ONE-DIMENSIONAL FEYNMAN–KAC REGULARITY UNDER THE ENGELBERT–SCHMIDT CONDITIONS

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Abstract. Let X be a time-homogeneous one-dimensional diffusion whose coefficients satisfy the Engelbert–Schmidt conditions, and let q be a bounded potential. We show that the associated Feynman–Kac semigroup is smooth in time and C^1 in space, with a locally absolutely continuous first spatial derivative; the Kolmogorov equation holds almost everywhere. If the drift, second-order coefficient, and potential are continuous, the Feynman–Kac value is $C^{1,2}$ in the interior. Thus, in the time-homogeneous one-dimensional setting, continuity suffices for interior $C^{1,2}$ regularity. We identify the corresponding semigroups with the killed and reflected Feynman–Kac functionals, treat nonzero, time-independent Dirichlet data, and give examples showing the sharpness of the continuity assumptions and the failure of the corresponding statement in two dimensions.

1. Introduction

Consider, on an open interval $I = (\ell, r)$, the time-homogeneous stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x \in I, \tag{1.1}$$

up to its exit time ζ from I. We study the spatial regularity of the associated Feynman–Kac value under the low regularity allowed by the Engelbert–Schmidt conditions.

Classical Feynman–Kac and parabolic regularity results are often formulated under Hölder or stronger spatial regularity assumptions; see, for example, the classical treatment in [9, Chapter 6]. Similar assumptions appear in the mathematical-finance literature. Janson and Tysk [14] assume local Lipschitz continuity of the diffusion coefficients in their multidimensional setting and local 1/2-Hölder continuity in one dimension. Ekström and Tysk [6, Hypothesis 2.1 and Theorem 3.2] and Bayraktar and Xing [1, Theorem 2] likewise work under local 1/2-Hölder continuity in space. Karatzas and Ruf [15, Propositions 5.2 and 5.4] characterize the tail distribution of the explosion time of a one-dimensional diffusion as the minimal nonnegative classical solution of the associated Cauchy problem under local

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Hölder assumptions. For the natural-scale half-line problem, Chen, Huang, Song, and Zhu [3, Theorems 4.4 and 5.2 and Remark 5.4] obtain classicality under local δ-Hölder continuity of σ for any $\delta > 0$ and, under mere continuity, represent the stochastic solution as a limit of classical approximations.

Ekström, Janson, and Tysk [5, Theorem 8.1 and Corollary 8.3] consider natural-scale generalized diffusions on $\mathbb{R}$. For continuous initial data satisfying a global growth condition, they prove smoothness in time and show that absolute continuity of the speed measure yields C^1 spatial regularity, while continuity of its density yields C^2 spatial regularity. Consequently, for a natural-scale diffusion with zero potential, their results already give positive-time $C^{1,2}$ regularity when σ is continuous and nonzero. Compared with this natural-scale, zero-potential result, we allow general Engelbert–Schmidt coefficients before passage to scale, bounded Borel potentials, arbitrary intervals, and the canonical killed and reflected realizations. Within the L^2 framework, the initial datum is arbitrary and need not be continuous or satisfy a pointwise growth condition.

We work under the Engelbert–Schmidt conditions

$$\sigma \neq 0, \qquad \frac{1}{\sigma^2}, \ \frac{b}{\sigma^2} \in L^1_{\mathrm{loc}}(I),$$

which ensure weak existence and uniqueness in law up to the exit time; see [16, Theorem 5.5.15] and [7]. Our argument passes to natural scale and uses the associated symmetric L^2 operator. For positive times, the semigroup maps L^2 data into functions that are C^1 in space with locally absolutely continuous first derivative, and the Kolmogorov equation holds almost everywhere. If the drift, second-order coefficient, and bounded potential are continuous, this yields interior $C^{1,2}$ regularity.

We also consider boundary behavior. The zero-boundary form domain corresponds to the killed diffusion, while the maximal form domain gives the symmetric reflected extension; see [12, 10, 11]. We identify the corresponding semigroups with their Feynman–Kac representations and also treat nonzero, time-independent Dirichlet data by an affine function of the scale variable.

Example 5.1 shows that none of the separate continuity hypotheses on $a = \sigma^2/2$, b, and the potential q can be dropped without replacement: a discontinuity in any one of these coefficients can destroy twice differentiability of the Feynman–Kac value. In addition, using the construction in [8, proof of Theorem 4], we give a time-homogeneous two-dimensional example with $b = q = 0$ and a continuous uniformly elliptic coefficient matrix for which the stationary Feynman–Kac value is not C^2.

Regularity is separate from uniqueness. On unbounded intervals, additional growth or integrability conditions may be needed for uniqueness among classical solutions. Strict local martingales provide standard examples in which several classical solutions have the same initial data; see [4, 6, 1, 18].

The paper is organized as follows. Section 2 introduces the scale–speed and L^2 framework and proves the two regularity lemmas used below. Section 3

constructs the semigroup with bounded Borel potential, proves its interior regularity, records the Dirichlet and Neumann-type boundary behavior, and treats nonzero, time-independent Dirichlet data. Section 4 gives the corresponding Feynman–Kac representations for the killed and reflected diffusions and records the classical verification argument. Finally, Section 5 gives an example showing the sharpness of the continuity assumptions and a two-dimensional example showing that, already in dimension two, continuity and uniform ellipticity of the diffusion matrix do not imply interior $C^{1,2}$ regularity.

2. Engelbert–Schmidt diffusions and the symmetric framework

We first recall the Engelbert–Schmidt conditions, the transformation to natural scale, and the L^2 framework. We then introduce the form domains and prove the two regularity lemmas used throughout the paper.

Throughout, $I = (\ell, r)$ is a nonempty open interval, possibly with infinite endpoints. We impose the Engelbert–Schmidt conditions

$$\sigma(x) \neq 0 \quad (x \in I), \qquad \frac{1}{\sigma^2},\ \frac{b}{\sigma^2} \in L^1_{\mathrm{loc}}(I), \tag{2.1}$$

where b and σ are Borel functions. Equivalently,

$$a > 0, \qquad \frac{1}{a},\ \frac{b}{a} \in L^1_{\mathrm{loc}}(I), \qquad a = \frac{\sigma^2}{2}.$$

Under (2.1), for every $x \in I$, equation (1.1) admits a weak solution that is unique in law up to its possibly finite exit time ζ from I; see [16, Theorem 5.5.15].

Fix $x_0 \in I$. Define the scale density and scale function by

$$\rho(x) = \exp\!\left(-\int_{x_0}^{x} \frac{b(z)}{a(z)}\,\mathrm{d}z\right), \qquad s(x) = \int_{x_0}^{x} \rho(z)\,\mathrm{d}z.$$

Then ρ is strictly positive and belongs to $\mathrm{AC}_{\mathrm{loc}}(I)$, while $s \in C^1(I)$ is strictly increasing with $s' = \rho$, and $\rho' = -(b/a)\rho$ almost everywhere. Set now $J = s(I) = (\alpha, \beta)$, where either endpoint may be infinite. By the standard scale transformation for one-dimensional diffusions [16, Section 5.5.B], the process $Y_t = s(X_t)$, $0 \le t < \zeta$, satisfies

$$\mathrm{d}Y_t = \widetilde{\sigma}(Y_t)\,\mathrm{d}W_t, \qquad \widetilde{\sigma}(s(x)) = \rho(x)\sigma(x),$$

and is therefore in natural scale, with the same lifetime ζ. Its coefficient $\widetilde{\sigma}$ again satisfies the corresponding Engelbert–Schmidt condition.

We use the symmetrizing measure

$$\mu(\mathrm{d}x) = \frac{\mathrm{d}x}{a(x)\rho(x)} = \frac{2\,\mathrm{d}x}{\sigma^2(x)\rho(x)}.$$

Since ρ is strictly positive and continuous and $1/a \in L^1_{\mathrm{loc}}(I)$, the measure μ is locally finite and has full support on I. Its push-forward by s is $w(y)\,\mathrm{d}y$,

where

$$w(s(x)) = \frac{1}{a(x)\rho^2(x)} = \frac{2}{\sigma^2(x)\rho^2(x)}. \tag{2.2}$$

Hence w is strictly positive and belongs to $L^1_{\mathrm{loc}}(J)$.

For the symmetric L^2 formulation, let $H = L^2(J, w(y)\,\mathrm{d}y)$ and define

$$\mathcal{V}_{\max} = \left\{ v \in H : v \text{ has an } \mathrm{AC}_{\mathrm{loc}}(J) \text{ representative with } v' \in L^2(J) \right\}. \tag{2.3}$$

Since $w > 0$, this representative is unique, and we identify v with it. We call a linear subspace $\mathcal{V} \subset \mathcal{V}_{\max}$ admissible if $C_c^\infty(J) \subset \mathcal{V}$ and $\mathcal{V}$ is complete under the norm

$$\|v\|_{\mathcal{V}}^2 = \|v\|_H^2 + \|v'\|_{L^2(J)}^2.$$

In particular, $\mathcal{V}_{\max}$ itself is admissible. For any admissible $\mathcal{V}$, define

$$\mathcal{E}(v, z) = \int_J v'(y) z'(y)\,\mathrm{d}y, \qquad v, z \in \mathcal{V}.$$

Then $(\mathcal{E}, \mathcal{V})$ is a densely defined, closed, nonnegative symmetric form on H. By the representation theorem for closed symmetric forms [12, Theorem 1.3.1 and Corollary 1.3.1], there is a unique nonpositive self-adjoint operator A on H characterized by

$$\mathcal{E}(v, z) = -\langle Av, z\rangle_H, \qquad v \in \mathrm{Dom}(A),\ z \in \mathcal{V}. \tag{2.4}$$

For certain admissible choices of $\mathcal{V}$, $(\mathcal{E}, \mathcal{V})$ is a regular Dirichlet form and hence defines a symmetric Markov process. These processes are symmetric extensions of the minimal diffusion and agree with it up to the first boundary hit; see [10, 11]. The regularity argument below does not use this Markovian structure. For the probabilistic Feynman–Kac identifications in Section 4, we shall use the two canonical form domains $\mathcal{V}_0$ and $\mathcal{V}_{\max}$. The domain $\mathcal{V}_{\max}$ was defined in (2.3), while

$$\mathcal{V}_0 = \{ v \in \mathcal{V}_{\max} : v(\alpha) = 0 \text{ if } \alpha > -\infty,\ v(\beta) = 0 \text{ if } \beta < \infty \}.$$

Here the boundary values at finite endpoints of J are understood as limits from within J, which exist because $v' \in L^2(J)$. The space $\mathcal{V}_0$ is also admissible. In Section 4, the semigroups associated with $\mathcal{V}_0$ and $\mathcal{V}_{\max}$ are represented probabilistically by the killed and reflected diffusions, respectively.

We use two elementary lemmas. The first gives local spatial regularity for functions in the operator domain; the second shows that the semigroup maps H-data into the operator domain at positive times.

Lemma 2.1 (Operator-domain regularity)**.** *Let J be an open interval and let $w : J \to (0, \infty)$ be Borel with $w \in L^1_{\mathrm{loc}}(J)$. Let $\mathcal{V}$ be admissible in $H = L^2(J, w(y)\,\mathrm{d}y)$, and let $\mathcal{E}$ and A be as above. Then every $v \in \mathrm{Dom}(A)$ belongs to $C^1(J)$, with $v' \in \mathrm{AC}_{\mathrm{loc}}(J)$, and*

$$v'' = wAv, \quad \textit{almost everywhere.} \tag{2.5}$$

Moreover, for every compact interval $K \subset J$, the restriction map $\mathrm{Dom}(A) \to C^1(K)$*, where* $\mathrm{Dom}(A)$ *is equipped with the graph norm, is continuous.*

Proof. Let $v \in \mathrm{Dom}(A)$. Since $\mathrm{Dom}(A) \subset \mathcal{V} \subset \mathcal{V}_{\max}$, the function v is locally absolutely continuous and $v' \in L^2(J)$. Moreover, since $C_c^\infty(J) \subset \mathcal{V}$, (2.4) gives, for every $\varphi \in C_c^\infty(J)$,

$$\int_J v'(y)\varphi'(y)\,\mathrm{d}y = -\int_J Av(y)\varphi(y)w(y)\,\mathrm{d}y.$$

The right-hand side is locally integrable, since for every compact interval $K_0 \subset J$,

$$\int_{K_0} |Av(y)|w(y)\,\mathrm{d}y \le \left(\int_{K_0} |Av(y)|^2 w(y)\,\mathrm{d}y\right)^{1/2} \left(\int_{K_0} w(y)\,\mathrm{d}y\right)^{1/2}.$$

Thus the distributional derivative of v' is $wAv \in L^1_{\mathrm{loc}}(J)$. Consequently, $v' \in \mathrm{AC}_{\mathrm{loc}}(J)$, hence $v \in C^1(J)$, and (2.5) holds.

Choose now a nondegenerate compact interval $K_0 \subset J$ containing K and set $M_0 = \int_{K_0} w(y)\,\mathrm{d}y > 0$. For $x, y \in K_0$, Cauchy–Schwarz gives

$$|v(x)| \le |v(y)| + |K_0|^{1/2}\|v'\|_{L^2(K_0)}.$$

Integrating in y against $w(y)\,\mathrm{d}y/M_0$ and applying Cauchy–Schwarz once more yields

$$\|v\|_{L^\infty(K)} \le \|v\|_{L^\infty(K_0)} \le M_0^{-1/2}\|v\|_H + |K_0|^{1/2}\|v'\|_{L^2(J)}. \tag{2.6}$$

On the other hand, taking $z = v$ in (2.4) gives

$$\|v'\|^2_{L^2(J)} = \mathcal{E}(v,v) = -\langle Av, v\rangle_H \le \|Av\|_H\|v\|_H,$$

and hence

$$\|v'\|_{L^2(J)} \le \frac{1}{2}(\|Av\|_H + \|v\|_H). \tag{2.7}$$

Finally, since $v' \in \mathrm{AC}(K_0)$, we may choose $y_0 \in K_0$ such that $|v'(y_0)| \le |K_0|^{-1/2}\|v'\|_{L^2(K_0)}$. Hence, for $x \in K_0$,

$$|v'(x)| \le |K_0|^{-1/2}\|v'\|_{L^2(J)} + \int_{K_0} |v''(y)|\,\mathrm{d}y,$$

yielding

$$\|v'\|_{L^\infty(K)} \le \|v'\|_{L^\infty(K_0)} \le |K_0|^{-1/2}\|v'\|_{L^2(J)} + M_0^{1/2}\|Av\|_H. \tag{2.8}$$

Combining (2.6) and (2.8) with (2.7) proves the existence of $C_K > 0$ such that

$$\|v\|_{C^1(K)} \le C_K(\|v\|_H + \|Av\|_H), \qquad v \in \mathrm{Dom}(A),$$

hence the asserted continuity follows. □

Lemma 2.2 (Semigroup regularization into operator domains)**.** *Let H be a Hilbert space and let L be a self-adjoint operator on H that is bounded above. For $g, h \in H$, define*

$$z(t) = e^{tL} g - \int_0^t e^{rL} h \, \mathrm{d}r.$$

Then $z \in C^\infty((0,\infty); \mathrm{Dom}(L))$, where $\mathrm{Dom}(L)$ is equipped with its graph norm, and

$$z'(t) = Lz(t) - h, \qquad t > 0. \tag{2.9}$$

Proof. Since L is bounded above, for every $f \in H$, $t > 0$, and $k \in \mathbb{N}$, $\sup_{\lambda \in \sigma(L)} |\lambda|^k e^{t\lambda} < \infty$. Hence the spectral theorem gives $e^{tL} f \in \mathrm{Dom}(L^k)$. Fix now $f \in H$, $k \in \mathbb{N}_0$, and $t_0 > 0$. For $t > t_0/2$, we have

$$L^k e^{tL} f = e^{(t - t_0/2)L} L^k e^{(t_0/2)L} f.$$

The strong continuity of $(e^{tL})_{t \geq 0}$ therefore shows that $t \mapsto L^k e^{tL} f$ is continuous in H on $(0,\infty)$.

Next, for $\varepsilon > 0$, the semigroup property gives

$$\frac{e^{\varepsilon L} - \mathrm{Id}_H}{\varepsilon} \int_0^t e^{rL} h \, \mathrm{d}r = \frac{1}{\varepsilon} \left(\int_t^{t+\varepsilon} e^{rL} h \, \mathrm{d}r - \int_0^\varepsilon e^{rL} h \, \mathrm{d}r \right).$$

By strong continuity, the right-hand side converges in H to $e^{tL} h - h$ as $\varepsilon \downarrow 0$. Since L is the generator of $(e^{tL})_{t \geq 0}$, it follows that

$$\int_0^t e^{rL} h \, \mathrm{d}r \in \mathrm{Dom}(L), \qquad L \int_0^t e^{rL} h \, \mathrm{d}r = e^{tL} h - h;$$

in particular, $z(t) \in \mathrm{Dom}(L)$.

As an H-valued function, z is differentiable for $t > 0$, with

$$z'(t) = L e^{tL} g - e^{tL} h = Lz(t) - h.$$

This proves (2.9). Iterating gives, for every $m \in \mathbb{N}$,

$$z^{(m)}(t) = L^m e^{tL} g - L^{m-1} e^{tL} h. \tag{2.10}$$

By the spectral estimate and the continuity established above, (2.10) shows that, for every $m \in \mathbb{N}$, $z^{(m)}(t) \in \mathrm{Dom}(L)$ and both $z^{(m)}$ and $Lz^{(m)}$ are continuous as H-valued functions. For $m = 0$, (2.9) gives $Lz = z' + h$, which is continuous in H. Thus $z^{(m)}$ is graph-norm continuous for every $m \in \mathbb{N}_0$. Finally, (2.10) and (2.9) show that $Lz^{(m)}$ is differentiable in H with derivative $Lz^{(m+1)}$ for every $m \in \mathbb{N}_0$. Since $z^{(m)}$ has H-valued derivative $z^{(m+1)}$, it has graph-norm derivative $z^{(m+1)}$. Hence $z \in C^\infty((0,\infty); \mathrm{Dom}(L))$. □

3. Regularity and boundary conditions with bounded potentials

In this section, we construct the L^2 semigroup with bounded potential and establish its interior regularity. We then identify the boundary behavior for the canonical Dirichlet and Neumann-type conditions and finally treat nonzero, time-independent Dirichlet data.

Fix an admissible form domain $\mathcal{V}$ as in Section 2, and let A be its associated operator on $H = L^2(J, w(y)\,dy)$. Since the push-forward of μ under the scale function s is $w(y)\,dy$, the map

$$U : L^2(I,\mu) \longrightarrow H, \qquad Uf = f \circ s^{-1},$$

is unitary. Let $q : I \to \mathbb{R}$ be bounded and Borel and let Q denote the bounded self-adjoint multiplication operator on H by $q \circ s^{-1}$.

The semigroup is obtained by the following bounded self-adjoint perturbation.

Proposition 3.1 (L^2 semigroup with bounded potential)**.** *Assume (2.1), let $\mathcal{V}$ be admissible with associated operator A, and let $q : I \to \mathbb{R}$ be bounded and Borel. Let Q be multiplication on H by $q \circ s^{-1}$. Then*

$$B = A - Q, \qquad \mathrm{Dom}(B) = \mathrm{Dom}(A),$$

is self-adjoint and bounded above by $\|q^-\|_\infty$. Moreover,

$$T_t^q = U^{-1} e^{tB} U, \qquad t \geq 0, \tag{3.1}$$

defines a strongly continuous semigroup on $L^2(I,\mu)$ satisfying

$$\|T_t^q f\|_{L^2(I,\mu)} \leq e^{t\|q^-\|_\infty} \|f\|_{L^2(I,\mu)}, \qquad f \in L^2(I,\mu). \tag{3.2}$$

Proof. Since Q is bounded and self-adjoint and A is self-adjoint, $B = A - Q$ with $\mathrm{Dom}(B) = \mathrm{Dom}(A)$ is self-adjoint. Moreover, since $\langle Av, v\rangle_H = -\mathcal{E}(v,v) \leq 0$ and $-\langle Qv, v\rangle_H \leq \|q^-\|_\infty \|v\|_H^2$, B is bounded above by $\|q^-\|_\infty$.

Hence the spectral theorem shows that $(e^{tB})_{t\geq 0}$ is a strongly continuous semigroup on H and

$$\|e^{tB} h\|_H \leq e^{t\|q^-\|_\infty} \|h\|_H, \qquad h \in H. \tag{3.3}$$

Since U is unitary, (3.1) defines a strongly continuous semigroup on $L^2(I,\mu)$, and (3.3) yields (3.2). □

We next prove the interior regularity of the semigroup in Proposition 3.1.

Theorem 3.2 (Interior regularity under Engelbert–Schmidt conditions)**.** *Assume (2.1), let $\mathcal{V}$ be an admissible form domain with associated operator A, let $q : I \to \mathbb{R}$ be bounded and Borel, and recall the strongly continuous semigroup $(T_t^q)_{t\geq 0}$ from Proposition 3.1. Let $f \in L^2(I,\mu)$. For every $t > 0$, $T_t^q f$ admits a unique continuous representative, which we identify with $T_t^q f$. Furthermore, the following statements hold with $u(t,x) = T_t^q f(x)$.*

(i) *For every $k \in \mathbb{N}_0$, $\partial_t^k u$ and $\partial_x \partial_t^k u$ exist on $(0,\infty) \times I$ and are jointly continuous there.*

(ii) *For every $t > 0$, $\partial_x u(t,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(I)$, and for almost every $x \in I$,*

$$\partial_t u(t,x) = a(x)\partial_{xx} u(t,x) + b(x)\partial_x u(t,x) - q(x)u(t,x). \tag{3.4}$$

For every $k \in \mathbb{N}_0$, the same assertions hold with u replaced by $\partial_t^k u$.

(iii) *If a, b, and q are continuous on I, then*

$$u \in C^{1,2}((0,\infty)\times I)$$

and (3.4) holds pointwise on $(0,\infty)\times I$. Moreover, $\partial_t^k u \in C^{1,2}((0,\infty)\times I)$ for every $k \in \mathbb{N}_0$.

In addition, $u(t,\cdot) \to f$ in $L^2(I,\mu)$ as $t \downarrow 0$.

Proof. Fix $f \in L^2(I,\mu)$ and, for $t > 0$, set $v(t) = e^{tB}Uf$. Since B is self-adjoint and bounded above, Lemma 2.2, applied with $L = B$, $g = Uf$, and $h = 0$, gives $v \in C^\infty((0,\infty);\mathrm{Dom}(B))$ and $v' = Bv$. Hence, inductively, $v^{(k+1)} = Bv^{(k)}$ for every $k \in \mathbb{N}_0$.

Since $B = A - Q$ with Q bounded, the graph norms of A and B are equivalent. Therefore, Lemma 2.1 implies that, for every compact interval $K \subset J$, $v \in C^\infty((0,\infty);C^1(K))$. Write now $v(t,y) = v(t)(y)$. Then, for every $k \in \mathbb{N}_0$, the functions $\partial_t^k v$ and $\partial_y\partial_t^k v$ are jointly continuous on $(0,\infty)\times J$.

Set next $u(t,x) = v(t,s(x))$. By (3.1), $u(t,\cdot)$ represents $T_t^q f$ in $L^2(I,\mu)$. Since $u(t,\cdot)$ is continuous and μ has full support on I, this continuous representative is unique. Since s does not depend on t, for every $k \in \mathbb{N}_0$, $\partial_t^k u(t,x) = \partial_t^k v(t,s(x))$. Moreover, since $s' = \rho$,

$$\partial_x\partial_t^k u(t,x) = \partial_y\partial_t^k v(t,s(x))\rho(x).$$

The joint continuity established above for $\partial_t^k v$ and $\partial_y\partial_t^k v$, together with the continuity of s and ρ, proves (i).

For (ii), Lemma 2.1 gives, for every $t > 0$, $\partial_y v(t,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(J)$ and

$$\partial_{yy} v(t,\cdot) = wAv(t,\cdot) = w\big(\partial_t v(t,\cdot) + (q\circ s^{-1})v(t,\cdot)\big) \tag{3.5}$$

almost everywhere.

On each compact subinterval of I, the map s is bi-Lipschitz. Consequently, $\partial_y v(t,s(\cdot)) \in \mathrm{AC}_{\mathrm{loc}}(I)$. Since $\rho \in \mathrm{AC}_{\mathrm{loc}}(I)$, it follows that

$$\partial_x u(t,\cdot) = \partial_y v(t,s(\cdot))\rho \in \mathrm{AC}_{\mathrm{loc}}(I).$$

Moreover, since s^{-1} is locally Lipschitz, the preimage under s of every Lebesgue-null subset of J is a Lebesgue-null subset of I. Thus the almost-everywhere identity in (3.5) remains valid after composition with s. The chain and product rules, together with (2.2) and $\rho' = -(b/a)\rho$ almost everywhere, therefore give, for almost every x,

$$\begin{aligned}
\partial_{xx} u(t,x) &= \partial_{yy} v(t,s(x))\rho^2(x) + \partial_y v(t,s(x))\rho'(x) \\
&= \frac{\partial_t v(t,s(x)) + q(x)v(t,s(x))}{a(x)} - \frac{b(x)}{a(x)}\partial_x u(t,x) \\
&= \frac{\partial_t u(t,x) + q(x)u(t,x) - b(x)\partial_x u(t,x)}{a(x)}. \qquad (3.6)
\end{aligned}$$

This is equivalent to (3.4).

For every $k \in \mathbb{N}_0$, the first part of the proof gives $\partial_t^k v(t,\cdot) \in \mathrm{Dom}(A)$ for every $t > 0$, and $\partial_t^{k+1} v(t,\cdot) = B\partial_t^k v(t,\cdot)$. Applying the same argument with v

and u replaced by $\partial_t^k v$ and $\partial_t^k u$, respectively, shows that $\partial_x \partial_t^k u(t,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(I)$ and that (3.4) holds almost everywhere with u replaced by $\partial_t^k u$. This proves (ii).

If a, b, and q are continuous, the expression for $\partial_{xx} u$ in (3.6) is then jointly continuous in (t,x). Since $\partial_x u(t,\cdot)$ is locally absolutely continuous and has this function as its almost-everywhere derivative, the fundamental theorem of calculus shows that $\partial_{xx} u$ exists everywhere and is jointly continuous. Hence $u \in C^{1,2}((0,\infty)\times I)$, and the equation holds pointwise on $(0,\infty)\times I$. Applying the same argument to $B^k v = \partial_t^k v$ proves the assertion for every time derivative and hence (iii).

The final assertion follows from the strong continuity of $(T_t^q)_{t\geq 0}$. □

Remark 3.3. *When $I = \mathbb{R}$, $b = q = 0$, σ is continuous and nonzero, and the initial datum is continuous and satisfies their global growth condition, the positive-time $C^{1,2}$ conclusion of Theorem 3.2(iii) follows from* [5, Theorem 8.1 and Corollary 8.3]. *Compared with that result, Theorem 3.2 allows arbitrary $L^2(I,\mu)$ initial data, general Engelbert–Schmidt coefficients before passage to scale, bounded Borel potentials, arbitrary intervals, and arbitrary admissible form domains.*

Theorem 3.2 is formulated for an arbitrary admissible form domain $\mathcal{V}$. We now record the boundary behavior for the two canonical choices $\mathcal{V}_0$ and $\mathcal{V}_{\max}$. The domain $\mathcal{V}_0$ gives the homogeneous Dirichlet condition at finite scale endpoints. For $\mathcal{V}_{\max}$, a finite speed measure near an endpoint yields a vanishing scale derivative; if ρ is bounded there, the derivative in the original coordinate also vanishes. At endpoints of finite scale with finite speed measure, this is the classical Neumann-type boundary behavior for the associated symmetric reflected diffusion; see [12, 10, 11].

Proposition 3.4 (Boundary behavior for canonical form domains)**.** *Let u denote the solution from Theorem 3.2. The following boundary conditions hold.*

(i) *For $\mathcal{V} = \mathcal{V}_0$,*

$$\lim_{x\downarrow \ell} u(t,x) = 0 \quad \text{if } s(\ell+) > -\infty, \qquad \lim_{x\uparrow r} u(t,x) = 0 \quad \text{if } s(r-) < \infty,$$

for every $t > 0$.

(ii) *For $\mathcal{V} = \mathcal{V}_{\max}$, let $e \in \{\ell, r\}$ and suppose that μ has finite mass near e. Then, for every $t > 0$,*

$$\lim_{x\to e} \frac{\partial_x u(t,x)}{\rho(x)} = 0,$$

where the limit is taken within I. If, in addition, ρ is bounded near e, then $\lim_{x\to e} \partial_x u(t,x) = 0$.

Proof. Fix $f \in L^2(I,\mu)$ and $t > 0$, and set $v_t = e^{tB} U f$. By Lemma 2.2, $v_t \in \mathrm{Dom}(B) = \mathrm{Dom}(A)$, where A is the operator associated with the

form domain under consideration. Moreover, by the proof of Theorem 3.2, $u(t,x) = v_t(s(x))$.

For (i), if $\mathcal{V} = \mathcal{V}_0$, then $v_t \in \mathrm{Dom}(A) \subset \mathcal{V}_0$. Hence $v_t(y) \to 0$ as $y \downarrow \alpha$ when $\alpha > -\infty$, and as $y \uparrow \beta$ when $\beta < \infty$. Since $s(x) \downarrow \alpha$ as $x \downarrow \ell$ and $s(x) \uparrow \beta$ as $x \uparrow r$, the asserted boundary limits for u follow.

For (ii), consider the left endpoint, i.e. $e = \ell$, and suppose that $\mu((\ell, c)) < \infty$ for some $c \in I$. Equivalently, $\int_\alpha^{s(c)} w(y)\, \mathrm{d}y < \infty$. Lemma 2.1 gives $v_t'' = wAv_t$ almost everywhere. Hence

$$\int_\alpha^{s(c)} |v_t''(y)|\, \mathrm{d}y \leq \|Av_t\|_H \left(\int_\alpha^{s(c)} w(y)\, \mathrm{d}y \right)^{1/2} < \infty.$$

Thus $v_t'(y)$ has a finite limit as $y \downarrow \alpha$.

If $\alpha = -\infty$, this limit must be zero because $v_t' \in L^2(J)$. If $\alpha > -\infty$, choose $z \in \mathcal{V}_{\max}$ such that $z = 1$ near α and $z = 0$ near $s(c)$. Using (2.4) and $v_t'' = wAv_t$, integration by parts gives

$$-\int_\alpha^{s(c)} v_t''(y) z(y)\, \mathrm{d}y = \int_\alpha^{s(c)} v_t'(y) z'(y)\, \mathrm{d}y = -\lim_{y \downarrow \alpha} v_t'(y) - \int_\alpha^{s(c)} v_t''(y) z(y)\, \mathrm{d}y.$$

Therefore

$$\lim_{x \downarrow \ell} \frac{\partial_x u(t,x)}{\rho(x)} = \lim_{x \downarrow \ell} v_t'(s(x)) = \lim_{y \downarrow \alpha} v_t'(y) = 0.$$

The argument at the right endpoint is analogous. This proves (ii). □

We next turn to nonzero, time-independent Dirichlet data, which are handled by subtracting a function that is affine in the scale variable. To this end, let $J = s(I) = (\alpha, \beta)$. Prescribe $\gamma_\ell \in \mathbb{R}$ if $\alpha > -\infty$ and $\gamma_r \in \mathbb{R}$ if $\beta < \infty$, and define $\xi : I \to \mathbb{R}$ by

$$\xi(x) = \begin{cases} \gamma_\ell + (\gamma_r - \gamma_\ell) \dfrac{s(x) - \alpha}{\beta - \alpha}, & \alpha > -\infty,\ \beta < \infty, \\ \gamma_\ell, & \alpha > -\infty,\ \beta = \infty, \\ \gamma_r, & \alpha = -\infty,\ \beta < \infty, \\ 0, & \alpha = -\infty,\ \beta = \infty. \end{cases} \tag{3.7}$$

By construction, $\xi \in C^1(I)$, $\xi' \in \mathrm{AC}_{\mathrm{loc}}(I)$, and, since $s' = \rho$ and $\rho' = -(b/a)\rho$ almost everywhere,

$$a\xi'' + b\xi' = 0 \qquad \text{almost everywhere on } I.$$

Proposition 3.5 (Nonzero, time-independent Dirichlet data)**.** *Assume (2.1), let $q : I \to \mathbb{R}$ be bounded and Borel, and let $f : I \to \mathbb{R}$ be Borel. Suppose that*

$$f - \xi \in L^2(I, \mu), \qquad q\xi \in L^2(I, \mu).$$

Then there exists a function $u : (0,\infty)\times I \to \mathbb{R}$ *satisfying Theorem 3.2*(i)–(iii). *Moreover, for every* $t > 0$, $u(t,\cdot) - \xi \in L^2(I,\mu)$,

(3.8)
$$\lim_{x\downarrow \ell} u(t,x) = \gamma_\ell \quad \text{if } s(\ell+) > -\infty, \qquad \lim_{x\uparrow r} u(t,x) = \gamma_r \quad \text{if } s(r-) < \infty,$$

and $u(t,\cdot) - f \to 0$ *in* $L^2(I,\mu)$ *as* $t \downarrow 0$.

Proof. For $\mathcal{V} = \mathcal{V}_0$, let A and B be as in Proposition 3.1. Set $g = U[f-\xi] \in H$ and $h = U[q\xi] \in H$, and define, for $t > 0$,

$$z(t) = e^{tB}g - \int_0^t e^{rB}h \, \mathrm{d}r, \qquad u(t,x) = \xi(x) + z(t,s(x)).$$

By Lemma 2.2, $z \in C^\infty((0,\infty); \mathrm{Dom}(B))$ and $z' = Bz - h$. Since $\mathrm{Dom}(B) = \mathrm{Dom}(A)$ and the graph norms of A and B are equivalent, the argument in the proof of Theorem 3.2, with z in place of v, gives (i) and $\partial_x u(t,\cdot) \in \mathrm{AC}_{\mathrm{loc}}(I)$ for every $t > 0$.

Moreover, for every $t > 0$ and almost every $x \in I$,

$$\partial_t[z(t,s(x))] = a(x)\partial_{xx}[z(t,s(x))] + b(x)\partial_x[z(t,s(x))] - q(x)z(t,s(x)) - q(x)\xi(x).$$

Since $a\xi'' + b\xi' = 0$ almost everywhere on I, it follows that (3.4) holds for u. For every $k \in \mathbb{N}$, differentiating $z' = Bz - h$ in time gives $\partial_t^{k+1} z = B\partial_t^k z$. The argument in the proof of Theorem 3.2(ii) therefore gives the remaining assertions in (ii). If a, b, and q are continuous on I, then $\xi \in C^2(I)$, and the argument in the proof of Theorem 3.2(iii) gives (iii).

Since $z(t) \in H$, $u(t,\cdot) - \xi \in L^2(I,\mu)$ for every $t > 0$. Moreover, $z(t,\cdot) \in \mathrm{Dom}(A) \subset \mathcal{V}_0$, so $z(t,y) \to 0$ as $y \downarrow s(\ell+)$ whenever $s(\ell+) > -\infty$, and $z(t,y) \to 0$ as $y \uparrow s(r-)$ whenever $s(r-) < \infty$. Together with the boundary values of ξ, this gives (3.8). Finally,

$$\|z(t) - g\|_H \le \|e^{tB}g - g\|_H + te^{t\|q^-\|_\infty}\|h\|_H \longrightarrow 0$$

as $t \downarrow 0$. Since $U[u(t,\cdot) - \xi] = z(t)$ and $U[f - \xi] = g$, the unitarity of U yields $u(t,\cdot) - f \to 0$ in $L^2(I,\mu)$ as $t \downarrow 0$. □

4. Feynman–Kac representations

In this section, we identify the semigroups constructed in Section 3 with their Feynman–Kac representations. In Section 4.1, we treat killed diffusion and nonzero, time-independent Dirichlet data. In Section 4.2, we treat the reflected diffusion associated with the maximal form domain. The regularity and boundary properties of these probabilistic representations then follow from the results of Section 3.

4.1. **Killed diffusion and Dirichlet data.**

Throughout this subsection, take $\mathcal{V} = \mathcal{V}_0$, let $q : I \to \mathbb{R}$ be bounded and Borel, and let A, B, and $(T_t^q)_{t\ge 0}$ be the corresponding objects from Section 3. By [10, Theorem 3.2], $(\mathcal{E}, \mathcal{V}_0)$ is a regular, strongly local Dirichlet form on H, and by [10, Theorem 3.3] its

associated diffusion is the natural-scale diffusion Y killed on leaving J. In particular, e^{tA} is its L^2 transition semigroup.

Write $\mathbb{E}_x$ for expectation with respect to the law of X started from $x \in I$. For $t > 0$ and $f \in L^2(I,\mu)$, choose a Borel representative $\widehat{f}$ of f and set

$$S_t^q f(x) = \mathbb{E}_x\left[\exp\left(-\int_0^t q(X_r)\ \mathrm{d}r\right)\widehat{f}(X_t);\ t < \zeta\right], \qquad x \in I. \tag{4.1}$$

The following proposition shows that the definition is independent of the chosen representative and identifies S_t^q with T_t^q.

Proposition 4.1 (Feynman–Kac identification)**.** *Assume (2.1), and let $q : I \to \mathbb{R}$ be bounded and Borel. For every $t > 0$ and every $f \in L^2(I,\mu)$, the expectation in (4.1) is absolutely integrable for every $x \in I$ and is independent of the chosen Borel representative $\widehat{f}$. Moreover, $S_t^q f$ is the unique continuous representative of $T_t^q f$.*

Proof. Fix $t > 0$ and $f \in L^2(I,\mu)$, and let $\widehat{f}$ be a Borel representative of f. Let $p_t(x,z)$ denote the transition density of the killed diffusion with respect to μ. It can be chosen symmetric and jointly continuous on $(0,\infty) \times I \times I$; see [13, Chapter IV, Section 4.11]. By symmetry and the Chapman–Kolmogorov identity,

$$\int_I p_t(x,z)^2\,\mu(\mathrm{d}z) = \int_I p_t(x,z)p_t(z,x)\,\mu(\mathrm{d}z) = p_{2t}(x,x) < \infty$$

for every $x \in I$. Hence

$$\begin{aligned}\mathbb{E}_x\left[\exp\left(-\int_0^t q(X_r)\ \mathrm{d}r\right)|\widehat{f}(X_t)|;\ t < \zeta\right] &\le e^{t\|q^-\|_\infty}\int_I p_t(x,z)|\widehat{f}(z)|\,\mu(\mathrm{d}z)\\ &\le e^{t\|q^-\|_\infty} p_{2t}(x,x)^{1/2}\|f\|_{L^2(I,\mu)}.\end{aligned} \tag{4.2}$$

This proves absolute integrability. If $\widetilde{f}$ is another Borel representative of f, applying (4.2) with $\widehat{f} - \widetilde{f}$ in place of $\widehat{f}$ shows that $S_t^q f$ does not depend on this choice.

Write now $\mathbb{E}_y^Y$ for expectation with respect to the law of Y started from $y \in J$. Set $c = \|q^-\|_\infty$ and $R = q \circ s^{-1} + c \ge 0$. The positive continuous additive functional

$$C_t^R = \int_0^{t\wedge\zeta} R(Y_r)\ \mathrm{d}r$$

has Revuz measure $\nu(\mathrm{d}y) = R(y)w(y)\ \mathrm{d}y$. Indeed, for nonnegative bounded Borel functions φ and ψ with compact support, symmetry of $(e^{tA})_{t\ge 0}$ gives

$$\int_J \psi(y)\mathbb{E}_y^Y\left[\int_0^t \varphi(Y_r)\ \mathrm{d}C_r^R\right] w(y)\ \mathrm{d}y = \int_0^t\int_J e^{rA}\psi(y)\varphi(y)R(y)w(y)\ \mathrm{d}y\ \mathrm{d}r.$$

After extending this identity to arbitrary nonnegative Borel φ and ψ by monotone convergence, the Revuz characterization [12, Theorem 5.1.3 (i)⇔(iii)] yields the claim.

Since R is bounded, [12, Theorem 6.1.1] shows that the process obtained from Y by additional killing with C^R has Dirichlet form $(\mathcal{E}^\nu, \mathcal{V}_0)$, where

$$\mathcal{E}^\nu(v,z) = \mathcal{E}(v,z) + \int_J vz \, \mathrm{d}\nu.$$

Let R also denote the multiplication operator by R on H. The operator associated with $(\mathcal{E}^\nu, \mathcal{V}_0)$ is $A - R$ with domain $\mathrm{Dom}(A)$. Indeed, for $v \in \mathrm{Dom}(A)$ and $z \in \mathcal{V}_0$,

$$\mathcal{E}^\nu(v,z) = -\langle Av, z\rangle_H + \langle Rv, z\rangle_H = -\langle (A-R)v, z\rangle_H.$$

Conversely, if v belongs to the operator domain associated with $\mathcal{E}^\nu$, then for some $g \in H$, $\mathcal{E}(v,z) = -\langle g + Rv, z\rangle_H$ for every $z \in \mathcal{V}_0$, hence $v \in \mathrm{Dom}(A)$.

It remains to identify the semigroup. By linearity, it suffices to consider nonnegative f. Choose $\widehat{f} \geq 0$ and set $h = \widehat{f} \circ s^{-1}$, so that $[h] = Uf$. By [12, Theorem 4.2.3(i)], the function

$$y \longmapsto \mathbb{E}_y^Y \left[\exp\left(-\int_0^t R(Y_r) \, \mathrm{d}r \right) h(Y_t); \ t < \zeta \right] \tag{4.3}$$

is an $(\mathcal{E}^\nu, \mathcal{V}_0)$-quasi-continuous representative of $e^{t(A-R)}[h]$. By [12, Example 2.1.2, p. 77], quasi-continuity here is indeed ordinary continuity. Hence the function in (4.3) is the continuous representative of $e^{t(A-R)}[h]$.

Since $R = q \circ s^{-1} + c$ and $B = (A - R) + c$, multiplying the function in (4.3) by e^{ct} and using $Y = s(X)$ on $[0, \zeta)$ shows that $S_t^q f \circ s^{-1}$ is the continuous representative of $e^{tB}Uf$. By (3.1), $S_t^q f$ is therefore a continuous representative of $T_t^q f$. Its uniqueness follows from Theorem 3.2. □

By Proposition 4.1 and Theorem 3.2, the function $u(t,x) = S_t^q f(x)$ satisfies all the conclusions of Theorem 3.2 for $\mathcal{V} = \mathcal{V}_0$. Moreover, Proposition 3.4(i) gives $\lim_{x \downarrow \ell} u(t,x) = 0$ if $s(\ell+) > -\infty$, and $\lim_{x \uparrow r} u(t,x) = 0$ if $s(r-) < \infty$, for every $t > 0$.

For the nonzero, time-independent Dirichlet data of Proposition 3.5, retain γ_ℓ, γ_r, and the function ξ from (3.7). The function ξ is bounded and has limits at both endpoints of I. On $\{\zeta < \infty\}$, write $X_{\zeta-} = \lim_{r \uparrow \zeta} X_r$ and interpret $\xi(X_{\zeta-})$ through these endpoint limits.

Corollary 4.2 (Feynman–Kac representation with nonzero, time-independent Dirichlet data)**.** *Under the assumptions of Proposition 3.5, for $t > 0$ and $x \in I$, define*

$$\begin{aligned} F(t,x) &= \mathbb{E}_x \left[\exp\left(-\int_0^t q(X_r) \, \mathrm{d}r \right) f(X_t); \ t < \zeta \right] \\ &\quad + \mathbb{E}_x \left[\exp\left(-\int_0^\zeta q(X_r) \, \mathrm{d}r \right) \xi(X_{\zeta-}); \ \zeta \leq t \right]. \end{aligned} \tag{4.4}$$

Both expectations are absolutely integrable, and, if u denotes the solution constructed in Proposition 3.5, then $F(t,x) = u(t,x)$ for all $t > 0$ and $x \in I$.

Proof. Since $f = (f-\xi)+\xi$, (4.2), applied to $f-\xi$, together with the boundedness of ξ, shows that the first expectation in (4.4) is absolutely integrable. The second expectation is absolutely integrable because, on $\{\zeta \le t\}$, its integrand has absolute value at most $e^{t\|q^-\|_\infty}\|\xi\|_\infty$.

By the proof of Proposition 3.5 and the unitarity of U,

$$[u(t,\cdot)-\xi] = T_t^q[f-\xi] - \int_0^t T_r^q[q\xi]\,\mathrm{d}r.$$

By Proposition 4.1, dominated convergence, (3.2), and Fubini's theorem, the continuous representative of the right-hand side is

$$\begin{aligned}&\mathbb{E}_x\bigg[\exp\Big(-\int_0^t q(X_v)\,\mathrm{d}v\Big)\,(f-\xi)(X_t);\ t<\zeta\bigg]\\&\quad-\int_0^t \mathbb{E}_x\bigg[\exp\Big(-\int_0^r q(X_v)\,\mathrm{d}v\Big)\,q(X_r)\xi(X_r);\ r<\zeta\bigg]\,\mathrm{d}r.\end{aligned}$$

Since $u(t,\cdot)-\xi$ is continuous and μ has full support, this equals $u(t,x)-\xi(x)$ for every $x\in I$.

Let $(\tau_n)_{n\in\mathbb{N}}$ be exit times from an increasing sequence of compact subintervals of I with $\tau_n \uparrow \zeta$. Since $\xi\in C^1(I)$, $\xi'\in \mathrm{AC}_{\mathrm{loc}}(I)$, and $a\xi''+b\xi'=0$ almost everywhere, the generalized Itô formula [16, Theorem 3.7.1], applied to $\xi(X_{\cdot\wedge\tau_n})$, and the product rule give, after letting $n\uparrow\infty$,

$$\begin{aligned}\xi(x) &= \mathbb{E}_x\bigg[\exp\Big(-\int_0^t q(X_v)\,\mathrm{d}v\Big)\,\xi(X_t);\ t<\zeta\bigg]\\&\quad+\mathbb{E}_x\bigg[\exp\Big(-\int_0^\zeta q(X_v)\,\mathrm{d}v\Big)\,\xi(X_{\zeta-});\ \zeta\le t\bigg]\\&\quad+\int_0^t \mathbb{E}_x\bigg[\exp\Big(-\int_0^r q(X_v)\,\mathrm{d}v\Big)\,q(X_r)\xi(X_r);\ r<\zeta\bigg]\,\mathrm{d}r,\end{aligned}$$

where dominated convergence and Fubini's theorem were used. Adding the last two displays gives $F(t,x)=u(t,x)$. □

Remark 4.3 (Classical verification)**.** *The preceding results identify the Feynman–Kac values with the solutions in Theorem 3.2 and Proposition 3.5, but do not imply uniqueness among all classical solutions. Let v be a classical solution with the same initial and boundary data. For the homogeneous Dirichlet problem, apply Itô's formula to $r\mapsto \exp(-\int_0^r q(X_w)\,\mathrm{d}w)\,v(t-r,X_r)$ up to the exit times from compact subintervals of I. If the resulting terminal variables are uniformly integrable, passage to the limit gives $v(t,x)=S_t^q f(x)$; compare* [14, Theorem 2.5]*. With nonzero, time-independent Dirichlet data, the same argument gives the exit payoff in (4.4) and identifies v with F.*

If I is bounded and v extends continuously to $[0,T]\times\overline{I}$, the localized terminal variables are bounded, so uniform integrability is automatic. On an unbounded interval, additional integrability may be needed to pass to the localization limit. Such a condition cannot in general be omitted: strict local martingales provide the standard obstruction and may lead to nonuniqueness

of classical solutions; see [4, 6, 1, 18]. *For uniqueness of weak solutions to related Cauchy problems under Engelbert–Schmidt conditions, see also* [2].

4.2. **Maximal form domain and reflected diffusion.** Throughout this subsection, take $\mathcal{V} = \mathcal{V}_{\max}$, let $q : I \to \mathbb{R}$ be bounded and Borel, and let A, B, and $(T_t^q)_{t\geq 0}$ be the corresponding objects from Section 3.

Call the left endpoint regular if

$$s(\ell+) > -\infty \qquad \text{and} \qquad \mu((\ell, c)) < \infty$$

for some $c \in I$, and the right endpoint regular if

$$s(r-) < \infty \qquad \text{and} \qquad \mu((c, r)) < \infty$$

for some $c \in I$. Let I^{r} be obtained from I by adjoining exactly the regular endpoints, and let J^{r} be the corresponding extension of J. The scale function extends to a homeomorphism from I^{r} onto J^{r}, again denoted by s. Extend $w(y)\,\mathrm{d}y$ by zero mass at the added endpoints, and denote the resulting measure by m^{r}. It has full support on J^{r}, and $L^2(J^{\mathrm{r}}, m^{\mathrm{r}})$ is naturally identified with H.

By [10, Theorem 5.2 and the discussion following (5.6)] and [11, Theorem 2.2(i), (iii), and Section 2.3], $(\mathcal{E}, \mathcal{V}_{\max})$, regarded as a form on J^{r}, is a regular, strongly local, irreducible Dirichlet form. Let Y^{r} be its associated diffusion. Killing Y^{r} on leaving J gives Y; at the added endpoints, Y^{r} is instantaneously reflecting: it is not killed and spends zero Lebesgue time there. Pulling Y^{r} back through the scale function gives a diffusion X^{r} on I^{r}, whose killing on leaving I gives X. We call X^{r} the reflected diffusion and write ζ^{r} for its lifetime, which may still be finite if X^{r} reaches a non-adjoined endpoint in finite time. Write $\mathbb{E}_x^{\mathrm{r}}$ for expectation under its law when started from $x \in I^{\mathrm{r}}$.

For $t > 0$ and $f \in L^2(I, \mu)$, choose a Borel representative $\widehat{f}$ of f. Extend q and $\widehat{f}$ to a bounded Borel function q^{r} on I^{r} and to a Borel function $\widehat{f}^{\mathrm{r}}$ on I^{r}, respectively, by assigning arbitrary finite values at the added endpoints. Set

$$S_t^{q,\mathrm{r}} f(x) = \mathbb{E}_x^{\mathrm{r}}\left[\exp\left(-\int_0^t q^{\mathrm{r}}(X_v^{\mathrm{r}})\,\mathrm{d}v\right)\widehat{f}^{\mathrm{r}}(X_t^{\mathrm{r}});\ t < \zeta^{\mathrm{r}}\right], \qquad x \in I^{\mathrm{r}}. \tag{4.5}$$

The next proposition shows that neither extension nor the chosen Borel representative affects this definition.

Proposition 4.4 (Feynman–Kac representation for the reflected diffusion)**.** *Assume* (2.1)*, and let $q : I \to \mathbb{R}$ be bounded and Borel. For every $t > 0$ and every $f \in L^2(I, \mu)$, the expectation in* (4.5) *is absolutely integrable for every $x \in I^{\mathrm{r}}$ and is independent of the choices of $\widehat{f}$, q^{r}, and $\widehat{f}^{\mathrm{r}}$. Moreover, $S_t^{q,\mathrm{r}} f \in C(I^{\mathrm{r}})$, and its restriction to I is the unique continuous representative of $T_t^q f$.*

Proof. Since X^{r} spends zero Lebesgue time at the added endpoints, the time integral in (4.5) is independent of the values assigned to q^{r} there. By [13, Chapter IV, Section 4.11], Y^{r} has a symmetric, jointly continuous transition

density $p^{\mathrm{r}}_t(y,z)$ with respect to m^{r}. The transition-density estimate used in the proof of Proposition 4.1, with p_t and μ replaced by p^{r}_t and m^{r}, respectively, proves absolute integrability in (4.5). It also proves independence of $\widehat{f}$ and $\widehat{f}^{\mathrm{r}}$, since any two resulting functions on J^{r} agree m^{r}-almost everywhere.

The continuity assertion and identification with $T^q_t f$ follow by the same arguments as in the proof of Proposition 4.1, with the Dirichlet realization replaced by the maximal realization. □

By Proposition 4.4 and Theorem 3.2, for every $f \in L^2(I,\mu)$ the function $u(t,x) = S^{q,\mathrm{r}}_t f(x)$, $t>0$ and $x \in I$, satisfies all the conclusions of Theorem 3.2 for $\mathcal{V} = \mathcal{V}_{\max}$. Proposition 3.4(ii) gives, at every regular endpoint $e \in \{\ell, r\}$,

$$\lim_{x\to e} \frac{\partial_x u(t,x)}{\rho(x)} = 0, \qquad t>0,$$

where the limit is taken within I. This is the homogeneous Neumann condition in the scale coordinate. If ρ is bounded near e, then also $\lim_{x\to e} \partial_x u(t,x) = 0$.

5. Sharpness and a two-dimensional counterexample

The three cases of Example 5.1 lie in the Engelbert–Schmidt framework and show that discontinuity of any one of a, b, or q can destroy twice differentiability. In that example, A and B denote the operators associated with the Dirichlet form domain $\mathcal{V}_0$. By Proposition 4.1, the corresponding semigroup values are the killed Feynman–Kac values. Example 5.2 shows that the one-dimensional regularity conclusion does not extend to two dimensions under continuity and uniform ellipticity alone.

Example 5.1 (Discontinuous coefficients)**.** *Consider the following three choices of I, a, b, q, and F.*

(i) *Let*

$$I = \left(-\frac{\pi}{2}, \frac{\pi}{\sqrt{2}}\right), \qquad b = q = 0, \qquad a(x) = \begin{cases} 1, & x<0, \\ 2, & x \geq 0, \end{cases}$$

and define

$$F(x) = \begin{cases} \cos x, & x \leq 0, \\ \cos(x/\sqrt{2}), & x \geq 0. \end{cases}$$

(ii) *Let*

$$I = \left(-\frac{3\pi}{4}, \frac{2\pi}{3\sqrt{3}}\right), \qquad a = 1, \qquad q = 0, \qquad b(x) = \mathbf{1}_{[0,\infty)}(x),$$

and define

$$F(x) = \begin{cases} \cos x - \sin x, & x \leq 0, \\ e^{-x/2}\left(\cos\frac{\sqrt{3}x}{2} - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}x}{2}\right), & x \geq 0. \end{cases}$$

(iii) *Let*

$$I = \left(-\frac{3\pi}{4}, 1\right), \qquad a = 1, \qquad b = 0, \qquad q(x) = \mathbf{1}_{[0,\infty)}(x),$$

and define

$$F(x) = \begin{cases} \cos x - \sin x, & x \le 0, \\ 1 - x, & x \ge 0. \end{cases}$$

In every case, the Engelbert–Schmidt conditions hold with $\sigma = \sqrt{2a}$. *Moreover,* $F \in C^1(I)$, F *vanishes at both endpoints,* $F' \in \mathrm{AC}(I)$, *and* $aF'' + bF' - qF = -F$ *almost everywhere. On the other hand,* $F''(0-) = -1$ *in every case, whereas* $F''(0+) = -1/2$ *in* (i) *and* $F''(0+) = 0$ *in* (ii) *and* (iii). *Thus* F *is not twice differentiable at* 0.

In each case, set $G = UF = F \circ s^{-1}$. *In each case,* a, b, *and* $1/a$ *are bounded, hence so is* b/a. *Since* I *is bounded,* ρ *is bounded above and away from zero. Since* F *and* F' *are bounded and* F *vanishes at the endpoints, a change of variables gives* $G \in \mathcal{V}_0$. *Moreover,* $G' = (F'/\rho) \circ s^{-1} \in \mathrm{AC}(J)$, *and the chain rule, together with* $\rho' = -(b/a)\rho$ *and* (2.2), *gives*

$$G''(s(x)) = \frac{F''(x)\rho(x) - F'(x)\rho'(x)}{\rho^3(x)} = \frac{a(x)F''(x) + b(x)F'(x)}{a(x)\rho^2(x)} = w(s(x))(q(x)-1)F(x)$$

almost everywhere. Since q *is bounded,* $(Q - I)G \in H$ *and integration by parts in* (2.4) *yields* $G \in \mathrm{Dom}(A)$ *and* $AG = QG - G$. *Hence* $G \in \mathrm{Dom}(B)$ *and* $BG = -G$. *Consequently,*

$$T_t^q F = U^{-1} e^{tB} G = e^{-t} F, \qquad t \ge 0.$$

Thus $T_t^q F$ *is not twice differentiable at* 0 *for any* $t > 0$. *Cases* (i), (ii), *and* (iii) *show, respectively, that continuity of* a, b, *and* q *cannot be omitted from Theorem 3.2*(iii), *even when the other two coefficients are continuous.*

Example 5.2 (Failure of $C^{1,2}$ regularity in two dimensions)**.** *The one-dimensional regularity conclusion of Proposition 3.5 and corollary 4.2 does not extend to two dimensions under mere continuity of the coefficient matrix. We use the construction in the proof of* [8, Theorem 4]. *For* $0 < r \le 1$ *set*

$$L(r) = 4 - \log r, \qquad \alpha(r) = \frac{4L(r) - 1}{L(r)^2 - 3L(r) + 1},$$

and, for $x \neq 0$, *define*

$$M(x) = I_2 + \frac{\alpha(|x| \wedge 1)}{|x|^2} \begin{pmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{pmatrix},$$

and set $M(0) = I_2$, *where* I_2 *denotes the* 2×2 *identity matrix. The matrix* M *is continuous and uniformly elliptic, but is not Hölder continuous at the origin (this is already seen along the* x_1*-axis).*

For $x \neq 0$, *define*

$$\Phi(x) = x_1 x_2 \left(4 - \log |x|\right)^2,$$

and set $\Phi(0) = 0$. *For* $0 < |x| < 1$, *a direct calculation in polar coordinates gives*

$$\operatorname{tr}\Big(M(x)D^2\Phi(x)\Big) = \frac{2x_1x_2}{|x|^2}\Big(1 - 4L(|x|) + \alpha(|x|)(L(|x|)^2 - 3L(|x|) + 1)\Big) = 0.$$

Thus, $\operatorname{tr}(MD^2\Phi) = 0$ *almost everywhere on the unit disc* $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$. *The function* Φ, *its first derivatives, and its almost-everywhere defined second derivatives belong to* $L^p(B_1)$ *for every* $p < \infty$. *Its mixed second derivative is nevertheless unbounded in every neighbourhood of the origin; in particular,* $\Phi \notin C^2(B_1)$.

We next turn to the parabolic Feynman–Kac interpretation. Since M *is bounded, continuous, and uniformly elliptic, the martingale problem for* $f \mapsto \operatorname{tr}(MD^2f)$ *is well posed; see* [19, Theorem 7.2.1]. *Let* Z *be the corresponding diffusion, equivalently a weak solution of*

$$\mathrm{d}Z_t = \sqrt{2M(Z_t)}\,\mathrm{d}W_t,$$

where W *now denotes a two-dimensional Brownian motion and* $\sqrt{2M(Z_t)}$ *is the symmetric positive-definite square root of* $2M(Z_t)$. *Let* $\tau = \inf\{t \geq 0 : Z_t \notin B_1\}$. *By the Itô–Krylov formula* [17, Theorem 2.10.1], *with* $\mathbb{E}_x^Z$ *denoting the expectation for the process* Z *started at* x,

$$\Phi(x) = \mathbb{E}_x^Z[\Phi(Z_{t\wedge\tau})], \qquad t \geq 0, \quad x \in B_1.$$

Thus the Feynman–Kac value with initial condition Φ *and boundary values* $\Phi|_{\partial B_1}$ *is stationary, i.e.,* $u(t,x) = \Phi(x)$. *In particular, it is not twice differentiable at the origin for any* $t > 0$.

Consequently, the parabolic initial-boundary value problem

$$\partial_t u = \operatorname{tr}(M(x)D^2u) \quad \text{on } (0,T]\times B_1, \qquad u(0,\cdot) = \Phi, \qquad u = \Phi \quad \text{on } (0,T]\times\partial B_1,$$

does not admit a solution in $C([0,T]\times\overline{B}_1)\cap C^{1,2}((0,T]\times B_1)$. *Indeed, if such a solution existed, applying Itô's formula with localization inside* B_1 *would identify it with the Feynman–Kac value above and hence with* Φ, *contradicting* $\Phi \notin C^2(B_1)$. *Thus, already for a time-homogeneous two-dimensional equation, continuity and uniform ellipticity of the coefficient matrix do not suffice for classical regularity.*

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