Pricing corporate bonds in an arbitrary jump-diffusion model based on an improved Brownian-bridge algorithm

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Abstract

We provide an efficient and unbiased Monte-Carlo simulation for the computation of bond prices in a structural default model with jumps. The algorithm requires the evaluation of integrals with the density of the first-passage time of a Brownian bridge as the integrand. Metwally and Atiya (2002) suggest an approximation of these integrals. We improve this approximation in terms of precision. From a modeler's point of view, we show that a structural model with jumps is able to endogenously generate stochastic recovery rates. It is well known that allowing a sudden default by a jump results in a positive limit of credit spreads at the short end of the term structure. We provide an explicit formula for this limit, depending only on the Lévy measure of the logarithm of the firm-value process, the recovery rate, and the distance to default.

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1 Introduction

The pricing of corporate bonds, being exposed to default and recovery risk, requires a model for the time and severity of default. The first structural default models were published in the seminal papers of Black and Scholes (1973), Merton (1974), and Black and Cox (1976). These approaches rely on geometric Brownian motion as a model for the value process of the firm. Following Black and Cox (1976), most of today's structural default models define the time of default as the first-passage time of the firm-value process below some threshold level.

A shortcoming of several model specifications is a predictable default time, which turns out to imply vanishing credit spreads for bonds with a short time to maturity, see for example the discussion in Jarrow and Protter (2004). This contradicts the empirical observation of credit spreads having a positive limit at the short end of the term structure. To overcome this problem, Zhou (2001) suggested modelling the logarithm of the firm-value process as the superposition of a diffusion and a jump component with normally distributed jumps. Additionally, Zhou (2001) provides an algorithm for pricing bonds within this framework. However, this algorithm is computationally expensive and implies a systematic bias.

To improve the situation, we interpret bond prices as options on the value of the firm and apply an elegant Monte-Carlo engine, originally developed by Metwally and Atiya (2002) for the pricing of barrier options in a jump-diffusion setting. One aim of this paper is to further improve Metwally and Atiya's (2002) algorithm and to adapt it to the pricing of corporate bonds. First of all, we include the option of allowing stochastic recovery rates. Secondly, we enhance the precision of an approximation to an integral which has to be evaluated as part of the algorithm. Due to the close relation of barrier options and structural default models, our improved approximation could easily be adapted for the pricing of barrier options.

Being able to work with an arbitrary jump-size distribution, we observe that the limit of credit spreads at time zero merely depends on the Lévy measure of the logarithm of the firm-value process, the recovery rate, and the distance to default. This observation is theoretically justified and an explicit formula for the limit of credit spreads is obtained.

Zhou's (2001) model and related structural default models with jumps have recently been discussed, extended, and empirically analyzed. The interested reader is referred to the following papers and references therein. Wong and Hodges (2002) incorporate a systematic risk component while relying on a framework close to that one of Zhou (2001). Joro and Na (2002) include a second jump component to model catastrophic events. An empirical analysis of structural default models is presented in Cremers et al. (2006) and Cserna and Imbierowicz (2008).

This paper is organized as follows. We introduce the model in Section 2. In Section 3, we compute the local default rate of a jump-diffusion process and derive the limit of credit spreads at the short end of the term structure. We discuss the Brownian-bridge pricing technique in Section 4 and a useful approximation in Section 5. We conclude with several numerical experiments concerning the runtime and accuracy of the algorithms in Section 6. The Appendix contains an outline of the proof of the approximation from Section 5.

2 Model description

Structural default models rely on a value based interpretation of default. Default occurs when the considered company cannot meet its financial obligations, to wit, when the firm value falls below a certain threshold. The company's liabilities are often used as the threshold level. Other interpretations are weighted averages of short- and long-term liabilities (Crosbie and Bohn (2003)), a minimum firm value required to operate the company (Black and Cox (1976)), or a default threshold tactically set by the equity owners (Leland (1994), Leland and Toft (1996)). Based on this model, default probabilities, bond and equity prices, and prices for credit derivatives are derived.

In our framework, we model the value of the respective company as a stochastic process $V = \{V_t\}_{t\geq 0}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $V_t = V_0 \exp(X_t)$ with $V_0 > 0$. Throughout this paper we work under the pricing measure $\mathbb{P}$. We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the natural filtration of the firm-value process, that is, $\mathcal{F}_t = \sigma(V_s : 0 \leq s \leq t)$, augmented to satisfy the usual conditions of completeness and right continuity. The process $X = \{X_t\}_{t\geq 0}$ is a jump-diffusion process given by

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

with $W = \{W_t\}_{t\geq 0}$ a Brownian motion. For simplicity, we assume a non-degenerate diffusion component, that is, $\sigma > 0$. The counting process $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\lambda \geq 0$. The sequence of jump sizes $\{Y_i\}_{i\geq 1}$ is i.i.d. with distribution $\mathbb{P}_Y$. Jump sizes $\{Y_i\}_{i\geq 1}$, the Poisson process N, and the Brownian motion W are mutually independent. We assume the company to default when its value process falls below the threshold d, that is, the default time is defined by $\tau := \inf\{t > 0 : V_t \leq d\}$.

Allowing the firm-value process to jump motivates a natural model of the default severity, which is therefore endogenously specified through the model. If a company defaults by a jump, its value process falls below the threshold d. This random undershoot might be used to specify the default severity, and, hence, the recovery rate. Therefore, we model the recovery rate w as a function of the firm-value to debt-level ratio at the default time. The bond holder receives the fraction $w(V_\tau/d) \in [0;1]$ of the face value in case of a default, where w is a positive, non-decreasing, and measurable function, defined on the unit interval $[0;1]$. Choosing a suitable w may, for instance, help consider default costs.

3 Pricing formula and credit spreads

In what follows, we focus w.l.o.g. on the pricing of zero-coupon bonds. Note that coupon bonds can be replicated through an appropriate portfolio of zero-coupon bonds and CDS contracts can be priced similarly. We denote the risk-free interest rate by r. The fair price $\phi(t,T)$, at time $t < \tau$, of a defaultable zero-coupon bond with maturity T and unit principal is given as the expectation of its discounted payoff with respect to the pricing measure $\mathbb{P}$. Thus, we have the pricing formula

$$\phi(t,T) \quad = \quad e^{-r(T-t)}\mathbb{P}(\tau > T|\mathcal{F}_t) + \mathbb{E}\left[e^{-r(\tau-t)} w\left(V_\tau/d\right) \mathbf{1}_{\{t<\tau\leq T\}} \middle| \mathcal{F}_t \right]. \qquad (1)$$

The credit spread which corresponds to $\phi(0,T)$ is denoted by η_T. It is implicitly defined as the real number that solves the equation $\phi(0,T) = \exp(-(r+\eta_T)T)$. As stated before, credit spreads for short-maturity bonds in a traditional pure diffusion model are smaller than the observed credit spreads on the market. However, we show that including negative jumps results in credit spreads which depend on the local default rate, defined as $\mathrm{LDR}_\tau := \lim_{h\downarrow 0} \mathbb{P}(\tau \leq h)/h$, and do not vanish as maturity decreases to zero.

We start by explicitly computing the local default rate. The next theorem shows that for absolutely continuous jump-size distributions, the local default rate is determined by the Lévy measure of the logarithm of the firm-value process and the distance to default, defined as $x_0 := \log(V_0/d)$. For the remainder of this paper, we assume $x_0 > 0$, that is, the company has not yet defaulted.

Theorem 3.1 (Local default rate).
The local default rate LDR_τ satisfies

$$LDR_\tau = \lambda\mathbb{P}(Y < -x_0) + \lambda\frac{1}{2}\mathbb{P}(Y = -x_0).$$

If the jump-size distribution is absolutely continuous, this simplifies to

$$LDR_\tau = \nu([-\infty; -x_0]),$$

where ν denotes the Lévy measure of the jump diffusion process X.

Proof. We condition on the number N_h of jumps occurring in $[0;h]$ and denote the first jump time by $\tau(h)$. We obtain

$$
\begin{aligned}
&\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(\tau\le h)\\
&\quad= \lim_{h\downarrow 0}\frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h=n)\mathbb{P}\left(\inf_{0\le s\le h}X_s\le -x_0\middle|N_h=n\right)\\
&\quad= \lim_{h\downarrow 0}\frac{e^{-\lambda h}}{h}\mathbb{P}\left(\inf_{0\le s\le h}(\gamma s+\sigma W_s)\le -x_0\right)+\\
&\qquad \lim_{h\downarrow 0}\lambda e^{-\lambda h}\mathbb{P}\left(\inf_{0\le s\le h}\left(\gamma s+\sigma W_s+\mathbf{1}_{\{s\ge\tau(h)\}}Y_1\right)\le -x_0\right)+\\
&\qquad \lim_{h\downarrow 0}\frac{1}{h}\sum_{n=2}^{\infty}\frac{e^{-\lambda h}(\lambda h)^n}{n!}\mathbb{P}\left(\inf_{0\le s\le h}\left(\gamma s+\sigma W_s+\sum_{j=1}^{N_s}Y_j\right)\le -x_0\middle|N_h=n\right).
\end{aligned}
$$

The first limit, representing a pure diffusion setup, is zero by l'Hospital's rule. The probabilities in the last limit are bounded by one, such that the limit also equals zero by the representation of an exponential function as a Taylor series.

We now examine the second limit, the case of exactly one jump. Writing $B_s := \gamma s+\sigma W_s$, $A_t(x) := \{\omega\in\Omega : \inf_{0\le s<t}B_s(\omega)\le x\}$, and $A_t^C(x) := \Omega\setminus A_t(x)$, we obtain by conditioning

$$
\begin{aligned}
&\mathbb{P}\left(\inf_{0\le s\le h}\left(B_s+\mathbf{1}_{\{s\ge\tau(h)\}}Y_1\right)\le -x_0\right)\\
&\quad= \mathbb{P}\left(A_{\tau(h)}(-x_0)\right)+\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{A^C_{\tau(h)}(-x_0)\cap\tilde{A}_{h-\tau(h)}(-x_0-Y_1-B_{\tau(h)})}\middle|B_{\tau(h)},Y_1\right]\right],
\end{aligned}
$$

where $\tilde{A}_t(x)$ is defined as $A_t(x)$ with B being replaced by the Brownian motion $\tilde{B}_s := B_{\tau(h)+s}-B_{\tau(h)}$. Since $\tau(h)\le h$ holds, the limit of the first term tends to zero with h. If $Y_1>-x_0$, the conditional expectation tends to zero, since $\mathbb{P}(B_{\tau(h)}\le -x_0-y)$ decreases to zero for all $y>-x_0$ and h tending to zero, due to the continuity of the diffusion part. If $Y_1<-x_0$, the conditional expectation tends to one, since so does $\mathbb{P}(B_{\tau(h)}\le -x_0-y)$ for all $y<-x_0$ and h tending to zero. If $Y_1=-x_0$ then the conditional expectation tends to zero if $B_{\tau(h)}>0$, and to one if $B_{\tau(h)}\le 0$, with h tending to zero. □

This result might be interpreted from an economic point of view. If a negative jump of the firm-value process exceeds the distance to default with a positive probability, that is, $\mathbb{P}(Y_1\le -x_0)>0$, the local default rate LDR_τ is positive, resembling a positive default intensity. Next, we derive the exact limit of credit spreads, as maturity decreases to zero, and show how the local default rate LDR_τ is involved therein.

Theorem 3.2 (Credit spreads at time zero)**.**
We assume the function w *, specifying the recovery rate, to be continuous. Furthermore we assume that either* $\mathbb{P}(Y_1 = -x_0) = 0$ *or* w *is constant. Then, the limit of credit spreads at time zero is given by*

$$\lim_{h \downarrow 0} \eta_h = \left(1 - \mathbb{E}\left[w\left(\frac{V_0 \exp(Y_1)}{d}\right) \middle| Y_1 \leq -x_0 \right]\right) LDR_\tau. \tag{2}$$

Proof. We observe from the proof of Theorem 3.1 that an immediate default of a solvent company only happens through a jump, to wit, for sufficiently small $h > 0$,

$$\mathbb{P}(\tau \leq h) = \mathbb{P}(\{N_h = 1\} \cap \{Y_1 \leq -x_0\}) + o(h),$$

assuming $\mathbb{P}(Y_1 = -x_0) = 0$. Thus, we obtain, with $B_s := \gamma s + \sigma W_s$,

$$\begin{aligned}
\lim_{h \downarrow 0} \mathbb{E}\left[e^{-r\tau} w\left(\frac{V_\tau}{d}\right) \middle| \tau \leq h \right] &= \lim_{h \downarrow 0} \mathbb{E}\left[e^{-r\tau} w\left(\frac{V_0 \exp(B_\tau + Y_1)}{d}\right) \middle| \{N_h = 1\} \cap \{Y_1 \leq -x_0\} \right] \\
&\leq \lim_{h \downarrow 0} \mathbb{E}\left[\sup_{0 \leq t \leq h} w\left(\frac{V_0 \exp(B_t + Y_1)}{d}\right) \middle| Y_1 \leq -x_0 \right] \\
&= \mathbb{E}\left[w\left(\frac{V_0 \exp(Y_1)}{d}\right) \middle| Y_1 \leq -x_0 \right]
\end{aligned}$$

due to the independence of the Poisson process N, the continuity of w, and dominated convergence. We have formally set $w(x) = w(1)$ for $x > 1$. A lower estimate with $\exp(-rh)$ replacing 1 and an infimum replacing the supremum yields the same bound, so that equality follows. For constant w, similar computations hold.

Using the definition of credit spreads and the pricing formula of Equation (1), we obtain

$$\begin{aligned}
\eta_h &= -\frac{1}{h} \log\left(\phi(0,h)\right) - r \\
&= -\frac{1}{h} \log\left(e^{-rh} \left(1 - \mathbb{P}(\tau \leq h) + \mathbb{P}(\tau \leq h) e^{rh} \mathbb{E}\left[e^{-r\tau} w\left(\frac{V_\tau}{d}\right) \middle| \tau \leq h \right] \right) \right) - r \\
&= -\frac{1}{h} \log\left(1 - \mathbb{P}(\tau \leq h) \left(1 - e^{rh} \mathbb{E}\left[e^{-r\tau} w\left(\frac{V_\tau}{d}\right) \middle| \tau \leq h \right] \right) \right).
\end{aligned}$$

Applying the inequalities $\log(x) \leq x - 1$ and $\log(x) \geq (x-1)/x$ yields a lower and an upper bound, which both have the expression of Equation (2) as their limit. This concludes the proof. □

Theorem 3.2 establishes that the limit of credit spreads at the short end of the term structure is the product of the local default rate and the expected loss given default. This is reasonable from an economic point of view since the local default rate approximates the probability of a default within a small time interval. Therefore, credit spreads of bonds with a small maturity merely depend on the probability of a sudden default, in other words, credit spreads are increasing in the local default rate.

4 Brownian-bridge pricing technique

Zhou (2001) presents an algorithm based on a Monte-Carlo simulation for estimating bond prices in a jump-diffusion framework. His idea is to discretize the time to maturity and to sample trajectories of the firm-value process on this grid. Then, on each grid point, it is checked whether the company defaults or not. However, we show that this algorithm produces biased bond prices and is very time-consuming. Metwally and Atiya (2002) suggest another algorithm, which is again based on a Monte-Carlo simulation. This algorithm is designed for pricing barrier options in a jump-diffusion model for the respective underlying. In contrast to Zhou's (2001) algorithm, it not only produces unbiased results, but is also significantly faster. The principal idea of this ansatz is to condition on the number of jumps, the jump times, and the values of the jump-diffusion process at these times.

Below, we provide an algorithm which allows us to include stochastic recovery rates depending on the value of the company at the default time. First, we generate the number and locations of the jumps. Then, we generate the value of X immediately before and after each jump. More precisely, if $(\tau_1, \tau_2, \ldots)$ denotes the sequence of jump times, the value of X immediately before the first jump is a sample drawn from a Gaussian distribution with mean $\gamma\tau_1$ and variance $\sigma^2\tau_1$. The value at the first jump time is obtained by adding a realization of the jump-size distribution to this number. For the value immediately before the second jump time, we add another sample drawn from a Gaussian distribution with mean $\gamma(\tau_2 - \tau_1)$ and variance $\sigma^2(\tau_2 - \tau_1)$, and so on. We can then check whether the company defaults at one of these jump times. The probability of the company defaulting between two jumps is given by the probability of a Brownian bridge not crossing a certain barrier b. This probability is calculated by Metwally and Atiya (2002). We simplify it for the readers' convenience. For that, let X denote a Brownian bridge over $[t_0; t_1]$ with volatility σ, pinned at X_{t_0} and X_{t_1}. Let $b \in \mathbb{R}$ denote an arbitrary barrier and S_b the first passage time of b by X. Then, we have for the density g of the passage time S_b for all $t \in (t_0; t_1]$

$$
\begin{aligned}
g(t)dt &:= \mathbb{P}(S_b \in dt | X_{t_0}, X_{t_1}) \qquad (3) \\
&= \mathbf{1}_{\{X_{t_0} > b\}} \frac{X_{t_0} - b}{2y\pi\sigma^2(t-t_0)^{3/2}(t_1-t)^{1/2}} \exp\left(-\frac{(X_{t_1}-b)^2}{2(t_1-t)\sigma^2} - \frac{(X_{t_0}-b)^2}{2(t-t_0)\sigma^2} \right) dt,
\end{aligned}
$$

where

$$y = \frac{1}{\sqrt{2\pi\sigma^2(t_1 - t_0)}} \exp\left(-\frac{(X_{t_1} - X_{t_0})^2}{2\sigma^2(t_1 - t_0)}\right). \quad (4)$$

By integration, we obtain the probability of X falling below the barrier b:

$$\begin{aligned} \tilde{\Phi}_b^{BB}(X_{t_0}, X_{t_1}, t_1 - t_0) &:= \mathbb{P}\left(\min_{t_0 \le s \le t_1} X_s \le b \middle| X_{t_0}, X_{t_1}\right) \\ &= \mathbf{1}_{\{X_{t_0} \le b \text{ or } X_{t_1} \le b\}} + \\ &\quad \mathbf{1}_{\{X_{t_0} > b \text{ and } X_{t_1} > b\}} \exp\left(-\frac{2(X_{t_0} - b)(X_{t_1} - b)}{(t_1 - t_0)\sigma^2}\right). \end{aligned}$$

Theorem 4.1 constitutes the theoretical justification of the Monte-Carlo simulation which we introduce below. The theorem is a direct consequence of Equation (1).

Theorem 4.1 (Price of a zero-coupon bond)**.**
The zero-coupon bond price of Equation (1) can be expressed as

$$\begin{aligned} \phi(0,T) &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tau > T\}} e^{-rT} + w(V_\tau/d)\mathbf{1}_{\{\tau \le T\}} e^{-r\tau} \middle| \mathcal{F}^*\right]\right] \quad (5) \\ &= \sum_{k=0}^{\infty} \int_{\substack{(\tau_1,\dots,\tau_k) \\ \in [0;T]^k}} \int_{\substack{(x_1,\dots,x_{k+1}) \\ \in(-\infty;\infty)^{k+1}}} \int_{\substack{(y_1,\dots,y_k) \\ \in(-\infty;\infty)^k}} \mathbb{E}\left[\mathbf{1}_{\{\tau > T\}} e^{-rT} + w(V_\tau/d)\mathbf{1}_{\{\tau \le T\}} e^{-r\tau} \middle| \mathcal{F}^*\right] \cdot \\ &\quad \prod_{j=1}^{k} \mathbb{P}_Y(dy_j) \cdot \prod_{j=1}^{k+1} \varphi_{\gamma\Delta\tau_j, \sigma^2\Delta\tau_j}(x_j) dx_j \cdot \\ &\quad \mathbf{1}_{\{0 < \tau_1 < \ldots < \tau_k < T\}} \frac{k!}{T^k} d(\tau_1, \dots, \tau_k) \cdot \frac{(\lambda T)^k}{k!} e^{-\lambda T}, \end{aligned}$$

where

$$\mathcal{F}^* := \sigma\left\{N_T;\ 0 < \tau_1 < \ldots < \tau_{N_T} < T;\ X_{\tau_1 -}, X_{\tau_1}, \dots, X_{\tau_i -}, X_{\tau_i}, \dots, X_T\right\}$$

is the σ-algebra representing the information from the number of jumps, their location, and the values of X immediately before the jump times, at the jump times, and at maturity. The function $\varphi_{\gamma\Delta\tau_j, \sigma^2\Delta\tau_j}$ represents the probability-density function of the normal distribution with mean $\gamma(\tau_j - \tau_{j-1})$ and variance $\sigma^2(\tau_j - \tau_{j-1})$, where $\tau_0 = 0$ and $\tau_{N_T+1} = T$.

For $b = \log(d/V_0)$, the conditional expectation satisfies

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}_{\{\tau > T\}} e^{-rT} + w(V_\tau/d)\mathbf{1}_{\{\tau \le T\}} e^{-r\tau} \middle| \mathcal{F}^*\right] \\ &\quad = w(1) \sum_{i=1}^{U} \prod_{j=1}^{i-1} \Phi_b^{BB}(j) \int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s) ds + \\ &\qquad w(V_{\tau_I}/d)\mathbf{1}_{\{I \ne 0\}} e^{-r\tau_I} \prod_{j=1}^{I} \Phi_b^{BB}(j) + \mathbf{1}_{\{I=0\}} e^{-rT} \prod_{j=1}^{N_T+1} \Phi_b^{BB}(j), \quad (6) \end{aligned}$$

where
$$I := \min\{i \in \{1,\ldots,N_T\} : X_{\tau_i} \le b\}, \quad \min\emptyset := 0,$$
denotes the index of the first jump time such that X_{τ_I} *crosses the barrier,*

$$U := \begin{cases} I & \text{if } I \neq 0, \\ N_T+1 & \text{if } I = 0, \end{cases}$$

$\Phi_b^{BB}(j) := 1-\tilde{\Phi}_b^{BB}(X_{\tau_{j-1}}, X_{\tau_j -}, \tau_j - \tau_{j-1})$ *represents the probability of the company not defaulting within the interval* $(\tau_{j-1};\tau_j)$ *, and* $g_i(t)dt = \mathbb{P}(S_b \in dt | X_{\tau_{i-1}}, X_{\tau_{i-}})$ *is defined, as in Equation (3), as the density of the company defaulting at time* t *for* $t \in (\tau_{i-1};\tau_i)$ *.*

We remark that the approach outlined in the last theorem can be interpreted as the variance-reduction technique Conditional Monte Carlo, see Boyle et al. (1997), Section 2.8. It allows us to replace several simulation steps by analytic formulas. Based on the last theorem, we now formally introduce our Brownian-bridge pricing algorithm.

Algorithm 4.1 (Brownian-bridge pricing algorithm)**.**
Choose the number of simulation runs K *and approximate* $\phi(0,T)$ *by*

$$\phi(0,T) \approx \frac{1}{K}\sum_{j=1}^{K}\phi_j(0,T),$$

where $\phi_j(0,T)$ *is calculated in simulation run* j *by these steps:*

1. *Simulate the number of jumps* N_T *from a* $Poi(\lambda T)$ *distribution.*
2. *Simulate the jump times* $\tau_1 < \tau_2 < \ldots < \tau_{N_T}$ *. Conditional on* N_T *, these jumps are distributed as order statistics on* $[0;T]$ *, see Sato (1999), p. 17.*
3. *Generate two series of mutually independent random variables* $x_1,\ldots,x_{N_T+1}$ *and* $y_1,\ldots,y_{N_T}$ *, independent from* N_T *, with*
$$\begin{aligned} x_i &\sim \mathcal{N}\left(\gamma(\tau_i-\tau_{i-1}), \sigma^2(\tau_i-\tau_{i-1})\right) \quad \text{and} \\ y_i &\sim \mathbb{P}_Y. \end{aligned}$$
4. *Calculate inductively* $X_0,\, X_{\tau_1-},\, X_{\tau_1},\, X_{\tau_2-},\ldots,X_{\tau_{N_T}},\, X_{\tau_{N_T+1}-} = X_{\tau_{N_T+1}}$ *by*
$$\begin{aligned} X_{\tau_0} &= 0, \\ X_{\tau_i-} &= X_{\tau_{i-1}} + x_i, \quad \forall\, i \in \{1,\ldots,N_T+1\}, \\ X_{\tau_i} &= X_{\tau_i-} + y_i, \quad \forall\, i \in \{1,\ldots,N_T\}. \end{aligned}$$

5. *Determine* I *,* U *, and* b *as in Theorem 4.1.*
6. *Calculate*

$$\phi_j(0,T) = I\!E\left[\mathbf{1}_{\{\tau>T\}}e^{-rT} + w(V_\tau/d)\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}\,\middle|\,\mathcal{F}^*\right]$$

as in Equation (6) of Theorem 4.1.

The expected runtime of this algorithm depends about linearly on the expected number of jumps, that is λT. The larger the jump intensity, the more samples have to be drawn and the more integrals have to be calculated. We illustrate this relation in Section 6.1, where we compare the runtime of the algorithm for different parameter sets.

5 Accelerating the algorithm

The most time-consuming step of Algorithm 4.1 is the computation of the integrals $\int_{\tau_{i-1}}^{\tau_i}\exp(-rs)g_i(s)ds$. Metwally and Atiya (2002) suggest an approximation of these integrals, which we improve below. The core idea is to calculate the Laplace transform of the integral, which admits a representation as the convolution of two functions. Then, this Laplace transform is expanded into a Taylor series in r. In the next step, the Laplace inverse of the second-order approximation is obtained. Our calculations yield a different result from that in the original paper of Metwally and Atiya (2002). However, numerical experiments which we present in Section 6.2 indicate that our approximation might be closer to the correct value. An outline of the proof of the next theorem is given in the appendix.

Theorem 5.1 (Approximation of the integral)**.**
We assume that $X_{\tau_{i-1}} > b$*. The integral in Equation (6) can be approximated by*

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs}g_i(s)ds = e^{-r\tau_{i-1}}\left(\exp\left(-\frac{2(X_{\tau_{i-1}}-b)(X_{\tau_i-}-b)}{\Delta\tau_i\sigma^2}\right) + \frac{r(X_{\tau_{i-1}}-b)}{4\sigma}(A_1+C_1B)\right) + O(r^3) \tag{7}$$

if $X_{\tau_i-} > b$ *and by*

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs}g_i(s)ds = e^{-r\tau_{i-1}}\left(1 + \frac{r(X_{\tau_{i-1}}-b)}{4\sigma}(A_2+C_2B)\right) + O(r^3) \tag{8}$$

if $X_{\tau_i -} \le b$ *, where* $\Delta\tau_i = \tau_i - \tau_{i-1}$ *,* $\Delta X_i = X_{\tau_i -} - X_{\tau_{i-1}}$ *,*

$$
\begin{aligned}
A_1 &= -\frac{r}{\sigma}\Delta\tau_i \Delta X_i \exp\left(-\frac{2(X_{\tau_{i-1}} - b)(X_{\tau_i -} - b)}{\Delta\tau_i \sigma^2}\right), \\
C_1 &= -\sqrt{2\pi\Delta\tau_i}\exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right)\Phi\left(\frac{2b - X_{\tau_i -} - X_{\tau_{i-1}}}{\sqrt{\Delta\tau_i\sigma^2}}\right), \\
B &= 4 - r\Delta\tau_i - \frac{r}{\sigma^2}\Delta X_i \left(X_{\tau_i -} + X_{\tau_{i-1}} - 2b\right), \\
A_2 &= \frac{r}{\sigma}\Delta\tau_i \left(X_{\tau_i -} + X_{\tau_{i-1}} - 2b\right), \\
C_2 &= -\sqrt{2\pi\Delta\tau_i}\exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right)\Phi\left(\frac{\Delta X_i}{\sqrt{\Delta\tau_i\sigma^2}}\right)
\end{aligned}
$$

with Φ *denoting the cumulative normal distribution function.*

It is important to note that using this approximation in Step 6 of Algorithm 4.1 introduces a small bias. While, in contrast to Zhou's (2001) algorithm, the Brownian-bridge pricing technique of the last section is unbiased, this approximation method relies on a Taylor series expansion. More precisely, the error terms in Equations (7) and (8) can be shown to be negative, which causes the approximated bond prices to be slightly larger and the corresponding credit spreads to be slightly lower.

We remark that the approximation of Theorem 5.1 can also be used when pricing barrier options with continuously monitored barriers. The same integrals appear in this case, see Metwally and Atiya (2002).

6 Numerical experiments

Section 6.1 contains the results of a numerical comparison of Zhou's (2001) algorithm and the Brownian-Bridge pricing technique. Section 6.2 compares Metwally and Atiya's (2002) approximation with ours of the integral $\int_{\tau_{i-1}}^{\tau_i} \exp(-rs) g_i(s) ds$ appearing in Equation (6). We implemented all algorithms in C, using the NAG-software library, see `www.nag.co.uk`, to generate the required samples and to evaluate the required integrals. We worked on a Sun computer equipped with an UltraSPARC-III+ processor (900MHz). To provide a benchmark of the runtime, the output user time of the Unix command `timex` was chosen.

6.1 Runtime and precision

In this section, we provide a numerical comparison of all aforementioned algorithms. Concerning Zhou's (2001) algorithm, we use two different discretizations.

The number of grid points is set to $12T$ and $250T$, respectively, where T denotes the bond's maturity in years. This corresponds to checking whether the bond defaulted once per month and once per trading day, respectively. As parameters, we set $r = 0.04$, $\gamma = 0.045$, $\sigma = 0.05$, and $T = 5$. Jump sizes are assumed to be two-sided exponentially distributed, that is,

$$\mathbb{P}_Y(dx) = p\lambda_{\oplus} e^{-\lambda_{\oplus} x} \mathbf{1}_{\{x \geq 0\}} dx + (1-p)\lambda_{\ominus} e^{\lambda_{\ominus} x} \mathbf{1}_{\{x<0\}} dx$$

with $p = 0.5$. This jump structure has been theoretically analyzed by Kou and Wang (2003). The ratio d/V_0 is set to 80%. We run all simulations in four different scenarios. In the first three scenarios, the recovery rate is constant with $w(x) \equiv 40\%$. In the scenario titled "Low", we expect only $\lambda = 0.5$ jumps per year but they are typically large, that is, $\lambda_{\ominus} = \lambda_{\oplus} = 10$. The scenario "Middle" corresponds to $\lambda = 2$ and $\lambda_{\ominus} = \lambda_{\oplus} = 20$. In the scenario "High", $\lambda = 8$ jumps per year are expected with $\lambda_{\ominus} = \lambda_{\oplus} = 40$. The scenario "Stochastic" has the same jump structure as the scenario "Middle", but the recovery rate is stochastic with $w(x) = 0.5x$. It can be shown that the volatility and expectation of the underlying Lévy process X remain the same in all scenarios. For each algorithm and scenario, we estimate bond prices using one million simulation runs. Since the Brownian-bridge pricing technique generates an unbiased estimate of the price, we additionally run this method using ten million simulation runs and interpret the respective results as the correct price in the different scenarios.

Each algorithm is started twice. In the first round, we measure the required runtime. In the second round, we use the same generated samples for the algorithms presented in Sections 4 and 5 in order to compare the approximation of the integrals. We present the obtained credit spreads (in bps) of the second round in Table 1, which also includes the relative error of the credit spread which we define as $(\text{spread} - \text{generated spread})/\text{spread}$. More precisely, while "spread" denotes the credit spread obtained from the Brownian-bridge simulation with ten million simulation runs, "generated spread" represents the credit spread from the corresponding algorithm.

The results show that Zhou's (2001) algorithm produces a significant bias. When simulating with only 12 grid points per year, the relative error exceeds 7%. Even with 250 grid points per year, the relative error is at least 2.8%, which is still above typically observed bid-ask spreads. However, using Brownian-bridge based techniques brings the relative error down to less than 1%. The lowest errors are about 0.1% to 0.2% percent and are obtained by using the Brownian-bridge technique of Section 4 and the approximation of Section 5. Bond prices are roughly between 0.76 and 0.78 in all scenarios. The sample standard deviations of the bond price samples are approximately 0.13, 0.14, 0.15, and 0.12 for the scenarios "Low", "Middle", "High", and "Stochastic", respectively. They are almost identical for all methods. These numbers can be used to approximate standard deviations of derived credit spreads by means of the so-called Delta method, see Van der Vaart (1998), Chapter 3. These computed standard deviations do not depend much on the method used, either. Divided by the square root of one million to adjust for the sample size, they are approximately 0.34, 0.36, 0.39, and 0.30 bps. The spread computed by Zhou's (2001) algorithm with one million simulation runs is therefore at least nine standard deviations off its true value.

As the discretization gets finer, the runtime of Zhou's (2001) algorithm increases dramatically but does not depend on the expected number of jumps. In contrast, the runtime of the methods based on the Brownian-bridge technique increase approximately linearly in the expected number of jumps. The reason therefore is the dependence of the number of random variables that have to be drawn and the number of integrals which have to be calculated on the number of jumps. However, even in the scenario "High" with eight expected jumps per year, the Brownian-bridge pricing algorithm, which does not use any approximations to improve runtime, is more than twice as fast as Zhou's (2001) approximation with daily discretization. We also observe that the approximation of the integrals significantly reduces the runtime by a factor of approximately 6.

6.2 A comparison with Metwally and Atiya (2002)

Our approximation of the integral $\int_{\tau_{i-1}}^{\tau_i} \exp(-rs)g_i(s)ds$ differs from the approximation in Metwally and Atiya (2002). They multiply the second term in the sum of Equations (7) and (8) by the factor $\exp(\Delta\tau_i)$, which does not appear in our formulas, and they evaluate Φ at a different position. More precisely, in Metwally and Atiya (2002) Φ is evaluated at $(2b - X_{\tau_i-} - X_{\tau_{i-1}})/(\sqrt{2\Delta\tau_i\sigma^2})$ (resp. $(\Delta X_i)/(\sqrt{2\Delta\tau_i\sigma^2})$) in C_1 (resp. C_2). We perform two kinds of simulations to compare both approximations. Firstly, we compare the approximations for randomly generated parameters. Secondly, we compare bond prices obtained by both approximations.

Step 1: We generate 500,000 random numbers for r, σ, τ_1, b, and $X_{\tau_1} - b$. More precisely, in each simulation run we draw a uniformly distributed random variable on $[0; 0.1]$ (resp. $[0.1; 0.5]$, $[0.5; 2.0]$, $[-0.2; -0.01]$, $[-0.2; 0.2]$) for r (resp. σ, τ_1, b, $X_{\tau_1} - b$). For every such parameter set, we calculate the relative error of the approximation to $\int_0^{\tau_1} \exp(-rs)g_1(s)ds$ suggested by Metwally and Atiya (2002) and of our approximation. After 500,000 simulation runs, the average relative error of the original (resp. our) approximation was found to be 1.553% (resp. 0.001%).

Step 2: To test the effect of the improved approximation on estimated bond prices,

		Low	Middle	High	Stochastic
	Spread in bps	104.5	117.0	125.1	97.1
Zhou (12)	Rel. error in %	7.35	9.81	11.10	9.54
	Runtime in h	0:07:32	0:07:57	0:10:17	0:08:11
	Spread in bps	109.6	125.8	136.2	103.0
Zhou (250)	Rel. error in %	2.85	3.04	3.25	4.01
	Runtime in h	2:33:09	2:31:48	2:34:49	2:32:48
	Spread in bps	112.7	129.8	141.0	107.4
Brown. bridge	Rel. error in %	0.14	-0.07	-0.14	-0.04
	Runtime in h	0:06:39	0:19:05	0:58:47	0:19:08
	Spread in bps	113.5	130.3	141.1	108.0
Taylor (orig.)	Rel. error in %	-0.59	-0.45	-0.23	-0.62
	Runtime in h	0:00:44	0:02:28	0:09:24	0:02:28
	Spread in bps	112.5	129.7	140.9	107.2
Taylor (our)	Rel. error in %	0.24	0.02	-0.06	0.08
	Runtime in h	0:00:45	0:02:25	0:09:17	0:02:25
Brown. bridge (10 Mil.)	Spread in bps	112.8	129.7	140.8	107.3

Table 1: This table compares Zhou's (2001) algorithm using a monthly and daily discretization, the Brownian-bridge pricing algorithm, Metwally and Atiya's (2002), and our approximation. All results are based on one million simulation runs. We interpret as true spread the results of an unbiased Brownian-bridge pricing technique relying on ten million simulation runs, summarized in the last row. The table lists computed credit spreads, the relative error, and runtime for each algorithm and four different sets of parameters.

we implement both algorithms in the context of the aforementioned Monte-Carlo simulation and estimate bond prices for different parameter sets and interest rates. We use the first three scenarios of Table 1 with parameters $\gamma = 0.025$, $\sigma = 0.05$, and two-sided exponentially distributed jumps with $p = 0.5$. The recovery rate is set to $w(x) \equiv 40\%$, the maturity to $T = 5$. The interest rate r is varied between 2.5% and 25%. We are aware of the fact that $r = 25\%$ is not a realistic assumption. Nevertheless, simulations with high interest rates illustrate how the original approximation becomes inaccurate. For each scenario, approximation, and interest rate, we run Algorithm 4.1 with ten million simulation runs. Based on the estimated bond prices, we compute the relative pricing error $(p_u - p_a)/p_u$, where p_u (resp. p_a) represents the unbiased (resp. approximated) bond price. Table 2 exhibits the results. Our simulations show that the approximation of Section 5 implies a lower relative pricing error than the one of Metwally and Atiya (2002) almost always, except for the scenario "High" when r equals 2.5% and the relative pricing errors are 0.0085% and -0.0086%, respectively.

Since both approximations rely on a Taylor series expansion around r we expect the approximation error to increase as r increases. This is confirmed by the numerical results listed in Table 2. The approximation performs best for low interest rates. For very high interest-rate scenarios, Metwally and Atiya's (2002) approximation can be outperformed by a factor between approximately 20 and 120, depending on the expected number of jumps. But already for $r = 5\%$, their approximation can be outperformed by a factor of 3 or more, without any additional computational needs.

	Low		Middle		High	
r in %	Original	Our	Original	Our	Original	Our
2.5	0.0726	-0.0065	0.0356	-0.0082	0.0085	-0.0086
5.0	0.1836	-0.0072	0.0893	-0.0087	0.0276	-0.0092
10.0	0.5505	-0.0091	0.2392	-0.0104	0.0745	-0.0106
15.0	1.2287	-0.0128	0.4624	-0.0121	0.1391	-0.0110
20.0	2.4000	-0.0214	0.7884	-0.0143	0.2212	-0.0138
25.0	4.4144	-0.0361	1.2532	-0.0176	0.3305	-0.0150

Table 2: This table summarizes the relative pricing errors of bonds (in percentage points) using the approximations of Metwally and Atiya (2002) ("Original") and the approximation of Section 5 ("Our"). The scenarios are as in Table 1. The first column contains the assumed interest rates.

7 Conclusion

In this paper, we discussed a structural default model based on a jump-diffusion process as a model for the value of the respective company. We explained how stochastic recovery rates are generated through the model. Then, we showed that credit spreads remain positive for bonds with small maturity, a property that matches empirical observations and overcomes a major shortfall of most pure diffusion models. Moreover, we calculated the exact limit of credit spreads as maturity tends to zero. In order to price defaultable bonds and more complex credit derivatives within our framework, the distribution of the first-passage time of the jump-diffusion process which models the value of the firm is required. For discontinuous processes, this distribution is not known in general. Therefore, several Monte-Carlo pricing algorithms have been suggested. Our algorithm allows stochastic recovery rates and an arbitrary jump-size distribution. We also showed that our algorithm is unbiased and noticeably faster than algorithms which rely on simulations of complete trajectories of the firm-value process. To further accelerate the algorithm, we presented an improved approximation technique of Metwally and Atiya (2002), which was originally introduced for the pricing of barrier options.

Appendix: Proof of Theorem 5.1

Note that the following computations for the proof of Theorem 5.1 are related to the derivation in Metwally and Atiya (2002).

Proof. A substitution and some calculations show that the integral can be written as the convolution of two functions. We find

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rx} g_i(x)dx = e^{-r\tau_{i-1}} \int_0^{\Delta\tau_i} f(x)h(\Delta\tau_i - x)dx,$$

where

$$f(x) = \frac{e^{-rx}\Delta_0}{\sqrt{2\pi\sigma^2}} x^{-\frac{3}{2}} \exp\left(-\frac{\Delta_0^2}{2x\sigma^2}\right)$$

and

$$h(x) = \frac{1}{\sqrt{2\pi\sigma^2 y}} x^{-\frac{1}{2}} \exp\left(-\frac{\Delta_1^2}{2x\sigma^2}\right)$$

with y as in Equation (4), $\Delta_0 := X_{\tau_{i-1}} - b$, and $\Delta_1 := X_{\tau_i -} - b$. For later use, we define

$$\alpha := \frac{1}{\sqrt{2}\sigma y} = \sqrt{\pi\Delta\tau_i} \exp\left(\frac{(\Delta X_i)^2}{2\sigma^2\Delta\tau_i}\right),$$
$$\beta := 1 - \Phi\left(\frac{\Delta_0 + |\Delta_1|}{\sigma\sqrt{\Delta\tau_i}}\right).$$

We now calculate the Laplace transform of the integral. Being a convolution, its Laplace transform is the product of the Laplace transforms of f and h. Formulas 5.28 and 5.30 of Oberhettinger and Badii (1973), Chapter 1.5, yield for the Laplace transform

$$\begin{aligned} l_r(s) &:= \left(\mathcal{L}\left(\int_0^t f(x)h(t-x)dx\right)\right)(s) \\ &= \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sigma y\sqrt{2s}} \exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right). \end{aligned}$$

The second-order Taylor expansion of $l_r(s)$ around zero is given by

$$l_r(s) = l_0(s) + r\left(\frac{\delta}{\delta r} l_r(s)\right)\Bigg|_{r=0} + \frac{r^2}{2}\left(\frac{\delta^2}{\delta r^2} l_r(s)\right)\Bigg|_{r=0} + \frac{r^3}{6}\left(\frac{\delta^3}{\delta r^3} l_r(s)\right)\Bigg|_{r=r_s^*}, \quad (9)$$

where $r_s^* \in (0; r)$ depends on s. We use Formulas 5.87, 5.89, 5.92, and 5.94 of Oberhettinger and Badii (1973), Chapter 2.5, to calculate the Laplace inverses of the derivatives:

$$\begin{aligned}
\left(\mathcal{L}^{-1}\left(l_0\right)\right)(\Delta\tau_i) &= \exp\left(\frac{(\Delta X_i)^2}{2\sigma^2\Delta\tau_i}\right)\cdot\exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right) \\
&= \begin{cases} 1, & X_{\tau_i-}\leq b, \\ \exp\left(\frac{-2(X_{\tau_{i-1}}-b)(X_{\tau_i-}-b)}{\sigma^2\Delta\tau_i}\right), & X_{\tau_i-}>b, \end{cases} \\
\left(\mathcal{L}^{-1}\left(\frac{\delta}{\delta r}l_r|_{r=0}\right)\right)(\Delta\tau_i) &= -\sqrt{2}\frac{\Delta_0}{\sigma}\alpha\beta, \\
\left(\mathcal{L}^{-1}\left(\frac{\delta^2}{\delta r^2}l_r|_{r=0}\right)\right)(\Delta\tau_i) &= \frac{2\sqrt{\Delta\tau_i}\exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right)}{\sqrt{\pi}}\left(\frac{\Delta_0^2\alpha}{2\sigma^2}-\frac{\Delta_0+|\Delta_1|}{\sqrt{2}\sigma}\frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma}\right)+ \\
&\quad 2\beta\left(\frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma}\left(\Delta\tau_i+\frac{(\Delta_0+|\Delta_1|)^2}{\sigma^2}\right)-\frac{\Delta_0^2\alpha}{2\sigma^2}\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\right) \\
&= \frac{\sqrt{\Delta\tau_i}\exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right)\alpha\Delta_0(\Delta_0-|\Delta_1|)}{2\sqrt{\pi}\sigma^2}+ \\
&\quad \beta\frac{\Delta_0\alpha}{\sqrt{2}\sigma}\Delta\tau_i+\beta\frac{\Delta_0\alpha}{\sqrt{2}\sigma^3}(\Delta_1^2-\Delta_0^2).
\end{aligned}$$

By combining these equations with the representation of $l_r(s)$ in Equation (9) we obtain the approximation of the integral. See Ruf (2006) for the proof that the error is of order $O(r^3)$. □

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