ONLINE APPENDIX TO: NONPARAMETRIC IDENTIFICATION OF THE MIXED HAZARD MODEL USING MARTINGALE-BASED MOMENTS

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Appendix B. The Proofs of Theorem 2.2 and Proposition 2.3

The proof of Theorem 2.2 requires two analytic results, which we provide in the following two lemmas.

Lemma B.1. *Let $(c, c') \in \mathbb{R}^2 \setminus \{(0,0)\}$, $V, W \in \mathcal{V}$, and let $f, g : [0,\infty) \mapsto [0,\infty)$ denote two right-continuous functions. Assume that there exist some $t \geq 0$ and $\varepsilon > 0$ such that*

$$f_t < f_s, \qquad \text{for all } s \in (t, t+\varepsilon) \tag{B.1}$$

and define the two right-continuous functions

$$\widetilde{f}_s = f_s + c\mathbf{1}_{s \in [t, t+\varepsilon)}, \qquad s \geq 0; \tag{B.2}$$

$$\widetilde{g}_s = g_s + c'\mathbf{1}_{s \in [t, t+\varepsilon)}, \qquad s \geq 0. \tag{B.3}$$

Then we have the equivalence

$$(\mathcal{L}_V, f, \widetilde{f}) = \left(\mathcal{L}_{\frac{c'}{c}W}, \frac{c}{c'}g, \frac{c}{c'}\widetilde{g}\right) \qquad \text{if and only if} \qquad (\mathcal{L}_V \circ f, \mathcal{L}_V \circ \widetilde{f}) = (\mathcal{L}_W \circ g, \mathcal{L}_W \circ \widetilde{g}).$$

Proof. The "only if" direction is trivial; hence let us assume that $(\mathcal{L}_V \circ f, \mathcal{L}_V \circ \widetilde{f}) = (\mathcal{L}_W \circ g, \mathcal{L}_W \circ \widetilde{g})$. Since this implies that

$$\mathcal{L}_V(f_s) - \mathcal{L}_V\left(\widetilde{f}_s\right) = \mathcal{L}_W(g_s) - \mathcal{L}_W\left(\widetilde{g}_s\right)$$

we then have $c \neq 0$ and $c' \neq 0$. By swapping f with $\widetilde{f}$ (and g with $\widetilde{g}$) we may assume that $c > 0$. The monotonicity of $\mathcal{L}_V$ and $\mathcal{L}_W$ then also yields that $c' > 0$. Next, define the functions

$$\varphi : (\mathcal{L}_V(\infty), 1] \to \mathbb{R}; \quad s \mapsto \mathcal{L}_V\left(\mathcal{L}_V^{-1}(s) + c\right);$$
$$\psi : (\mathcal{L}_W(\infty), 1] \to \mathbb{R}; \quad s \mapsto \mathcal{L}_W\left(\mathcal{L}_W^{-1}(s) + c'\right)$$

and observe

$$\varphi\left(\mathcal{L}_V(f_s)\right) = \mathcal{L}_V\left(\widetilde{f}_s\right) = \mathcal{L}_W\left(\widetilde{g}_s\right) = \psi\left(\mathcal{L}_W(g_s)\right) = \psi\left(\mathcal{L}_V(f_s)\right), \quad s \in [t, t+\varepsilon).$$

Now, (B.1) guarantees that the range of the function $[t, t+\varepsilon) \ni s \mapsto \mathcal{L}_V(f_s)$ contains an open non-empty interval. This then yields that $\varphi = \psi$ since the two functions φ and ψ are analytic; see Chapter II.5 in Widder (1946).

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Next, note that $\mathcal{L}_V(0) = \mathcal{L}_W(0)$ and proceed by induction as follows. Assume that we have argued $\mathcal{L}_V((n-1)c) = \mathcal{L}_W((n-1)c')$ for some $n \in \mathbb{N}$. Then we get

$$\mathcal{L}_V(nc) = \varphi(\mathcal{L}_V((n-1)c)) = \psi(\mathcal{L}_W((n-1)c')) = \mathcal{L}_W(nc'), \qquad n \in \mathbb{N}.$$

Now, consider the random variable $W' = (c'/c)W \geq 0$ with Laplace transform $\mathcal{L}_{W'}(x) = \mathcal{L}_W(xc'/c)$ for all $x \geq 0$. We thus have

$$\mathcal{L}_V(nc) = \mathcal{L}_{W'}(nc), \qquad n \in \mathbb{N}.$$

Since, moreover, $\sum_{n\in\mathbb{N}} 1/(cn) = \infty$, the Müntz theorem yields $\mathcal{L}_V = \mathcal{L}_{W'} = \mathcal{L}_{(c'/c)W}$; see Feller (1968), in particular, Theorem 2 and the representation of (1.7) in that paper. We then also get that

$$f = \mathcal{L}_V^{-1}(\mathcal{L}_V(f)) = \mathcal{L}_V^{-1}(\mathcal{L}_W(g)) = \mathcal{L}_V^{-1}(\mathcal{L}_{(c/c')V}(g)) = \mathcal{L}_V^{-1}\left(\mathcal{L}_V\left(\frac{c}{c'}g\right)\right) = \frac{c}{c'}g,$$

and similarly, that $\widetilde{f} = (c/c')\widetilde{g}$. Hence, the statement follows. □

The proof of Lemma B.1 is inspired by Brinch (2007).

Lemma B.2. *Recall the setup and notation of Lemma B.1. Then there exists $n \in \mathbb{N}$ such that, for all right-continuous functions $\overline{f}, \overline{g} : [0,\infty) \mapsto [0,\infty)$ satisfying*

$$\sup_{s\in[t,t+\varepsilon)} \left(|c + f_s - \overline{f}_s| + |c' + g - \overline{g}_s|\right) < \frac{1}{n} \tag{B.4}$$

and

$$(\mathcal{L}_V, f, \overline{f}) \neq \left(\mathcal{L}_{\kappa W}, \frac{1}{\kappa}g, \frac{1}{\kappa}\overline{g}\right) \quad \textit{for each constant } \kappa > 0,$$

we have

$$(\mathcal{L}_V \circ f, \mathcal{L}_V \circ \overline{f}) \neq (\mathcal{L}_W \circ g, \mathcal{L}_W \circ \overline{g}),$$

Proof. We may assume in the proof that $\mathcal{L}_V \circ f = \mathcal{L}_W \circ g$; otherwise nothing is to be argued. We recall the right-continuous functions $\widetilde{f}$ and $\widetilde{g}$ from (B.2) and (B.3). Let us first assume that $\mathcal{L}_V \circ \widetilde{f} = \mathcal{L}_W \circ \widetilde{g}$ on $[t, t+\varepsilon)$. Then we also have $\mathcal{L}_V \circ \widetilde{f} = \mathcal{L}_W \circ \widetilde{g}$. Lemma B.1 now yields that $\mathcal{L}_V = \mathcal{L}_{(c'/c)W}$. This then yields the statement.

Hence, let us now assume that $\mathcal{L}_V \circ \widetilde{f} \neq \mathcal{L}_W \circ \widetilde{g}$ on $[t, t+\varepsilon)$. Thanks to the continuity properties of the two functions $\mathcal{L}_V$ and $\mathcal{L}_W$, there exists $n \in \mathbb{N}$ such that $\mathcal{L}_V \circ \overline{f} \neq \mathcal{L}_W \circ \overline{g}$ whenever (B.4) holds. This concludes the proof. □

Proof of Theorem 2.2. We only need to argue one direction. Hence, let us assume that $(\mathcal{L}_V, F) \neq (\mathcal{L}_{\kappa W}, G/\kappa)$ for each constant $\kappa > 0$. In the notation of Assumption (R), fix $\omega \in A_1 \cap A_2$, set $f = F(\omega), g = G(\omega), t = \rho(\omega), c = C(\omega), c' = C'(\omega)$, and $\varepsilon = \mathcal{E}(\omega)$. Then Lemma B.2 yields the existence of $N(\omega)$ such that

$$(\mathcal{L}_V \circ f, \mathcal{L}_V \circ F(\widetilde{\omega})) \neq (\mathcal{L}_W \circ g, \mathcal{L}_W \circ G(\widetilde{\omega})), \qquad \widetilde{\omega} \in O_{f,g,t,c,c',\varepsilon,N(\omega)}. \tag{B.5}$$

Setting $N(\omega) = 1$ for all $\omega \notin A_1 \cap A_2$ then yields a mapping $N : \Omega \to \mathbb{N}$. It can be checked that N is measurable, hence N is a random variable.

To proceed with the argument, let us consider the product space $(\Omega\times\Omega, \mathscr{H}\times\mathscr{H}, \mathsf{P}\times\mathsf{P})$ and identify all random variables $X : \Omega \to \mathbb{R}$ with random vectors $(X^1, X^2) : \Omega\times\Omega \to \mathbb{R}^2$ by $X^1(\omega_1, \omega_2) = X(\omega_1)$ and $X^2(\omega_1,\omega_2) = X(\omega_2)$ for all $(\omega_1,\omega_2) \in \Omega \times \Omega$. Next, define the set

$$B = \widetilde{A} \cap \{(\omega_1, \omega_2) : \omega_2 \in O_{F^1,G^1,\rho^1,C^1,C'^1,\mathcal{E}^1,N^1}\} \subset \Omega \times \Omega, \tag{B.6}$$

where

$$\widetilde{A} = \{(\omega_1, \omega_2) : \omega_1 \in A_1 \cap A_2\} \in \mathscr{H} \times \mathscr{H}.$$

Similar as in the proof of Lemma C.1 below, we can see that $B \in \mathscr{H} \times \mathscr{H}$.

Next, note that Assumption (R) guarantees that $(\mathsf{P} \times \mathsf{P})[B] > 0$. Moreover, on B we have

$$|\mathcal{L}_V(F^1) - \mathcal{L}_W(G^1)| + |\mathcal{L}_V(F^2) - \mathcal{L}_W(G^2)| > 0,$$

thanks to (B.5). Since the distribution of (F, G) under P is the same as the one of (F^1, G^1) under the product measure $\mathsf{P} \times \mathsf{P}$, as well as the one of (F^2, G^2), we now obtain

$$\mathsf{E}[|\mathcal{L}_V(F) - \mathcal{L}_W(G)|] = \frac{1}{2}\mathsf{E}_{\mathsf{P}\times\mathsf{P}}\big[|\mathcal{L}_V(F^1) - \mathcal{L}_W(G^1)| + |\mathcal{L}_V(F^2) - \mathcal{L}_W(G^2)|\big] > 0,$$

which proves that $\mathcal{L}_V(F) \neq \mathcal{L}_W(G)$, and hence the statement follows. □

Remark B.3. In Assumption (R), each $\omega \in A_1 \cap A_2$ is matched with an event that has positive probability. After reading the proof of Theorem 2.2, the diligent reader might possibly wonder why it does not suffice to consider those paths that can be paired with a single path $\widetilde{\omega}$ (instead of an event with positive probability). To require now that the family of those ω's has positive probability is less restrictive than requiring that the event $A_1 \cap A_2$ in Assumption (R) has positive probability. Instead of considering the product measure in the proof of Theorem 2.2 one could conjecture that it suffices to use

$$|\mathcal{L}_V(F(\omega)) - \mathcal{L}_W(G(\omega))| + |\mathcal{L}_V(F(\widetilde{\omega})) - \mathcal{L}_W(G(\widetilde{\omega}))| > 0$$

for all such pairs $(\omega, \widetilde{\omega})$. However, we were not able to construct a measurable selection to pick the "right partner." Indeed, if $\omega \in \Omega$ can be paired with some candidate $\widetilde{\omega}$, it can also usually be paired with uncountably many other candidates $\widetilde{\omega}$ and it is not clear how to pick one in a measurable way. □

Proof of Proposition 2.3. Fix some continuous functions $\alpha, \beta \in \mathcal{A}$ and set

$$F_t = \int_0^t \alpha(s, Z_s)\mathrm{d}s \qquad \text{and} \qquad G_t = \int_0^t \beta(s, Z_s)\mathrm{d}s$$

By assumption there exist $x, y \in \mathcal{Z}$ and $t_1 > 0$ such that $\alpha(t_1, x) \neq \alpha(t_2, y)$. Set now $\rho = t_1$ and $\mathcal{E} = t_2 - t_1$. We directly get that $\mathsf{P}[A_1] = 1$ since the function α was assumed to be strictly positive.

Next, observe that the continuity of α implies that there exists $\delta \in (0, t_1)$ such that

$$\int_{t_1-\delta}^{t_1} \alpha(s, x)\mathrm{d}s \neq \int_{t_1-\delta}^{t_1} \alpha(s, y)\mathrm{d}s.$$

Hence, we can construct, in a measurable way, a random variable $\overline{Z}$, taking values in $\{x, y\}$, such that

$$C = \int_{t_1-\delta}^{t_1} \big(\alpha(s, \overline{Z}) - \alpha(s, Z_s)\big)\,\mathrm{d}s \neq 0, \tag{B.7}$$

where the inequality is with probability 1. Hence, C is $\mathbb{R} \setminus \{0\}$–valued. Similarly, we define the random variable

$$C' = \int_{t_1-\delta}^{t_1} \big(\beta(s, \overline{Z}) - \beta(s, Z_s)\big)\,\mathrm{d}s.$$

We now want to argue that $\mathsf{P}[A_2] > 0$ with this choice of random variables $\rho, \mathcal{E}$, and (C, C'). Indeed, we will argue that $\mathsf{P}[A_2] = 1$. To this end, fix $n \in \mathbb{N}$ and $\omega \in \Omega$ such that $z = Z(\omega)$ is in the support of the process Z, which happens with probability one. Moreover, set

$$f = F(\omega) = \int_0^\cdot \alpha(s, z_s)\mathrm{d}s, \qquad g = G(\omega) = \int_0^\cdot \beta(s, z_s)\mathrm{d}s,$$

$t = \rho(\omega)$, $c = C(\omega)$, $c' = C'(\omega)$, and $\varepsilon = \mathcal{E}(\omega)$. Next, let $\widetilde{z}$ denote a $\mathcal{Z}$–valued path with

$$\widetilde{z}_s = z_s \mathbf{1}_{s<t_1-\delta \text{ or } s>t_1} + \overline{Z}(\omega)\mathbf{1}_{t_1-\delta\leq s\leq t_1}, \qquad s \geq 0.$$

Hence $\widetilde{z}$ equals the given path z outside of the interval $[t_1 - \delta, t_1]$. On that interval, $\widetilde{z}$ is constant and takes value x or y. We now want to prove that the event

$$\begin{aligned} O_{f,g,t,c,c',\varepsilon,n} &= \left\{\widetilde{\omega} \in \Omega : \sup_{s\in[t,t+\varepsilon)} \big(|c + f_s - F_s| + |c' + g_s - G_s|\big) < \frac{1}{n}\right\} \\ &= \left\{\widetilde{\omega} \in \Omega : \sup_{s\in[t,t+\varepsilon)} \bigg(\left|\int_0^s \left(\alpha(u,\widetilde{z}_u) - \alpha(u, Z_u(\widetilde{\omega}))\right)\mathrm{d}u\right| \right. \\ &\qquad\qquad \left. + \left|\int_0^s \left(\beta(u,\widetilde{z}_u) - \beta(u, Z_u(\widetilde{\omega}))\right)\mathrm{d}u\right|\bigg) < \frac{1}{n}\right\} \end{aligned}$$

has positive probability. Indeed, with this representation, Lemma B.4 below yields that $\mathsf{P}[O_{f,g,t,c,c',\varepsilon,n}] > 0$, which concludes the proof. □

Lemma B.4. *Suppose Assumptions (P) and (A) hold along with* (2.5)*; i.e., we are in the mixed hazard setup. Assume, moreover, that* $\alpha \in \mathcal{A}$ *and that* z *be a* $\mathcal{Z}$*–valued function in the support of the observation process* Z*. Fix* $n \in \mathbb{N}$*,* $\overline{z} \in \mathcal{Z}$*,* $\delta \in (0, t_1)$*, and* $T > 0$*, and define the* $\mathcal{Z}$*–valued function*

$$\widetilde{z}_s = z_s \mathbf{1}_{s<t_1-\delta \text{ or } s>t_1} + \overline{z}\mathbf{1}_{t_1-\delta\leq s\leq t_1}, \qquad s \geq 0.$$

Then we have

$$\mathsf{P}\left[\sup_{s\in[0,T]} \left(\int_0^s |\alpha(u,\widetilde{z}_u) - \alpha(u, Z_u)|\,\mathrm{d}u\right) < \frac{1}{n}\right] > 0.$$

Proof. Thanks to the continuity of the function α, it suffices to show for some appropriately chosen sufficiently small $\varepsilon > 0$ that the event

$$\left\{\sup_{s\in[0,t_1-\varepsilon)} |Z_s - z_s| < \varepsilon\right\} \cap \left\{\sup_{s\in(t_1+\varepsilon,t_2-\varepsilon)} |Z_s - \overline{z}| < \varepsilon\right\} \cap \left\{\sup_{s\in(t_2+\varepsilon,T]} |Z_s - z_s| < \varepsilon\right\}$$

has positive probability. Any of the three conditions in Assumption (P) yields this. Indeed, if the covariate process Z is piecewise constant with Poisson update times this probability can be written as the product of positive probabilities of the following three events: (1) the event that up to time $t_1 - \varepsilon$ the observations stay close to z; (2) the event that a jump occurs around time t_1 to a neighbourhood of $\overline{z}$, and that another jump occurs around time t_2 to a neighbourhood of z; (3) the event that after time $t_2 + \varepsilon$ the observations stay close to z.

If the covariate process is a diffusion, the result follows from the support theorem; see, for example, Stroock and Varadhan (1972). In the case of one-dimensonsal Markov processes, a similar argument yields the claim; see for example Bruggeman and Ruf (2016). □

Appendix C. Some Results on Measurability

Assumption (R) implicitly uses two subtle but essential facts:

1. The set $O_{f,g,t,c,c',\varepsilon,n}$ is measurable such that the probability in (2.6) is well-defined.
2. The set $A_1 \cap A_2$ is measurable.

In this subsection, we provide the necessary arguments to justify these facts.

Lemma C.1. *With the notation of Assumption (R), the set $O_{f,g,t,c,c',\varepsilon,n}$ is an event; more precisely,*

$$O_{f,g,t,c,c',\varepsilon,n} \in \mathscr{F}_\infty.$$

Proof. Consider the right-continuous $\mathscr{F}_\infty$–measurable process

$$H_s = \mathbf{1}_{s\in[t,t+\varepsilon)}\big(|c+f_s-F_s| + |c'+g_s-G_s|\big), \qquad s \geq 0.$$

Then we have

$$O_{f,g,t,c,c',\varepsilon,n} = \bigcap_{q\in\mathbb{Q}\cap[0,\infty)} \{H_q < 1/n\} \in \mathscr{F}_\infty,$$

which concludes the argument. □

Lemma C.2. *With the notation of Assumption (R), we have $A_1 \cap A_2 \in \mathscr{H}$.*

Proof. It suffices to fix $n \in \mathbb{N}$ and check the measurability of the following set:

$$A^{(n)} = A_1 \cap \left\{\mathsf{P}\left[O_{f,g,t,c,c',\varepsilon,n}\right]\big|_{f=F,g=G,t=\rho,c=C,c'=C',\varepsilon=\mathcal{E}} > 0\right\},$$

where $A_1 \in \mathscr{H}$. We now use the same argument as in the proof of Theorem 2.2. Consider the product space $(\Omega\times\Omega, \mathscr{H}\times\mathscr{H}, \mathsf{P}\times\mathsf{P})$ and define, as in (B.6),

$$B^{(n)} = \overline{A} \cap \{(\omega_1,\omega_2) : \omega_2 \in O_{F^1,G^1,\rho^1,C^1,C'^1,\mathcal{E}^1,n}\} \in \mathscr{H}\times\mathscr{H},$$

where

$$\overline{A} = \{(\omega_1,\omega_2) : F^1_\rho < F^1_s, \quad \text{for each } s \in (\rho^1, \rho^1+\mathcal{E}^1)\} \in \mathscr{H}\times\mathscr{H}.$$

Next, let Y describe the conditional expectation of $\mathbf{1}_{B^{(n)}}$ given the sigma algebra $\mathscr{H}\times\{\emptyset,\Omega\}$; to wit,

$$\begin{aligned} Y &= \mathsf{E}\left[\mathbf{1}_{B^{(n)}} | \mathscr{H}\times\{\emptyset,\Omega\}\right] \\ &= \mathbf{1}_{\overline{A}} \times \mathsf{P}\left[O_{f,g,t,c,c',\varepsilon,n}\right]\big|_{f=F^1,g=G^1,t=\rho^1,c=C^1,c'=C'^1,\varepsilon=\mathcal{E}^1}. \end{aligned}$$

Then we see that

$$A^{(n)} = \{\omega_1 : Y(\omega_1,\omega_2) > 0\}.$$

Since we assumed that the sigma algebra $\mathscr{H}$ is complete, we get $A^{(n)} \in \mathscr{H}$, concluding the proof. □

Appendix D. Some Martingale Properties Of The Hazard Model

Theorem D.1. *Fix a pair $(W,G) \in \mathcal{V}\times\mathcal{F}$, assume that $\int_0^\infty F^\circ_t \mathrm{d}t < \infty$, and define the process $M = 1/\mathcal{L}_W(G)\mathbf{1}_{[\![0,\tau[\![}$. Then the following conditions are equivalent:*

(i) $\mathcal{L}_W(G) = \mathcal{L}_{V^\circ}(F^\circ)$.
(ii) *M is a uniformly integrable $(\mathscr{G}_t)$–martingale.*

Proof. Assume that (i) holds. Now fix $s, t \in [0, \infty]$ with $s < t$ and $A \in \mathscr{G}_s$. Then we can write $A \cap \{\tau > s\} = B \cap \{\tau > s\}$ for some $B \in \mathscr{F}_s$. Hence, we get

$$\begin{aligned}\mathsf{E}[M_t \mathbf{1}_A] &= \mathsf{E}[M_t \mathbf{1}_B] = \mathsf{E}\left[\frac{1}{\mathcal{L}_{V^\circ}(F_t^\circ)} \lim_{u \uparrow t} \mathbf{1}_{\{\tau > u\} \cap B}\right] = \mathsf{E}\left[\frac{1}{\mathcal{L}_{V^\circ}(F_t^\circ)} \mathbf{1}_B \mathrm{e}^{-V^\circ F_t^\circ}\right] = \mathsf{P}[B] \\ &= \mathsf{E}\left[\frac{1}{\mathcal{L}_{V^\circ}(F_s^\circ)} \mathbf{1}_B \mathrm{e}^{-V^\circ F_s^\circ}\right] = \mathsf{E}\left[\frac{1}{\mathcal{L}_{V^\circ}(F_s^\circ)} \mathbf{1}_{\{\tau > s\} \cap B}\right] = \mathsf{E}[M_s \mathbf{1}_B] = \mathsf{E}[M_s \mathbf{1}_A].\end{aligned}$$

Here, we used (2.5). Thus, M is indeed a uniformly integrable $(\mathscr{G}_t)$–martingale.

Let us now assume that (ii) holds, but $\mathcal{L}_W(G) \neq \mathcal{L}_{V^\circ}(F^\circ)$. Then (2.5) yields the existence of $t > 0$ such that

$$\mathsf{P}\left[\{\tau > t\} \cap \{\mathcal{L}_W(G) \neq \mathcal{L}_{V^\circ}(F^\circ)\}\right] > 0.$$

Thus, we have

$$M \neq \frac{1}{\mathcal{L}_{V^\circ}(F^\circ)} \mathbf{1}_{[\![0,\tau[\![}.$$

The left-hand side is a $(\mathscr{G}_t)$–martingale by assumption, the right-hand side by the implication from (i) to (ii). This, however, contradicts the uniqueness of the multiplicative decomposition of the nonnegative $(\mathscr{G}_t)$–supermartingale $\mathbf{1}_{[\![0,\tau[\![}$ as a product of a local martingale and a predictable nonincreasing process:

$$\mathbf{1}_{[\![0,\tau[\![} = M \times \mathcal{L}_W(G); \qquad \mathbf{1}_{[\![0,\tau[\![} = \frac{1}{\mathcal{L}_{V^\circ}(F^\circ)} \mathbf{1}_{[\![0,\tau[\![} \times \mathcal{L}_{V^\circ}(F^\circ);$$

see also Yoeurp (1976) and Appendix B in Perkowski and Ruf (2015). Hence, we have the implication from (ii) to (i). □

Example D.2. The choice of filtration is essential in the statement of Theorem D.1, even if there is no unobserved factor. To see this, we provide now a setup that satisfies Assumption (R). However, in this specific setup there exists $G \in \mathcal{F}$ such that the process $M = 1/\mathcal{L}_{V^\circ}(G) \mathbf{1}_{[\![0,\tau[\![}$ is a uniformly integrable $(\mathscr{E}_t)$–martingale, but $\mathcal{L}_{V^\circ}(G) \neq \mathcal{L}_{V^\circ}(F^\circ)$. Here, $(\mathscr{E}_t) \subset (\mathscr{G}_t)$ denotes the filtration generated by M itself.

Suppose that $\Omega = \{w_1, w_2\} \times [0, \infty)$, that $\zeta(w, r) = r$ for all $(w, r) \in \Omega$ and that

$$\mathsf{P}\left[\{w_1\} \times (t, \infty)\right] = \frac{1}{2} \mathrm{e}^{-t} = \mathsf{P}\left[\{w_2\} \times (t, \infty)\right], \qquad t \geq 0.$$

In particular, ζ is exponentially distributed. Moreover, let $V^\circ = 1$ and $\mathcal{F} = \{F^\circ, G\}$, where

$$\begin{aligned} F_t^\circ(w_1, r) &= t \wedge 3; & G_t(w_1, r) &= \log\left(\frac{2}{1 + \mathrm{e}^{-(1 \wedge t)}}\right) + (t-1)^+ \wedge 2, & t, r &\geq 0; \\ F_t^\circ(w_2, r) &= (t-2)^+ \wedge 1; & G_t(w_2, r) &= \log\left(\frac{2}{1 + \mathrm{e}^{-(1 \wedge t)}}\right) + (t-2)^+ \wedge 1, & t, r &\geq 0. \end{aligned}$$

Next, define τ in the same way as in Footnote 3. Then the basic setup is satisfied. It is also easy to check that Assumption (R) is satisfied with $\rho = C = 2$ and $\mathcal{E} = C' = 1$. However, clearly we have $\mathcal{L}_{V^\circ}(G) \neq \mathcal{L}_{V^\circ}(F^\circ)$.

Let us observe that

$$\begin{aligned} M_t &= \frac{2}{1 + \mathrm{e}^{-t}} \mathbf{1}_{[\![0,\tau[\![}, & t &\in [0, 1]; \\ M_t &= \frac{2}{(1 + \mathrm{e}^{-1}) \mathrm{e}^{F_1^\circ}} \times \mathrm{e}^{F_t^\circ} \mathbf{1}_{[\![0,\tau[\![}, & t &\geq 1. \end{aligned}$$

Next, let us check that M is a uniformly integrable $(\mathscr{E}_t)$–martingale. To this end, for $s, t \in [0,1]$ with $s < t$ we have, on the event $\{\tau > s\} \in \mathscr{E}_s$,

$$\mathsf{E}[M_t|\mathscr{E}_s] = \frac{2}{1+\mathrm{e}^{-t}}\mathsf{P}[\tau > t|\tau > s] = \frac{2}{1+\mathrm{e}^{-t}} \times \frac{1+\mathrm{e}^{-t}}{1+\mathrm{e}^{-s}} = M_s.$$

For $s, t \in [1, \infty]$ with $s < t$ it is also easy to check that $\mathsf{E}[M_t|\mathscr{E}_s] = M_s$. Hence, M is indeed a uniformly integrable $(\mathscr{E}_t)$–martingale.

Let us now double-check that M is indeed not a $(\mathscr{G}_t)$–martingale. The event $A = \{w_2\} \times [0, \infty)$ is $\mathscr{G}_{1/2}$–measurable, but we have

$$\begin{aligned}\mathsf{E}[M_1\mathbf{1}_A] &= \frac{2}{1+\mathrm{e}^{-1}}\mathsf{P}[\{\tau > 1\} \cap A] = \frac{2}{1+\mathrm{e}^{-1}}\mathsf{P}[A] = \frac{1}{1+\mathrm{e}^{-1}} \\ &> \frac{1}{1+\mathrm{e}^{-1/2}} = \frac{2}{1+\mathrm{e}^{-1/2}}\mathsf{P}[A] = \mathsf{E}[M_{1/2}\mathbf{1}_A].\end{aligned}$$

Here we used the fact that $\tau(\omega_2, r) = r + 2 \geq 2$, for all $r \geq 0$, on A. Thus, M is not a $(\mathscr{G}_t)$–martingale, which is consistent with the assertion of Theorem D.1. □

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