The Uniform Integrability of Martingales. On a Question by Alexander Cherny∗

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Abstract

Let \( X \) be a progressively measurable, almost surely right-continuous stochastic process such that \( X_\tau \in L^1 \) and \( \mathbb{E}[X_\tau] = \mathbb{E}[X_0] \) for each finite stopping time \( \tau \). In 2006, Cherny showed that \( X \) is then a uniformly integrable martingale provided that \( X \) is additionally nonnegative. Cherny then posed the question whether this implication also holds even if \( X \) is not necessarily nonnegative. We provide an example that illustrates that this implication is wrong, in general. If, however, an additional integrability assumption is made on the limit inferior of \( |X| \) then the implication holds. Finally, we argue that this integrability assumption holds if the stopping times are allowed to be randomized in a suitable sense.

Key words: Stopping time; Uniform integrability

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1 Introduction

We fix a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) with expectation operator \( \mathbb{E}[\cdot] \), where \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)} \) and \( \mathcal{F}_\infty = \bigvee_{t \in [0, \infty)} \mathcal{F}_t \subset \mathcal{F} \). Furthermore, we fix a progressively measurable, almost surely right-continuous process \( X \). We write \( Z \in L^1 \) if \( \mathbb{E}[|Z|] < \infty \) for some random variable \( Z \). For some \( \mathbb{F} \)-adapted process \( Y \) and some stopping time \( \eta \) we write \( Y^\eta \) to denote the process \( Y \) stopped at time \( \eta \); to wit, \( Y^\eta_t = Y_{\eta \wedge t} \) for each \( t \in [0, \infty) \). All identifications and statements in the following are in the almost-sure sense.

We consider the following five statements:

(I) \( X \) is a uniformly integrable martingale.

(II) \( X_\infty = \lim_{t \uparrow \infty} X_t \) exists, \( X_\tau \in L^1 \) and \( \mathbb{E}[X_\tau] = \mathbb{E}[X_0] \) for all stopping times \( \tau \).

(III) \( X_\tau \in L^1 \) and \( \mathbb{E}[X_\tau] = \mathbb{E}[X_0] \) for all finite stopping times \( \tau \) and \( \liminf_{t \uparrow \infty} |X_t| \in L^1 \).

(IV) \( X_\tau \in L^1 \) and \( \mathbb{E}[X_\tau] = \mathbb{E}[X_0] \) for all finite stopping times \( \tau \).

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The optional sampling theorem yields the implications \((\text{I}) \implies (\text{II}) \implies (\text{III}) \implies (\text{IV}) \implies (\text{V})\). The implication \((\text{II}) \implies (\text{I})\) follows from the following simple argument. Fix \(s, t \in [0, \infty]\) with \(s < t\) and \(A \in \mathcal{F}_s\). Let \(\tau_1 = s\) and \(\tau_2 = s1_{A^c} + t1_A\) denote two stopping times, where \(A^c = \Omega \setminus A\). Then, by assumption, \(\mathbb{E}[X_{\tau_1}] = \mathbb{E}[X_{\tau_2}]\), which yields \(\mathbb{E}[X_s 1_A] = \mathbb{E}[X_t 1_A]\). Thus we obtain the desired implication \((\text{II}) \implies (\text{I})\). However, if only \((\text{III})\) or \((\text{IV})\) is assumed, this argument only yields the martingale property of \(X\) but not its uniform integrability.

Cherny (2006) now asks the question whether also the implication \((\text{IV}) \implies (\text{I})\) holds. Before answering this question and discussing the role of \((\text{III})\), let us first briefly consider the statement in \((\text{V})\). Hulley (2009) shows that \((\text{V})\) is satisfied but \(X\) is not a martingale. Alternatively, if \(X\) denotes Brownian motion started in 0 then \((\text{V})\) holds but \((\text{IV})\) is not satisfied. To see this, we only need to let \(\tau\) denote the first hitting time of level 1 by \(X\). Thus, the implication \((\text{V}) \implies (\text{IV})\) does not hold in general.

We now return to discuss the missing implications, namely whether \((\text{III})\) or, more generally \((\text{IV})\), imply \((\text{I})\) (or equivalently, \((\text{II})\)). Cherny (2006) proves that these implications hold if the nonnegativity assumption on \(X\) is dropped. The following theorem proves that the implication \((\text{III}) \implies (\text{I})\) holds always, not only if \(X\) is nonnegative. However, the example of the next section shows that \((\text{IV})\) does not necessarily imply any of the statements \((\text{I}) \rightarrow (\text{III})\) if the nonnegativity assumption on \(X\) is dropped.

**Theorem 1.** The statements \((\text{I})\), \((\text{II})\), and \((\text{III})\) are equivalent.

**Proof.** We only need to show the implication \((\text{III}) \implies (\text{I})\). We start by arguing that we may assume, without loss of generality, that \(\mathbb{F}\) and \(X(\omega)\) are right-continuous for each \(\omega \in \Omega\). Towards this end, we set \(\mathcal{F}^+ = \bigcap_{s \geq t} \mathcal{F}_s\) and \(\mathbb{F}^+ = (\mathcal{F}^+_t)_{t \in [0, \infty]}\). Next, we observe that the \(\mathbb{F}\)-martingale \(X\) is also an \(\mathbb{F}^+\)-martingale due to Exercise 1.5.8 in Stroock and Varadhan (2006). Now, Lemma 1.1 in Föllmer (1972) yields the existence of a right-continuous version of \(X\), which we call again \(X\). Next, we fix a finite \(\mathbb{F}^+\)-stopping time \(\hat{\sigma}\) and set \(\sigma = \hat{\sigma} + 1\), which is a finite \(\mathbb{F}\)-stopping time by Theorem IV.57 in Dellacherie and Meyer (1978). Then, \(X^\sigma\) satisfies \((\text{II})\) and is therefore an uniformly integrable \(\mathbb{F}^+\)-martingale. The optional sampling theorem then also yields that \(X_{\hat{\sigma}} \in L^1\) and \(\mathbb{E}[X_{\hat{\sigma}}] = \mathbb{E}[X_0]\). Thus, \((\text{III})\) also holds for all finite \(\mathbb{F}^+\)-stopping times, and we shall assume from now on, throughout this proof, that \(\mathbb{F}\) and \(X(\omega)\) are right-continuous for each \(\omega \in \Omega\).

We now construct a nondecreasing sequence \((T_n)_{n \in \mathbb{N}}\) of \([0, \infty)\)-valued random variables such that \(\lim_{n \uparrow \infty} |X_{T_n}| = \lim_{n \uparrow \infty} \mathbb{E} [X_t | \mathcal{F}_t] \in \mathbb{F}_\infty\). For example, we can choose \(T_n\) as the first time that \(||X| - \liminf_{t \uparrow \infty} |X_t| \leq 1/n\). Then \(T_n\) is \(\mathbb{F}_\infty\)-measurable, due to the right-continuity of \(X\), for each \(n \in \mathbb{N}\) (but not necessarily a stopping time). Now we set \(Y = \liminf_{n \uparrow \infty} X_{T_n} \in \mathbb{F}_\infty\) and note that \(|Y| = \liminf_{t \uparrow \infty} |X_t|\), thus \(Y \in L^1\) by assumption.

Next, let us consider the martingale \(M\) given by \(M_t = X_t - \mathbb{E}[Y | \mathcal{F}_t]\) for each \(t \in [0, \infty)\), where we may use a right-continuous modification of the conditional expectation process thanks to \(\mathbb{F}\) being right-continuous; see again Lemma 1.1 in Föllmer (1972). Then \((\text{III})\) holds with \(X\) replaced by \(M\). We note that \(\liminf_{t \uparrow \infty} |M_t| = 0\) since \(\liminf_{n \uparrow \infty} M_{T_n} = \liminf_{n \uparrow \infty} X_{T_n} - Y = 0\), thanks to Lévy’s martingale convergence theorem and the fact that \(Y \in \mathbb{F}_\infty\).

It is sufficient to show that the martingale \(M\) is uniformly integrable, or, equivalently that \(\mathbb{E}[M] = 0\). Towards this end, we assume that there exists \(\epsilon \in (0, 1)\) such that \(\mathbb{P}(\mathbb{P}(\sup_{t \in [0, 1/\epsilon]} |M_t| > \epsilon) > \epsilon\). We then let \(\sigma_1\) be the first time that \(M\) is greater than or equal to \(\epsilon\) and \(\sigma_2\) the first time after time \(1/\epsilon\) that \(|M|\) is less
than or equal to $\varepsilon^2/4$. Then $\sigma_2$ is finite since $\lim \inf |M_t| = 0$ and, with $\tau = \sigma_1 \land \sigma_2$, we may assume that $M_\tau \in L^2$ and obtain

$$E[M_\tau] = E[M_{\sigma_1}1_{\{\sigma_1 \leq \sigma_2\}}] + E[M_{\sigma_2}1_{\{\sigma_1 > \sigma_2\}}] \geq \varepsilon P(\sigma_1 \leq 1/\varepsilon) - \varepsilon^2/4 \geq 3\varepsilon^2/4 \geq E[|M_{\sigma_2}|] \geq E[M_{\sigma_2}],$$

which contradicts (III) with $X$ replaced by $M$. Thus $M \leq 0$ and in the same manner, we can show that $M \geq 0$, which yields the statement. \qed

We briefly remark that if $X$ is nonnegative, then (IV) yields that $X$ is a martingale, thus a nonnegative supermartingale and therefore $\lim \inf_{t,\infty} |X_t| \in L^1$. Theorem 1 then yields Cherny’s result, namely the implication (IV) $\Rightarrow$ (I) provided that $X$ is nonnegative.

2 A counterexample

We now construct a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with a right-continuous martingale $X$ that satisfies (IV) and has a limit $X_\infty = \lim_{t\uparrow \infty} X_t$, but is not uniformly integrable.

Example 1. We let $\Omega = (\mathbb{N} \cup \{\infty\}) \times \{-1, 1\}$ and $\mathcal{F}$ the power set of $\Omega$. Next, we let $P$ be the probability measure on $(\Omega, \mathcal{F})$ such that $P((n, i)) = 1/(4n^2)$ for all $n \in \mathbb{N}$ and $i \in \{-1, 1\}$ and $P((\infty, i)) = (1 - \pi^2/12)/2$ for all $i \in \{-1, 1\}$. Since $\sum_{n\in\mathbb{N}} 1/(2n^2) = \pi^2/12 \in (0, 1)$, this yields indeed a probability measure on $(\Omega, \mathcal{F})$.

We let $\sigma : \Omega \to [0, \infty]$, $(\omega_1, \omega_2) \mapsto \omega_1$ denote the first component and $D : \Omega \to \{-1, 1\}$, $(\omega_1, \omega_2) \mapsto \omega_2$ the second component of each scenario $(\omega_1, \omega_2) \in \Omega$. Then $\sigma$ and $D$ are independent and $P(\sigma = x) = 1/(2x^2) 1_{x \in \mathbb{N}}$ for all $x \in [0, \infty]$; $P(\sigma = \infty) = 1 - \pi^2/12$, and $P(D = -1) = 1/2 = P(D = 1)$.

We now set $X \equiv D \sigma^2 1_{[\sigma, \infty]}$ and let $\mathbb{F}$ denote the natural filtration of $X$. To wit, $X$ is a martingale that at time $\sigma$ jumps to either $\sigma^2$ or $-\sigma^2$ provided that $\sigma$ is finite. In particular, we have

$$X_\infty = \lim_{t\uparrow \infty} X_t = D \sigma^2 1_{[\sigma, \infty]}.$$

Next, we observe that

$$E[|X_\infty|] = E[\sigma^2 1_{[\sigma, \infty]}] = \sum_{n\in\mathbb{N}} n^2 = \infty.$$

Hence, $X_\infty \notin L^1$, and $X$ is not a uniformly integrable martingale.

Now, we let $\tau$ be an arbitrary finite stopping time and set $u = \tau((\infty, -1)) \vee \tau((\infty, 1)) \in [0, \infty)$. Thus, $\{\sigma = \infty\} \subset \{\tau \leq u\} \subset \mathcal{F}_u$, which again yields $\{\sigma > u\} \subset \{\tau \leq u\}$ since $\{\sigma > u\}$ is the smallest event in $\mathcal{F}_u$ that contains $\{\sigma = \infty\}$. Thus, since

$$\{\tau \land \sigma \leq u\} = \{\sigma \leq u\} \cup \{\tau \leq u\} \supset \{\sigma \leq u\} \cup \{\sigma > u\} = \Omega,$$

the stopping time $\tau \land \sigma$ is uniformly bounded by $u$ and we obtain that $X_\tau = X_\tau^\sigma = X_{\tau \land \sigma} \in L^1$ and, by the optional sampling theorem again, that $E[X_\tau] = E[X_{\tau \land \sigma}] = 0 = E[X_0]$. Hence, $X$ satisfies (IV) but not (I) and therefore neither (II) nor (III).

We remark that Dellacherie (1970) discusses the filtration of a similar example.

In order to motivate the arguments in the next section, we now slightly modify Example 1 by extending the underlying filtration.
Example 2. We let $(\Omega, \mathcal{F}, \mathbb{P})$ be an arbitrary probability space that supports a right-continuous martingale $X = D\sigma^2 \mathbb{1}_{[\sigma, \infty[}$ with the same distribution as in Example 1 and an independent $\mathcal{F}_0$–measurable $(0, 1)$–uniformly distributed random variable $U$.

Our goal is to construct a finite stopping time $\tau$ such that $X_\tau \notin L^1$. Thus, under this enlarged filtration, the previous example is not a counterexample for the implication $[\text{IV}] \implies [\text{I}]$. Indeed, we note that $\tau = 1/U$ is a finite stopping time and $|X_\tau| = \sigma^2 \mathbb{1}_{\{\sigma \leq 1/U\}}$. Therefore,

$$
\mathbb{E}[|X_\tau|] = \mathbb{E}[\sigma^2 \mathbb{1}_{\{\sigma \leq 1/U\}}] = \mathbb{E} \left[ \sum_{n=1}^{[1/U]} n^2 \frac{1}{2n^2} \right] = \frac{1}{2} \mathbb{E} \left[ \left\lfloor \frac{1}{U} \right\rfloor \right] \geq \mathbb{E} \left[ \frac{1}{2U} \right] - \frac{1}{2} = \int_0^1 \frac{1}{2y} \, dy - \frac{1}{2} = \infty,
$$

where $\lfloor \cdot \rfloor$ denotes the Gauss brackets, by independence of $U$ and $X$.

Example 2 indicates that if “randomized stopping” is possible, a non-uniformly integrable martingale $X$ will not satisfy $[\text{IV}]$. In the next section, we will prove this assertion.

3 An additional randomization

In this section, we show that the implication $[\text{IV}] \implies [\text{I}]$ holds if we may randomize stopping times. More precisely, we shall make the following assumption:

There exists a $(0, 1)$–uniformly distributed random variable $U$ and a finite stopping time $\eta$, such that $U$ is $\mathcal{F}_\eta$–measurable. (9n)

We emphasize that $(9n)$ is an assumption on the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and not on the stochastic process $X$. We recall that we already argued that $X$ is a martingale if $[\text{IV}]$ holds. The conclusion that $X$ is also a uniformly integrable martingale, if additionally $(9n)$ holds, follows then from the following theorem.

**Theorem 2.** If $[\text{IV}]$ and $(9n)$ hold then so does $[\text{I}]$, to wit, $X$ is then a uniformly integrable martingale.

**Proof.** Exactly as in the proof of Theorem 1, we may assume, without loss of generality, that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. Lemmata 1 and 2 below then yield that $\liminf_{t \uparrow \infty} |X_t| \in L^1$ and the implication $[\text{III}] \implies [\text{I}]$ proven in Theorem 1 yields the assertion.

**Lemma 1.** Assume that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. If $X_\tau \in L^1$ for all finite stopping times $\tau$ and $(9n)$ holds then $\mathbb{E}[\liminf_{t \uparrow \infty} |X_t| | \mathcal{F}_\eta] < \infty$.

**Proof.** We define the event

$$
A = \left\{ \mathbb{E} \left[ \liminf_{t \uparrow \infty} |X_t| \big| \mathcal{F}_\eta \right] = \infty \right\} \in \mathcal{F}_\eta.
$$

We need to argue that $\mathbb{P}(A) = 0$. Towards this end, we assume that $\mathbb{P}(A) > 0$ and define the function $g : [0, 1] \to [0, \infty]$ by $t \mapsto 1/\mathbb{P}(A \cap \{U \leq t\})$, where $U$ is the uniformly distributed random variable of $(9n)$. We note that the function $1/g$ is continuous and nondecreasing and set $t_\infty = \sup\{t \in [0, 1] | g(t) = \infty\}$. Then we have $\mathbb{P}(A \cap \{U \leq t_\infty\}) = 0$ and $\mathbb{P}(A \cap \{U \leq t\}) > 0$ for all $t > t_\infty$, which yields that $1_A g(U)$ (with $0 \times \infty = 0$) is finite (almost surely).
We now let $\sigma$ denote the first time $t$ after $\eta$ such that $\mathbb{E}[|X_t||\mathcal{F}_\eta]$ is greater than or equal to $g(U)$ and note that $\sigma$ is a stopping time. Then, Fatou’s lemma yields that $\sigma$ is finite on $A$. We now set $\tau = \eta 1_{\Omega \setminus A} + \sigma 1_A$, which is again a finite stopping time, and observe

$$\mathbb{E}[|X_\tau|] \geq \mathbb{E}[1_A|X_\sigma|] = \mathbb{E}[1_A \mathbb{E}[|X_\sigma||\mathcal{F}_\eta]] \geq \mathbb{E}[1_A g(U)] \geq \sum_{n \in \mathbb{N}} \mathbb{P}(A \cap \{\mathbb{P}(A \cap \{U \leq t\})_{t=U} \leq \frac{1}{n}\}) = \infty.$$  

Here the last inequality follows from Tonelli’s theorem and the last equality follows from the fact that for each $n \geq 1/\mathbb{P}(A)$ the corresponding term in the sum equals $1/n$. To see this, fix $n \geq 1/\mathbb{P}(A)$, let $t_n = \sup\{t \in [0, 1] | g(t_n) \geq n\}$, and use the fact that $\mathbb{P}(A \cap \{U \leq t_n\}) = 1/n$. The last display contradicts the assumption and thus yields $\mathbb{P}(A) = 0$. \hfill \Box

**Lemma 2.** Assume that $\mathbb{F}$ and $X(\omega)$ are right-continuous for each $\omega \in \Omega$. If $X_\tau \in L^1$ for all finite stopping times $\tau$ and $\mathbb{E}[\lim \inf_{t \uparrow \infty} |X_t| |\mathcal{F}_\eta] < \infty$ holds for some finite stopping time $\eta$ then $\lim \inf_{t \uparrow \infty} |X_t| \in L^1$.

**Proof.** We let $Y = (Y_t)_{t \in [0, \infty)}$ denote the right-continuous modification of the finite-valued conditional expectation process

$$\mathbb{E} \left[ \lim \inf_{s \uparrow \infty} |X_s| \mid \mathcal{F}_\eta \right]_{t \in [0, \infty)}.$$  

For each $\kappa > 0$ the process $(Y_t 1_{\{Y_0 \leq \kappa\}})_{t \in [0, \infty)}$ is a uniformly integrable martingale under its natural filtration and Lévy’s martingale convergence theorem yields that $1_{\{Y_0 \leq \kappa\}} \lim_{t \uparrow \infty} Y_t = 1_{\{Y_0 \leq \kappa\}} \lim_{s \uparrow \infty} |X_s|$, and thus $\lim_{t \uparrow \infty} Y_t = \lim_{s \uparrow \infty} |X_s|$.

We now let $\tau$ denote the first time after time $\eta$ such that $|X|$ is greater than or equal to $Y_t - 1$, which is a stopping time. Moreover, $\tau$ is finite since $\lim \inf_{t \uparrow \infty} |X_t| > \lim_{t \uparrow \infty} Y_t - 1$. We then obtain

$$\mathbb{E} \left[ \lim \inf_{t \uparrow \infty} |X_t| \right] = \mathbb{E} [Y_\tau] \leq 1 + \mathbb{E} [|X_\tau|] < \infty,$$  

which yields the statement. \hfill \Box

**References**


