $(n_1^k)_{k\in\mathbb{N}}$ and $(n_2^k)_{k\in\mathbb{N}}$ with $\lim_{k\uparrow\infty} n_1^k = \infty = \lim_{k\uparrow\infty} n_2^k$ such that

$$\lim_{k\uparrow\infty} \int_0^t \sum_{i=1}^d h_i^{n_1^k,n_2^k}(\Lambda(u), \mu(u))\mathrm{d}\mu_i(u) = \int_0^t \sum_{i=1}^d h_i(\Lambda(u), \mu(u))\mathrm{d}\mu_i(u), \quad a.s., \tag{A.4}$$

for all $t \geq 0$.

Proof. Fix $i \in \{1, \cdots, d\}$ and write

$$\Theta_i^{n_1,n_2}(\cdot) = h_i^{n_1,n_2}(\Lambda(\cdot), \mu(\cdot)) - h_i(\Lambda(\cdot), \mu(\cdot)).$$

By (A.3) and the bounded convergence theorem, we have

$$\begin{aligned} 0 &= \mathbb{E}\left[\lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\int_0^\infty (\Theta_i^{n_1,n_2}(t))^2 \mathrm{d}\left[M_i, M_i\right](t)\right] \\ &= \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\mathbb{E}\left[\int_0^\infty (\Theta_i^{n_1,n_2}(t))^2 \mathrm{d}\left[M_i, M_i\right](t)\right] \\ &= \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\mathbb{E}\left[\left(\int_0^\infty \Theta_i^{n_1,n_2}(t)\mathrm{d}M_i(t)\right)^2\right], \end{aligned}$$

by Itô's isometry, and

$$0 = \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\mathbb{E}\left[\left(\int_0^\infty |\Theta_i^{n_1,n_2}(t)|\, \mathrm{d}|V_i(t)|\right)^2\right]. \tag{A.5}$$

Since $\int_0^\cdot \Theta_i^{n_1,n_2}(t)\mathrm{d}M_i(t)$ is a uniformly integrable martingale (as it is a local martingale with bounded quadratic variation), Doob's submartingale inequality yields

$$\mathbb{E}\left[\left(\sup_{t\geq 0}\left|\int_0^t \Theta_i^{n_1,n_2}(u)\mathrm{d}M_i(u)\right|\right)^2\right] \leq 4\mathbb{E}\left[\left(\int_0^\infty \Theta_i^{n_1,n_2}(t)\mathrm{d}M_i(t)\right)^2\right],$$

which implies

$$0 = \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\mathbb{E}\left[\left(\sup_{t\geq 0}\left|\int_0^t \Theta_i^{n_1,n_2}(u)\mathrm{d}M_i(u)\right|\right)^2\right]. \tag{A.6}$$

Therefore, (A.5), (A.6), and the triangle inequality yield

$$0 = \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty}\mathbb{E}\left[\left(\sup_{t\geq 0}\left|\int_0^t \Theta_i^{n_1,n_2}(u)\mathrm{d}\mu_i(u)\right|\right)^2\right].$$

Write

$$E_i^{n_1,n_2} = \mathbb{E}\left[\left(\sup_{t\geq 0}\left|\int_0^t \Theta_i^{n_1,n_2}(u)\mathrm{d}\mu_i(u)\right|\right)^2\right], \quad n_1, n_2 \in \mathbb{N},$$

and

$$E_i = \lim_{n_2\uparrow\infty}\lim_{n_1\uparrow\infty} E_i^{n_1,n_2}.$$

For each $n_2 \in \mathbb{N}$, denote $E_i^{n_2} = \lim_{n_1\uparrow\infty} E_i^{n_1,n_2}$. Then we can find a subsequence $(n_1(n_2))_{n_2\in\mathbb{N}}$ of $\mathbb{N}$ with $n_1(n_2) \uparrow \infty$ as $n_2 \uparrow \infty$ such that, for each $n_2 \in \mathbb{N}$,

$$\left|E_i^{n_1(n_2),n_2} - E_i^{n_2}\right| \leq \frac{1}{n_2}.$$

Since the triangle inequality yields

$$\left|E_i^{n_1(n_2),n_2} - E_i\right| \leq \frac{1}{n_2} + |E_i^{n_2} - E_i| \to 0 \quad \text{as } n_2 \uparrow \infty,$$

we have $\lim_{n_2\uparrow\infty} E_i^{n_1(n_2),n_2} = E_i = 0$. This implies

$$\lim_{n_2\uparrow\infty}\sup_{t\geq 0}\left|\int_0^t \sum_{i=1}^d h_i^{n_1(n_2),n_2}(\Lambda(u), \mu(u))\mathrm{d}\mu_i(u) - \int_0^t \sum_{i=1}^d h_i(\Lambda(u), \mu(u))\mathrm{d}\mu_i(u)\right| = 0$$

in L^2. Since convergence in L^2 implies almost sure convergence of a subsequence, we can find a random subsequence $(n_2^k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with $n_2^k \uparrow \infty$ as $k \uparrow \infty$ such that (A.4) holds with $n_1^k = n_1(n_2^k)$. □

Lemma 24. *Fix $l \in \mathbb{N}$; let $\overline{\Lambda}(\cdot)$ be an l-dimensional continuous process of finite variation; let $(\overline{\Upsilon}_{u,n}(\cdot))_{u\in\{1,\cdots,l\},n\in\mathbb{N}}$ be a family of processes with $(\overline{\Upsilon}_{u,n}(\cdot))_{n\in\mathbb{N}}$ uniformly bounded, for each $u \in \{1,\cdots,l\}$; and let $(\overline{\Theta}_n(\cdot))_{n\in\mathbb{N}}$ be a sequence of non-decreasing continuous processes. Define*

$$H_n(\cdot) = \int_0^\cdot \sum_{u=1}^{l} \overline{\Upsilon}_{u,n}(t)\mathrm{d}\overline{\Lambda}_u(t) + \overline{\Theta}_n(\cdot), \quad n \in \mathbb{N}.$$

If $\lim_{n\uparrow\infty} H_n(\cdot) = H(\cdot)$, a.s., then $H(\cdot)$ is of finite variation.

Proof. The following steps are partially inspired by the proof of Lemma 3.3 in Jaber et al. (2018).

Since $(\overline{\Upsilon}_{1,n}(\cdot))_{n\in\mathbb{N}}$ is uniformly bounded, the Komlós theorem (see Theorem 1.3 in Delbaen and Schachermayer (1999)) yields the following. For each $n \in \mathbb{N}$, there exists a convex combination $\Upsilon^1_{1,n}(\cdot) \in \mathrm{Conv}(\overline{\Upsilon}_{1,k}(\cdot), k \geq n)$ such that $(\Upsilon^1_{1,n}(\cdot))_{n\in\mathbb{N}}$ converges to some adapted bounded process $\Upsilon_1(\cdot)$. More precisely, for each $n \in \mathbb{N}$, we can find some random integer $N_n \geq 0$ and $(w^k_n)_{n\leq k\leq N_n} \subset [0,1]$ such that

$$\sum_{k=n}^{N_n} w^k_n = 1 \quad \text{and} \quad \Upsilon^1_{1,n}(\cdot) = \sum_{k=n}^{N_n} w^k_n \overline{\Upsilon}_{1,k}(\cdot).$$

For each $n \in \mathbb{N}$, define

$$H^1_n(\cdot) = \sum_{k=n}^{N_n} w^k_n H_n(\cdot), \quad \Theta^1_n(\cdot) = \sum_{k=n}^{N_n} w^k_n \overline{\Theta}_k(\cdot), \quad \text{and} \quad \Upsilon^1_{u,n}(\cdot) = \sum_{k=n}^{N_n} w^k_n \overline{\Upsilon}_{u,k}(\cdot),$$

for all $u \in \{2,\cdots,l\}$.

Since $\lim_{n\uparrow\infty} H_n(\cdot) = H(\cdot)$, a.s., we have

$$\left|H^1_n(\cdot) - H(\cdot)\right| = \left|\sum_{k=n}^{N_n} w^k_n H_k(\cdot) - H(\cdot)\right| \leq \sum_{k=n}^{N_n} w^k_n \left|H_k(\cdot) - H(\cdot)\right| \to 0$$

as $n \uparrow \infty$, which implies $\lim_{n\uparrow\infty} H^1_n(\cdot) = H(\cdot)$, a.s. Besides, $\Theta^1_n(\cdot)$ is non-decreasing, as it is a convex combination of non-decreasing processes.

Since $(\Upsilon^1_{2,n}(\cdot))_{n\in\mathbb{N}}$ is also uniformly bounded, by the Komlós theorem again, for each $n \in \mathbb{N}$, there exists another convex combination $\Upsilon^2_{2,n}(\cdot) \in \mathrm{Conv}(\Upsilon^1_{2,k}(\cdot), k \geq n)$ such that $(\Upsilon^2_{2,n}(\cdot))_{n\in\mathbb{N}}$ converges to some adapted bounded process $\Upsilon_2(\cdot)$. With the same convex combination for each $n \in \mathbb{N}$, define $\Upsilon^2_{u,n}(\cdot)$, for all $u \in \{1,3,\cdots,l\}$, $H^2_n(\cdot)$, and similarly $\Theta^2_n(\cdot)$. In particular, $(\Upsilon^2_{1,n}(\cdot))_{n\in\mathbb{N}}$ still converges to $\Upsilon_1(\cdot)$, as for each $n \in \mathbb{N}$, $\Upsilon^2_{1,n}(\cdot)$ is a convex combination of processes that converge to $\Upsilon_1(\cdot)$. Similarly, we have $\lim_{n\uparrow\infty} H^2_n(\cdot) = H(\cdot)$, a.s. Moreover, $\Theta^2_n(\cdot)$ is non-decreasing.

Iteratively, we construct sequences of processes $(\Upsilon^3_{u,n}(\cdot))_{n\in\mathbb{N}},\cdots,(\Upsilon^l_{u,n}(\cdot))_{n\in\mathbb{N}}$, for each $u \in \{1,\cdots,l\}$, and processes $H^3_n(\cdot),\cdots,H^l_n(\cdot)$ and $\Theta^3_n(\cdot),\cdots,\Theta^l_n(\cdot)$ in the same manner. In particular, $(\Upsilon^l_{u,n}(\cdot))_{n\in\mathbb{N}}$ converges to some adapted bounded process Υ_u, for each $u \in \{1,\cdots,l\}$, and we have $\lim_{n\uparrow\infty} H^l_n(\cdot) = H(\cdot)$, a.s. Moreover, $\Theta^l_n(\cdot)$ is non-decreasing.

By the dominated convergence theorem, we have

$$\lim_{n\uparrow\infty} \int_0^\cdot \sum_{u=1}^{l} \Upsilon^l_{u,n}(t)\mathrm{d}\overline{\Lambda}_u(t) = \int_0^\cdot \sum_{u=1}^{l} \Upsilon_u(t)\mathrm{d}\overline{\Lambda}_u(t), \quad \text{a.s.},$$

which is of finite variation. Therefore, we have

$$H(\cdot) = \lim_{n\uparrow\infty} H^l_n(\cdot) = \int_0^\cdot \sum_{u=1}^{l} \Upsilon_u(t)\mathrm{d}\overline{\Lambda}_u(t) + \lim_{n\uparrow\infty} \Theta^l_n(\cdot), \quad \text{a.s.}$$

Since $\Theta_n^l(\cdot)$ is non-decreasing and converges, it is of finite variation, which implies the assertion. □

A.2. **Proof of Theorem 8.**

Proof of Theorem 8. Assume that the semimartingale $\mu(\cdot)$ has the Doob-Meyer decomposition $\mu(\cdot) = \mu(0) + M(\cdot) + V(\cdot)$, where $M(\cdot)$ is a d-dimensional continuous local martingale and $V(\cdot)$ is a d-dimensional finite variation process with $M(0) = V(0) = 0$.

Let $(\mathcal{W}_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of open sets such that the closure of $\mathcal{W}_n$ is in $\mathcal{W}$, for all $n \in \mathbb{N}$. For each $\kappa \in \mathbb{N}$, we consider the stopping time

$$\begin{aligned}\tau_\kappa = \inf \Bigg\{ & t \geq 0;\ (\Lambda(t), \mu(t)) \notin \mathcal{W}_\kappa \\ & \text{or } \sum_{i,j=1}^d [M_i, M_j](t) + \sum_{i=1}^d \int_0^t \mathrm{d}|V_i(u)| + \sum_{v=1}^m \int_0^t \mathrm{d}|\Lambda_v(u)| \geq \kappa \Bigg\}\end{aligned} \tag{A.7}$$

with $\inf\{\emptyset\} = \infty$. Since $(\Lambda(\cdot), \mu(\cdot)) \in \mathcal{W}$, we have $\lim_{\kappa\uparrow\infty} \tau_\kappa = \infty$, a.s. As $\bigcup_{\kappa\in\mathbb{N}}\{\tau_\kappa > t\} = \Omega$, for all $t \geq 0$, to prove that G is regular (Lyapunov), it is equivalent to show that G is regular (Lyapunov) for $\Lambda(\cdot \wedge \tau_\kappa)$ and $\mu(\cdot \wedge \tau_\kappa)$, for all $\kappa \in \mathbb{N}$. Hence, without loss of generality, let us assume that $(\Lambda(\cdot), \mu(\cdot)) = (\Lambda(\cdot \wedge \tau_\kappa), \mu(\cdot \wedge \tau_\kappa))$, for some $\kappa \in \mathbb{N}$.

Without loss of generality, assume that $a_{ij}(\cdot)$ is a predictable and uniformly bounded process, for all $i, j \in \{1, \cdots, d\}$, such that

$$[\mu_i, \mu_j](t) = \int_0^t a_{ij}(u)\mathrm{d}A(u) \leq \kappa, \quad t \geq 0,$$

where $A(\cdot) = \sum_{i=1}^d [\mu_i, \mu_i](\cdot)$. Here, the equality holds according to the Kunita-Watanabe inequality (see also Proposition 2.9 in Jacod and Shiryaev (2003)) and the inequality due to (A.7).

Now, consider a mollification $(G_{n_1,n_2})_{n_1,n_2\in\mathbb{N}}$ of G defined as in (A.1). By Lemma 21(ii), for $n_1, n_2 \in \mathbb{N}$ large enough, Itô's lemma applied to G_{n_1,n_2} yields

$$\begin{aligned}G_{n_1,n_2}(\Lambda(t), \mu(t)) = G_{n_1,n_2}(\Lambda(0), \mu(0)) &+ \int_0^t \sum_{i=1}^d \frac{\partial G_{n_1,n_2}}{\partial x_i}(\Lambda(u), \mu(u))\mathrm{d}\mu_i(u) \\ &+ \int_0^t \Upsilon_{0,n_1,n_2}(u)\mathrm{d}A(u) + \int_0^t \sum_{v=1}^m \Upsilon_{v,n_1,n_2}(u)\mathrm{d}\Lambda_v(u),\end{aligned} \tag{A.8}$$

for all $t \geq 0$, where

$$\Upsilon_{0,n_1,n_2}(t) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 G_{n_1,n_2}}{\partial x_i \partial x_j}(\Lambda(t), \mu(t))a_{ij}(t) \quad \text{and} \quad \Upsilon_{v,n_1,n_2}(t) = \frac{\partial G_{n_1,n_2}}{\partial \lambda_v}(\Lambda(t), \mu(t)),$$

for all $v \in \{1, \cdots, m\}$.

For all $(\lambda, x) \in \mathcal{W}$ and $i \in \{1, \cdots, d\}$, if (bi) holds, Lemma 21(iv) yields

$$\lim_{n_2\uparrow\infty} \lim_{n_1\uparrow\infty} \frac{\partial G_{n_1,n_2}}{\partial x_i}(\lambda, x) = \frac{\partial G}{\partial x_i}(\lambda, x);$$

if (bii) holds, Lemma 22 yields

$$\lim_{n_2\uparrow\infty} \lim_{n_1\uparrow\infty} \frac{\partial G_{n_1,n_2}}{\partial x_i}(\lambda, x) = f_i(\lambda, x),$$

for some measurable function f_i. Moreover, thanks to (bi) or (bii), there exists a constant $L = L(\mathcal{W}_\kappa) \geq 0$ such that, for $n_1, n_2 \in \mathbb{N}$ large enough,

$$\left|\frac{\partial G_{n_1,n_2}}{\partial x_i}\right| \leq L, \quad i \in \{1, \cdots, d\}.$$

This follows from the Lipschitz continuity of G on the closure of $\mathcal{W}_\kappa$ in the second argument and a similar reasoning as in the proof of Lemma 21(iii). Then by Lemma 23, there exist random subsequences $(n_1^k)_{k\in\mathbb{N}}$ and $(n_2^k)_{k\in\mathbb{N}}$ with $\lim_{k\uparrow\infty} n_1^k = \infty = \lim_{k\uparrow\infty} n_2^k$ such that, if we write $G_k = G_{n_1^k,n_2^k}$, we have

$$\lim_{k\uparrow\infty}\int_0^t \sum_{i=1}^d \frac{\partial G_k}{\partial x_i}(\Lambda(u),\mu(u))\mathrm{d}\mu_i(u) = F(\Lambda(t),\mu(t)), \quad \text{a.s.,} \tag{A.9}$$

for all $t \geq 0$, where

$$F(\Lambda(t),\mu(t)) = \begin{cases} \int_0^t \sum_{i=1}^d \frac{\partial G}{\partial x_i}(\Lambda(u),\mu(u))\mathrm{d}\mu_i(u), & \text{if (bi) holds} \\ \int_0^t \sum_{i=1}^d f_i(\Lambda(u),\mu(u))\mathrm{d}\mu_i(u), & \text{if (bii) holds} \end{cases}.$$

To proceed, write

$$H_k(t) = G_k(\Lambda(0),\mu(0)) - G_k(\Lambda(t),\mu(t)) + \int_0^t \sum_{i=1}^d \frac{\partial G_k}{\partial x_i}(\Lambda(u),\mu(u))\mathrm{d}\mu_i(u),$$

for all $k \in \mathbb{N}$, and

$$H(t) = G(\Lambda(0),\mu(0)) - G(\Lambda(t),\mu(t)) + F(\Lambda(t),\mu(t)),$$

for all $t \geq 0$. Then, (A.8) with respect to the random subsequences $(n_1^k)_{k\in\mathbb{N}}$ and $(n_2^k)_{k\in\mathbb{N}}$ is of the form

$$H_k(t) = -\int_0^t \Upsilon_{0,k}(u)\mathrm{d}A(u) - \int_0^t \sum_{v=1}^m \Upsilon_{v,k}(u)\mathrm{d}\Lambda_v(u), \quad t \geq 0.$$

Note that by Lemma 21(i) and (A.9), $\lim_{k\uparrow\infty} H_k(t) = H(t)$, a.s., for all $t \geq 0$.

A measurable function DG in Condition 1 of Definition 5 is chosen with components

$$D_iG(\lambda,x) = \begin{cases} \frac{\partial G}{\partial x_i}(\lambda,x), & \text{if (bi) holds} \\ f_i(\lambda,x), & \text{if (bii) holds} \end{cases}, \quad i \in \{1,\cdots,d\}.$$

Then, as $\Gamma^G(\cdot) = H(\cdot)$ according to (3.2), it is enough to show that $H(\cdot)$ is of finite variation in the following four cases.

Case 1.

Assume that (ai) and (bi) hold. Then by Lemma 21(iii)&(v), the processes $(\Upsilon_{0,k}(\cdot))_{k\in\mathbb{N}}$ and $(\Upsilon_{v,k}(\cdot))_{v\in\{1,\cdots,m\},k\in\mathbb{N}}$ are uniformly bounded. With $l = m+1$, $\overline{\Lambda}_v(\cdot) = \Lambda_v(\cdot)$ and $(\overline{\Upsilon}_{v,k}(\cdot))_{k\in\mathbb{N}} = (\Upsilon_{v,k}(\cdot))_{k\in\mathbb{N}}$, for all $v \in \{1,\cdots,m\}$, $\overline{\Lambda}_{m+1}(\cdot) = A(\cdot)$, $(\overline{\Upsilon}_{m+1,k}(\cdot))_{k\in\mathbb{N}} = (\Upsilon_{0,k}(\cdot))_{k\in\mathbb{N}}$, and $(\overline{\Theta}_k(\cdot))_{k\in\mathbb{N}} = 0$, Lemma 24 yields that $H(\cdot)$ is of finite variation on compact sets.

Case 2.

Assume that (ai) and (bii) hold. By Lemma 21(iii), the processes $(\Upsilon_{v,k}(\cdot))_{v\in\{1,\cdots,m\},k\in\mathbb{N}}$ are uniformly bounded. Since G is concave in the second argument, for each $k \in \mathbb{N}$, G_k is also concave in the second argument. Using the negative semidefinite property of the Hessian of G_k and choosing the matrix-valued process $a(\cdot) = (a_{ij}(\cdot))_{i,j\in\{1,\cdots,d\}}$ to be symmetric and positive semidefinite, one can show that $\Upsilon_{0,k}(t) \leq 0$, for all $t \geq 0$. This implies that the processes

$$\overline{\Theta}_k(\cdot) = -\int_0^\cdot \Upsilon_{0,k}(t)\mathrm{d}A(t), \quad k \in \mathbb{N},$$

are non-decreasing. Similar to Case 1, but now with $l = m$, Lemma 24 yields again that $H(\cdot)$ is of finite variation.

Case 3.

Assume that (aii) and (bi) hold. By Lemma 21(v), the process $(\Upsilon_{0,k}(\cdot))_{k\in\mathbb{N}}$ is uniformly bounded. As G is non-increasing in the v-th dimension of the first argument, so is G_k, for all

$v \in \{1, \cdots, m\}$. Therefore, $\Upsilon_{v,k}(t) \leq 0$, for all $t \geq 0$, as $\Lambda(\cdot)$ is non-decreasing in the v-th dimension, for all $v \in \{1, \cdots, m\}$. This implies that the processes

$$\overline{\Theta}_k(\cdot) = -\int_0^\cdot \sum_{v=1}^m \Upsilon_{v,k}(t) \mathrm{d}\Lambda_v(t), \quad k \in \mathbb{N},$$

are non-decreasing. Similar to above, Lemma 24 implies that $H(\cdot)$ is of finite variation.

Case 4.

Assume that (aii) and (bii) hold. With

$$\overline{\Theta}_k(\cdot) = -\int_0^\cdot \Upsilon_{0,k}(t) \mathrm{d}A(t) - \int_0^\cdot \sum_{v=1}^m \Upsilon_{v,k}(t) \mathrm{d}\Lambda_v(t), \quad k \in \mathbb{N},$$

Lemma 24 implies again that $H(\cdot)$ is of finite variation. It is clear that G is Lyapunov. □

A.3. **Proof of Theorem 11.**

Proof of Theorem 11. The following steps are partially inspired by the proof of Theorem 3.8 in Karatzas and Ruf (2017). According to Theorem 2.3 in Banner and Ghomrasni (2008), for each $l \in \{1, \cdots, d\}$, one can find a measurable function $\boldsymbol{h}_l : \Delta^d \to (0, 1]$ and a finite variation process $\boldsymbol{B}_l(\cdot)$ with $\boldsymbol{B}_l(0) = 0$ such that

$$\mu_{(l)}(\cdot) = \mu_{(l)}(0) + \int_0^\cdot \sum_{i=1}^d \boldsymbol{h}_l(\mu(t)) \mathbf{1}_{\{\mu_{(l)}(t)=\mu_i(t)\}} \mathrm{d}\mu_i(t) + \boldsymbol{B}_l(\cdot). \tag{A.10}$$

Since $\boldsymbol{G}$ is regular for $\Lambda(\cdot)$ and $\boldsymbol{\mu}(\cdot)$, by Definition 5, there exist a measurable function $D\boldsymbol{G}$ and a finite variation process $\Gamma^{\boldsymbol{G}}(\cdot)$ such that

$$\boldsymbol{G}(\Lambda(\cdot), \boldsymbol{\mu}(\cdot)) = \boldsymbol{G}(\Lambda(0), \boldsymbol{\mu}(0)) + \int_0^\cdot \sum_{l=1}^d D_l \boldsymbol{G}(\Lambda(t), \boldsymbol{\mu}(t)) \mathrm{d}\mu_{(l)}(t) - \Gamma^{\boldsymbol{G}}(\cdot). \tag{A.11}$$

By (A.10), we have

$$\begin{aligned}\int_0^\cdot \sum_{l=1}^d D_l \boldsymbol{G}(\Lambda(t), \boldsymbol{\mu}(t)) \mathrm{d}\mu_{(l)}(t) = &\int_0^\cdot \sum_{l=1}^d D_l \boldsymbol{G}(\Lambda(t), \boldsymbol{\mu}(t)) \boldsymbol{h}_l(\mu(t)) \mathbf{1}_{\{\mu_{(l)}(t)=\mu_i(t)\}} \mathrm{d}\mu_i(t) \\ &+ \int_0^\cdot \sum_{l=1}^d D_l \boldsymbol{G}(\Lambda(t), \boldsymbol{\mu}(t)) \mathrm{d}\boldsymbol{B}_l(t).\end{aligned} \tag{A.12}$$

Now consider the measurable function $DG : \mathcal{W} \to \mathbb{R}^d$ with components

$$D_i G(\lambda, x) = \sum_{l=1}^d D_l \boldsymbol{G}(\lambda, \mathfrak{R}(x)) \boldsymbol{h}_l(x) \mathbf{1}_{x_{(l)}=x_i}, \quad i \in \{1, \cdots, d\},$$

and the finite variation process

$$\Gamma^G(\cdot) = \Gamma^{\boldsymbol{G}}(\cdot) - \int_0^\cdot \sum_{l=1}^d D_l \boldsymbol{G}(\Lambda(t), \boldsymbol{\mu}(t)) \mathrm{d}\boldsymbol{B}_l(t).$$

Then (A.11) and (A.12), together with $G(\lambda, x) = \boldsymbol{G}(\lambda, \mathfrak{R}(x))$, yield (3.2), i.e., G is regular for $\Lambda(\cdot)$ and $\mu(\cdot)$. □

A.4. **An alternative proof for a special case.** The proof technique of Theorem VII.31 in Dellacherie and Meyer (1982) suggests an alternative argument for the case that conditions (ai) and (bii) in Theorem 8 hold. We summarise these ideas in the following result.

Theorem 25. *If a function $f : \mathcal{W} \to \mathbb{R}$ is locally Lipschitz in the first argument and concave in the second argument, then the process $f(\Lambda(\cdot), \mu(\cdot))$ is a semimartingale.*

Proof. Assume that the semimartingale $\mu(\cdot)$ has the Doob-Meyer decomposition $\mu(\cdot) = \mu(0) + M(\cdot) + V(\cdot)$, where $M(\cdot)$ is a d-dimensional continuous local martingale and $V(\cdot)$ is a d-dimensional finite variation process with $M(0) = V(0) = 0$.

Let $(\mathcal{W}_n)_{n\in\mathbb{N}}$ be a non-decreasing sequence of open sets such that the closure of $\mathcal{W}_n$ is in $\mathcal{W}$, for all $n \in \mathbb{N}$. For each $\kappa \in \mathbb{N}$, we consider the stopping time τ_κ given in (A.7). Without loss of generality, let us assume again that $(\Lambda(\cdot), \mu(\cdot)) = (\Lambda(\cdot \wedge \tau_\kappa), \mu(\cdot \wedge \tau_\kappa))$, for some $\kappa \in \mathbb{N}$.

Since f is locally Lipschitz in both arguments (see Theorem 10.4 in Rockafellar (1970)), we can find a Lipschitz constant L such that, for all $s, t \geq 0$ with $s \leq t$, we have

$$\begin{aligned}
&|f(\Lambda(t), \mu(t)) - f(\Lambda(s), \mu(0) + M(t) + V(s))| \\
&\quad \leq L\left(\sum_{v=1}^{m} |\Lambda_v(t) - \Lambda_v(s)| + \sum_{i=1}^{d} |V_i(t) - V_i(s)|\right) \\
&\quad \leq L\left(\sum_{v=1}^{m} \int_s^t |\mathrm{d}\Lambda_v(u)| + \sum_{i=1}^{d} \int_s^t |\mathrm{d}V_i(u)|\right).
\end{aligned} \tag{A.13}$$

Let

$$Z(\cdot) = -f(\Lambda(\cdot), \mu(\cdot)) + L\left(\sum_{v=1}^{m} \int_0^{\cdot} |\mathrm{d}\Lambda_v(t)| + \sum_{i=1}^{d} \int_0^{\cdot} |\mathrm{d}V_i(t)|\right),$$

then $Z(\cdot)$ is bounded. Hence we have

$$\begin{aligned}
\mathbb{E}\left[Z(t) - Z(s) | \mathcal{F}(s)\right] &= \mathbb{E}\left[f(\Lambda(s), \mu(s)) - f(\Lambda(s), \mu(0) + M(t) + V(s)) | \mathcal{F}(s)\right] \\
&\quad + \mathbb{E}\Bigg[f(\Lambda(s), \mu(0) + M(t) + V(s)) - f(\Lambda(t), \mu(t)) \\
&\quad + L\left(\sum_{v=1}^{m} \int_s^t |\mathrm{d}\Lambda_v(u)| + \sum_{i=1}^{d} \int_s^t |\mathrm{d}V_i(u)|\right) \Bigg| \mathcal{F}(s)\Bigg] \\
&\geq \mathbb{E}\left[f(\Lambda(s), \mu(s)) - f(\Lambda(s), \mu(0) + M(t) + V(s)) | \mathcal{F}(s)\right] \geq 0,
\end{aligned}$$

where the first inequality is by (A.13) and the second inequality holds by Jensen's inequality. Therefore, $Z(\cdot)$ is a submartingale, which makes $f(\Lambda(\cdot), \mu(\cdot))$ a semimartingale. □

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