# OPTIMAL TRADING STRATEGIES AND THE BESSEL PROCESS[1]

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It is shown that delta hedging is the optimal trading strategy in terms of minimal required initial capital to replicate a given terminal payoff in a continuous-time Markovian context. This holds true in market models where no equivalent local martingale measure exists but only a square-integrable market price of risk. A new probability measure is constructed, which takes the place of an equivalent local martingale measure. In order to ensure the existence of the delta hedge, sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration. For a precise statement of the assumptions, proofs of the statements, further references and results we refer to Ruf (2010).

## 1. STOCK PRICE MODEL AND WEALTH PROCESSES

We use the notation $\mathbb{R}_+^n := \{s = (s_1, \ldots, s_n)^\mathsf{T} \in \mathbb{R}^n, s_i > 0, \text{ for all } i = 1, \ldots, n\}$, fix a time horizon $T$ and assume a market where the stock price processes are modelled as positive continuous Markovian semimartingales. That is, we consider a financial market $S(\cdot) = (S_1(\cdot), \ldots, S_n(\cdot))^\mathsf{T}$ of the form

$$dS_i(t) = S_i(t) \left( \mu_i(t, S(t))dt + \sum_{k=1}^{n} \sigma_{i,k}(t, S(t))dW_k(t) \right) \tag{1}$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$ starting at $S(0) \in \mathbb{R}_+^n$ and a money market $B(\cdot)$. Here $\mu : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}^n$ denotes the mean rate of return and $\sigma : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}^{n \times n}$ the volatility. Both functions are assumed to be measurable. For the sake of convenience we only look at discounted (forward) prices and set the interest rates constant to zero, that is, $B(\cdot) \equiv 1$. The flow of information is modelled by a right-continuous filtration $\mathcal{F}(\cdot)$ such that $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))^\mathsf{T}$ is an $n$-dimensional Brownian motion with independent components. We only consider mean rates of return $\mu$ and volatilities $\sigma$ which imply that the stock prices $S_1(\cdot), \cdots, S_n(\cdot)$ exist and are unique

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and strictly positive. We denote by $a(\cdot,\cdot) = \sigma(\cdot,\cdot)\sigma^{\mathsf{T}}(\cdot,\cdot)$ the covariance process of the stocks in the market.

Furthermore, we assume here that $\sigma(t, S(t))$ is invertible for all $t \in [0;T]$ and that the market price of risk

$$\theta(t, S(t)) := \sigma^{-1}(t, S(t))\mu(t, S(t))$$

satisfies the integrability condition $\int_0^T \|\theta(t, S(t))\|^2 dt < \infty$ almost surely.

Based upon the market price of risk, we are now ready to define the stochastic discount factor as

$$Z^\theta(t) := \exp\left(-\int_0^t \theta^{\mathsf{T}}(u, S(u))dW(u) - \frac{1}{2}\int_0^t \|\theta(u, S(u))\|^2 du\right)$$

for all $t \in [0,T]$. In classical no-arbitrage theory, $Z^\theta(\cdot)$ represents the Radon-Nikodym derivative which translates the "real-world" measure into the generic "risk-neutral" measure with the money market as the underlying. Since in this work we explicitly want to allow a "Free Lunch with Vanishing Risk", we shall not assume that the stochastic discount factor $Z^\theta(\cdot)$ is a true martingale. Thus, we can only rely on a local martingale property of $Z^\theta(\cdot)$.

We denote the number of stocks held by an investor with initial capital $v > 0$ at time $t$ by $\eta(t) = (\eta_1(t), \ldots, \eta_n(t))^{\mathsf{T}}$ and the corresponding wealth process by $V^{v,\eta}(\cdot)$. To wit,

$$dV^{v,\eta}(t) = \sum_{i=1}^n \eta_i(t) dS_i(t)$$

for all $t \in [0,T]$. We call $\eta$ a *trading strategy* or in short, a *strategy*. To ensure that $V^{v,\eta}(\cdot)$ is well-defined and to exclude doubling strategies we restrict ourselves to trading strategies which satisfy $V^{v,\eta}(t) \geq 0$ for all $t \in [0,T]$.

If $Y$ is a nonnegative $\mathcal{F}(T)$-measurable random variable such that $\mathbb{E}[Y|\mathcal{F}(t)]$ is a function of $S(t)$ for all $t \in [0,T]$, we use the Markovian structure of $S(\cdot)$ to denote conditioning on the event $\{S(t) = s\}$ by $\mathbb{E}^{t,s}[Y]$.

## 2. HEDGING

In the following, we shall call $(t,s) \in [0,T] \times \mathbb{R}^n_+$ a *point of support for* $S(\cdot)$ if there exists some $\omega \in \Omega$ such that $S(t,\omega) = s$. We define for any measurable function $p : \mathbb{R}^n_+ \to [0,\infty)$ a candidate $h^p : [0,T] \times \mathbb{R}^n_+ \to [0,\infty)$ for the hedging price of the corresponding European option:

$$h^p(t,s) := \mathbb{E}^{t,s}\left[\frac{Z^\theta(T)}{Z^\theta(t)} p(S(T))\right]. \tag{2}$$

Equation (2) has appeared as the "real-world pricing formula" in the Benchmark approach, compare Platen and Heath (2006), Equation (9.1.30). Applying Itô's rule to Equation (2) yields the following result. Here we write $D_i$ and $D^2_{i,j}$ for the partial derivatives with respect to the variable $s$.

**Theorem 2.1 (Markovian representation for non path-dependent European claims)** *Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that the function $h^p$ of Equation (2) is sufficiently differentiable, or more precisely, for all points of support $(t, s)$ for $S(\cdot)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of $(t, s)$. Then, with $\eta_i^p(t, s) := D_i h^p(t, s)$, for all $i = 1, \ldots, n$ and $(t, s) \in [0, T] \times \mathbb{R}_+^n$, and with $v^p := h^p(0, S(0))$, we get*

$$V^{v^p,\eta^p}(t) = h^p(t, S(t))$$

*for all $t \in [0, T]$. Furthermore, the strategy $\eta^p$ is optimal in the sense that for any $\tilde{v} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$ we have $\tilde{v} \geq v^p$. Furthermore, $h^p$ satisfies the PDE*

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j a_{i,j}(t, s) D_{i,j}^2 h^p(t, s) = 0 \tag{3}$$

*at all points of support $(t, s)$ for $S(\cdot)$.*

Next, we will provide sufficient conditions under which the function $h^p$ is sufficiently smooth. For that we need the following definition.

**Definition 2.1 (Locally Lipschitz and bounded)** *We call a function $f : [0, T] \times \mathbb{R}_+^n \to \mathbb{R}$ locally Lipschitz and bounded on $\mathbb{R}_+^n$ if for all $s \in \mathbb{R}_+^n$ the function $t \to f(\cdot, s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that*

$$\sup_{\substack{\frac{1}{M} \leq \|y\|, \|z\| \leq M \\ y \neq z}} \frac{|f(t, y) - f(t, z)|}{\|y - z\|} + \sup_{\frac{1}{M} \leq \|y\| \leq M} |f(t, y)| \leq C(M),$$

*for all $t \in [0, T]$.*

Using the theory of stochastic flows and Schauder estimates, we obtain the necessary differentiability of $h^p$.

**Theorem 2.2** *We assume that the functions $\theta_k$ and $\sigma_{i,k}$ are for all $i, k = 1, \ldots, n$ locally Lipschitz and bounded. We furthermore assume that for all points of support $(t, s)$ for $S(\cdot)$ there exist $C_1, C_2 > 0$ and some neighborhood $\mathcal{U}$ of $(t, s)$ such that $\sum_{i,j=1}^{n} a_{i,j}(u, y) \xi_i \xi_j > C_1 \|\xi\|^2$ for all $\xi \in \mathbb{R}^n$ and $h^p(u, y) \leq C_2$ for all $(u, y) \in \mathcal{U}$. Then, there exists for all points of support $(t, s)$ for $S(\cdot)$ some neighborhood $\tilde{\mathcal{U}}$ of $(t, s)$ such that the function $h^p$ defined in Equation (2) is in $C^{1,2}(\tilde{\mathcal{U}})$.*

## 3. CHANGE OF MEASURE

To simplify the computation of $h^p$, one can perform a change of measure after making some technical assumptions. For that, we rely on the techniques developed by Föllmer (1972), Meyer (1972), and Delbaen and Schachermayer (1995), Section 2.

**Theorem 3.1 (Generalized change of measure)** *There exists a measure $\mathbb{Q}$ such that for all $\mathcal{F}(T)$-measurable random variables $Y \geq 0$ we have*

$$\mathbb{E}^{\mathbb{P}}\left[Z^{\theta}(T)\, Y\right] = \mathbb{E}^{\mathbb{Q}}\left[Y \mathbf{1}_{\left\{1/Z^{\theta}(T)>0\right\}}\right]$$

*where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation with respect to the new measure $\mathbb{Q}$. That is, $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$. Under this measure $\mathbb{Q}$, the process $\widetilde{W}(\cdot) = \left(\widetilde{W}_1(\cdot), \ldots \widetilde{W}_n(\cdot)\right)^{\mathsf{T}}$ with*

$$\widetilde{W}_k(t \wedge \tau^{\theta}) := W_k(t \wedge \tau^{\theta}) + \int_0^{t\wedge\tau^{\theta}} \theta_k(u, S(u))du$$

*for all $k = 1, \ldots, n$ and $t \in [0, T]$ is an $n$-dimensional Brownian motion stopped at time $\tau^{\theta} := \lim_{i\to\infty} \inf\{t \in [0, T] : Z^{\theta}(t) \geq i\}$.*

Furthermore, it is now easy to show that we have, up to the stopping time $\tau^{\theta}$, the following dynamics for $S(\cdot)$ and $1/Z^{\theta}(\cdot)$ under $\mathbb{Q}$:

$$dS_i(t) = S_i(t) \sum_{k=1}^{n} \sigma_{i,k}(t, S(t)) d\widetilde{W}_k(t),$$
$$d\left(\frac{1}{Z^{\theta}(t)}\right) = \frac{1}{Z^{\theta}(t)} \sum_{k=1}^{n} \theta_k(t, S(t)) d\widetilde{W}_k(t),$$

for all $i = 1, \ldots, n$ and $t \in [0, T]$. One can also prove a generalization of Bayes' rule for Girsanov-type measure changes to the measure change suggested by Theorem 3.1.

## 4. THREE-DIMENSIONAL BESSEL PROCESS

We illustrate the techniques presented here with a toy model. Let $n = 1$ and $S(\cdot)$ be a three-dimensional Bessel process. To wit,

$$dS(t) = \frac{1}{S(t)} dt + dW(t)$$

for all $t \in [0, T]$. For any payoff function $p(\cdot) \geq 0$ we obtain from Theorem 3.1 that $h^p(t, s) := \mathbb{E}^{\mathbb{Q},t,s}[p(S(T))\mathbf{1}_{\{S(T)>0\}}]$, where $S(\cdot)$ is now a $\mathbb{Q}$-Brownian motion stopped at zero. For example, if $p(s) \equiv s$, that is, the stock itself, then $h^p(t, s) = \mathbb{E}^{\mathbb{Q},t,s}[S(T)] = s$. To wit, the hedging price of the stock is exactly its price and the optimal strategy is to hold the stock. However, if $p(s) \equiv 1$, then we compute

$$h^p(t, s) = \mathbb{Q}^{t,s}(S(T) > 0) = 2\Phi\left(\frac{s}{\sqrt{T-t}}\right) - 1 < 1.$$

There is a trading strategy $\eta^p$, which yields exactly one monetary unit at time $T$ and costs $h^p(0, s)$ at time $0$ if the stock price equals $s$. By Theorem 2.1, there is no other strategy which needs less initial capital and leads to a nonnegative wealth process. Furthermore, we have the representation

$$\eta^p(t,s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right),$$

where $\phi$ denotes the standard normal density function.

## 5. FURTHER RESULTS AND A VERY INCOMPLETE LIST OF REFERENCES

The hedging results of Theorem 2.1 also hold when the number of Brownian motions is larger than the number of stocks. However, in this case one has to pay attention to the choice of the market price of risk, which is no longer unique. The PDE (3) usually allows for several solutions satisfying the same boundary conditions and being of polynomial growth. The function $h^p$ can be characterized as the minimal nonnegative solution of that PDE.

This work is motivated by the desire to better understand the question of hedging in stochastic portfolio theory and in the Benchmark process. For an overview of the former, we recommend the survey paper by Fernholz and Karatzas (2009). Furthermore, in Fernholz and Karatzas (2010) optimal trading strategies to hold the market portfolio at time horizon $T$ are discussed. For an introduction to the Benchmark process, developed by Eckhard Platen and co-authors, we refer to the monograph by Platen and Heath (2006). In particular, Theorem 2.1 generalizes Platen and Hulley (2008), Proposition 3, where the same statement is shown for a one-dimensional market with a time-transformed squared Bessel process of dimension four modelling the stock price process.

The results presented here also yield optimal trading strategies for models where the stock price has a bubble. A stock is said to have a bubble if its price does not equal its “intrinsic value”. We refer to Jarrow et al. (2007) for a precise definition and further references.

Theorem 2.2 generalizes recent Feynman-Kac type theorems by Heath and Schweizer (2000), Janson and Tysk (2006), and Ekström and Tysk (2009) for the stock price models presented here.

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