HEDGING UNDER ARBITRAGE*

Johannes Ruf†
Columbia University
May 29, 2011

Abstract

It is shown that delta hedging provides the optimal trading strategy in terms of minimal required initial capital to replicate a given terminal payoff in a continuous-time Markovian context. This holds true in market models in which no equivalent local martingale measure exists but only a square-integrable market price of risk. A new probability measure is constructed, which takes the place of an equivalent local martingale measure. In order to ensure the existence of the delta hedge, sufficient conditions are derived for the necessary differentiability of expectations indexed over the initial market configuration. The phenomenon of "bubbles," which has recently been frequently discussed in the academic literature, is a special case of the setting in this paper. Several examples at the end illustrate the techniques described in this work.

KEY WORDS: Benchmark Approach, Stochastic Portfolio Theory, bubbles, local martingales, Föllmer measure, continuous time, diffusions, stochastic discount factor, market price of risk, trading strategies, arbitrage, pricing, hedging, options, put-call-parity, Black-Scholes PDE, stochastic flows, Schauder estimates, Bessel process

1 INTRODUCTION

In a financial market, an investor usually has several trading strategies at her disposal to obtain a given wealth at a specified point in time. For example, if the investor wanted to cover a short-position in a given stock tomorrow at the cheapest cost today, buying the stock today is generally not optimal, as there may be a trading strategy requiring less initial capital that still replicates the exact stock price tomorrow. In this paper, we show that optimal trading strategies, in the sense of minimal required initial capital, can be represented as delta hedges.

*I am indebted to two anonymous referees and an associate editor for their careful reading and constructive feedback. Many thanks go to Ioannis Karatzas and Hans Föllmer for sharing their insights and for their helpful comments on previous drafts of this work. I am grateful to Michael Agne, Adrian Banner, Daniel Fernholz, Robert Fernholz, Ashley Griffith, Tomoyuki Ichiba, Phi Long Nguen-Tranh, Sergio Pulido, Subhankar Sadhukhan, Emilio Seijo, Li Song, Winslow Strong, Johan Tysk, and Hao Xing for fruitful discussions on the subject matter of this paper. This work was partially supported by the National Science Foundation DMS Grant 09-05754 and by a Faculty Fellowship of Columbia University. Results in this paper are drawn in part from the second chapter of the author's doctoral thesis Ruf (2011) supervised by Ioannis Karatzas.

†Address correspondance to Johannes Ruf, Department of Statistics, Columbia University, 1255 Amsterdam Avenue, New York, NY 10027, USA; ruf@stat.columbia.edu.

This paper has been motivated by the problem of finding trading strategies to exploit relative arbitrage opportunities, which arise naturally in the framework of Stochastic Portfolio Theory (SPT). For that, we generalize the results of Fernholz and Karatzas (2010)'s paper "On optimal arbitrage," in which specifically the market portfolio is examined, to a wide class of terminal wealths which can be optimally replicated by delta hedges. For an overview of SPT and a discussion of relative arbitrage opportunities, we recommend the reader consult the monograph by Fernholz (2002) and the survey paper by Fernholz and Karatzas (2009). The problem investigated here is directly linked to the question of computing hedges of contingent claims, which has been studied within the Benchmark Approach (BA), developed by Eckhard Platen and co-authors. Indeed, we generalize some of the results in the BA here and provide tools to compute the so-called "real-world prices" of contingent claims under that approach. The monograph by Platen and Heath (2006) provides an excellent overview of the BA.

We shall not restrict ourselves only to markets satisfying the the "No free lunch with vanishing risk" (NFLVR) or, more precisely, the "No arbitrage for general admissible integrands" (NA) condition.[1] Thus, we cannot rely on the existence of an equivalent local martingale measure (ELMM), which we otherwise would have done. However, we shall construct another probability measure to take the place of the "risk-neutral" measure. We do not assume an ELMM a priori for several reasons. First, we cannot always assume the existence of a statistical test that relies upon stock price observations to determine whether an ELMM exists, as illustrated in Karatzas and Kardaras (2007), Example 3.7. Second, examining arbitrage opportunities, rather than excluding them a priori, is of interest in itself. Further arguments and empirical evidence supporting the consideration of models without an ELMM are discussed in Kardaras (2008), Section 0.1 and Platen and Hulley (2008), Section 1. A model of economic equilibrium for such models is provided in Loewenstein and Willard (2000a). In the spirit of these papers, we shall impose some restrictions on the arbitrage opportunities and exclude a priori models which imply "unbounded profit with bounded risk," which can be recognized by a typical agent.

There have been several recent papers treating the subject of "bubbles" within models guaranteeing NFLVR; a very incomplete list consists of the work by Loewenstein and Willard (2000b), Cox and Hobson (2005), Heston et al. (2007), Jarrow et al. (2007, 2010), Pal and Protter (2010), and Ekström and Tysk (2009). A bubble is usually defined as the difference between the market price of a tradeable asset and its smallest hedging price. The analysis here includes the case of bubbles, but is more general, as it also allows for models without an ELMM. To wit, while the bubbles literature concentrates on a single stock whose price process is modeled as a strict local martingale, we consider markets with several assets with the stochastic discount factor itself being represented by a (possibly strict) local martingale. In the case of an asset with a bubble, our contribution is limited to the explicit representation of the optimal replicating strategy as a delta hedge. We shall also discuss in this context the reciprocal of the three-dimensional Bessel process as the standard example for a bubble.

We set up our analysis in a continuous-time Markovian context; to wit, we focus on stock price processes whose mean rates of return and volatility coefficients only depend on time and on the current market configuration. Since we do not rely on a martingale representation theorem,

[1] We refer the reader to the monograph by Delbaen and Schachermayer (2006) for a thorough introduction to NA, NFLVR and other notions of arbitrage. Since we shall assume the existence of a square-integrable market price of risk, we implicitly impose the condition that NFLVR fails if and only if NA fails; see Karatzas and Kardaras (2007), Proposition 3.2.

we can allow for a larger number of driving Brownian motions than the number of stocks, which generalizes the ideas of Fernholz and Karatzas (2010) to not only a larger set of payoffs, but also to a broader set of models for the specific case of the market portfolio. We shall prove that a classical delta hedge yields the cheapest hedging strategy for European contingent claims. This is of course well-known in the case where an ELMM exists and is extended here to models which allow for arbitrage opportunities and that are not necessarily complete. In this context, we provide sufficient conditions to ensure the differentiability of the hedging price, generalizing results by Heath and Schweizer (2000), Janson and Tysk (2006), and Ekström and Tysk (2009). This set of conditions is also applicable to models satisfying the NFLVR assumption. Because the computations for the optimal trading strategy under the "real-world" measure are often too involved and because we cannot always rely on an ELMM, we derive a non-equivalent change of measure including a generalized Bayes' rule.

The next section introduces the market model and trading strategies. Section 3 provides a discussion about the market price of risk. Section 4 contains the precise representation of an optimal strategy to hedge a non path-dependent European claim and sufficient conditions for the differentiability of the hedging price. A modified put-call parity follows directly. We suggest in Section 5 a change to some non-equivalent probability measure that simplifies computations. Section 6 then provides several examples and Section 7 draws the conclusions.

2 MARKET MODEL AND TRADING STRATEGIES

In this section, we introduce the market model and trading strategies. We assume the perspective of a small investor who takes positions in a frictionless financial market with finite time horizon T. We shall use the notation $\mathbb{R}_+^d := \{s = (s_1, \ldots, s_d)^\mathsf{T} \in \mathbb{R}^d, s_i > 0, \text{ for all } i = 1, \ldots, d\}$ and assume a market in which the stock price processes are modeled as positive continuous Markovian semimartingales. That is, we consider a financial market $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))^\mathsf{T}$ of the form

$$dS_i(t) = S_i(t)\left(\mu_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t)\right) \tag{2.1}$$

for all $i = 1, \ldots, d$ and $t \in [0, T]$ starting at $S(0) \in \mathbb{R}_+^d$ and a money market $B(\cdot)$. Here $\mu : [0, T] \times \mathbb{R}_+^d \to \mathbb{R}^d$ denotes the mean rate of return and $\sigma : [0, T] \times \mathbb{R}_+^d \to \mathbb{R}^{d \times K}$ denotes the volatility. We assume that both functions are measurable.

For the sake of convenience we only consider discounted (forward) prices and set the interest rate constant to zero; that is, $B(\cdot) \equiv 1$. The flow of information is modeled as a right-continuous filtration $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$ such that $W(\cdot) = (W_1(\cdot), \ldots, W_K(\cdot))^\mathsf{T}$ is a K-dimensional Brownian motion with independent components. In Section 5, we impose more conditions on the filtration $\mathbb{F}$ and the underlying probability space Ω. The underlying measure and its expectation will be denoted by $\mathbb{P}$ and $\mathbb{E}$, respectively.

We only consider those mean rates of return μ and volatilities σ that imply the stock prices $S_1(\cdot), \cdots, S_d(\cdot)$ exist and are unique and strictly positive. More precisely, denoting the covariance process of the stocks by $a(\cdot, \cdot) = \sigma(\cdot, \cdot)\sigma^\mathsf{T}(\cdot, \cdot)$, we impose the almost sure integrability condition

$$\sum_{i=1}^{d} \int_0^T \left(|\mu_i(t, S(t))| + a_{i,i}(t, S(t))\right) dt < \infty.$$

Next, we introduce the notion of trading strategies and associated wealth processes to be able to describe formally delta hedging below. We denote the number of shares held by an investor at time t by $\eta(t) = (\eta_1(t), \dots, \eta_d(t))^\mathsf{T}$ and call $\eta(\cdot)$ a *trading strategy* or in short, a *strategy*. We assume that $\eta(\cdot)$ is progressively measurable with respect to $\mathbb{F}$ and self-financing. This yields for the corresponding wealth process $V^{v,\eta}(\cdot)$ of an investor with initial capital $v > 0$ the dynamics

$$dV^{v,\eta}(t) = \sum_{i=1}^{d} \eta_i(t) dS_i(t)$$

for all $t \in [0, T]$ and $V^{v,\eta}(0) = v$. To ensure that $V^{v,\eta}(\cdot)$ is well-defined and to exclude doubling strategies we restrict ourselves to trading strategies which satisfy $V^{v,\eta}(t) \geq 0$ for a given initial wealth $v > 0$, and the almost sure integrability condition

$$\sum_{i=1}^{d} \int_0^T \left(S_i(t) |\eta_i(t) \mu_i(t, S(t))| + S_i^2(t) \eta_i^2(t) a_{i,i}(t, S(t)) \right) dt < \infty.$$

3 MARKET PRICE OF RISK AND STOCHASTIC DISCOUNT FACTOR

This section discusses two important components of the market model. We assume that the market model of (2.1) implies a *market price of risk (MPR)*, which generalizes the concept of the Sharpe ratio to several dimensions. More precisely, an MPR is a progressively measurable process $\theta(\cdot)$, which maps the volatility structure σ onto the mean rate of return μ. That is,

$$\mu(t, S(t)) = \sigma(t, S(t)) \theta(t) \tag{3.1}$$

for all $t \in [0, T]$ holds almost surely. We further assume that $\theta(\cdot)$ is square-integrable, to wit,

$$\int_0^T \|\theta(t)\|^2 dt < \infty \tag{3.2}$$

almost surely. An MPR does not have to be uniquely determined. Uniqueness is intrinsically connected to completeness, which we need not assume. In general, infinitely many MPRs may exist. An example for non-uniqueness is given following Proposition 1 below.

The existence of an MPR is a central assumption in both the BA (see Platen and Heath, 2006, Chapter 10) and SPT (see Fernholz and Karatzas, 2009, Section 6). This assumption enables us to discuss hedging prices, as we do throughout this paper, since it excludes scalable arbitrage opportunities by guaranteeing 'no unbounded profit with bounded risk" (NUPBR) as demonstrated in Karatzas and Kardaras (2007). Similar assumptions have been discussed in the economic literature. For example, in the terminology of Loewenstein and Willard (2000a), the existence of a square-integrable MPR excludes "cheap thrills" but not necessarily "free snacks." Theorem 2 of Loewenstein and Willard (2000a) shows that a market with a square-integrable MPR is consistent with an equilibrium where agents prefer more to less.

Based upon the MPR, we can now define the *stochastic discount factor (SDF)* as

$$Z^\theta(t) := \exp\left(-\int_0^t \theta^\mathsf{T}(u) dW(u) - \frac{1}{2} \int_0^t \|\theta(u)\|^2 du \right) \tag{3.3}$$

for all $t \in [0, T]$. In classical no-arbitrage theory, $Z^\theta(\cdot)$ represents the Radon-Nikodym derivative which translates the "real-world" measure into the generic "risk-neutral" measure with the money market as the underlying. Since we do not want to impose NFLVR a priori in this work, but are rather interested in situations in which NFLVR does not necessarily hold, we shall not assume that the SDF $Z^\theta(\cdot)$ is a true martingale. Cases where $Z^\theta(\cdot)$ is only a local martingale have, for example, been discussed by Karatzas et al. (1991), Schweizer (1992), in the BA starting with Platen (2002) and Heath and Platen (2002a,b) and in SPT; see, for example, Fernholz et al. (2005) and especially, Fernholz and Karatzas (2010).

In this context, it is important to remind ourselves that $Z^\theta(\cdot)$ is a true martingale if and only if there exists an ELMM $\mathbb{Q}$, under which the stock price processes are local martingales. The question of whether $\mathbb{Q}$ is a martingale measure or only a local martingale measure is not connected to whether $Z^\theta(\cdot)$ is a strict local or a true martingale. A *bubble* is usually defined within a model in which $Z^\theta(\cdot)$ is a true martingale. Then, a wealth process is said to have a bubble if it is a strict local martingale under an ELMM.[2] Jarrow et al. (2007, 2010) suggest replacing the NFLVR condition by the stronger condition of "no dominance" first proposed by Merton (1973) to exclude bubbles. Here, we take the opposite approach. Instead of imposing a new condition, the goal of this analysis is to investigate a general class of models and study how much can be said in this more general framework without having the tool of an ELMM.

We observe that the existence of a square-integrable MPR implies the existence of a Markovian square-integrable MPR. To see this, we define $\theta(\cdot, \cdot) := \sigma^\mathsf{T}(\cdot, \cdot)(\sigma(\cdot, \cdot)\sigma^\mathsf{T}(\cdot, \cdot))^\dagger \mu(\cdot, \cdot)$, where $\dagger$ denotes the Moore-Penrose pseudo-inverse of a matrix. Given the existence of any MPR, we know from the theory of least-squares estimation that $\theta(\cdot, \cdot)$ is also a MPR. Furthermore, we have $\|\theta(t, S(t))\|^2 \leq \|\nu(t)\|^2$ for all $t \in [0, T]$ almost surely for any MPR $\nu(\cdot)$, which yields the square-integrability of $\theta(\cdot, \cdot)$. This observation has been pointed out to us by a referee.

The next proposition shows that any square-integrable Markovian MPR maximizes the random variable which will later be a candidate for a hedging price. We denote by $\mathcal{F}^S(\cdot)$ the augmented filtration generated by the stock price process. We emphasize that the next result only holds so long as the "terminal payoff" M is $\mathcal{F}^S(T)$-measurable.

Proposition 1 (Role of Markovian MPR)**.** *Let $M \geq 0$ be a random variable measurable with respect to $\mathcal{F}^S(T) \subset \mathcal{F}(T)$. Let $\nu(\cdot)$ denote any square-integrable MPR and $\theta(\cdot, \cdot)$ any Markovian square-integrable MPR. Then, with*

$$M^\nu(t) := \mathbb{E}\left[\frac{Z^\nu(T)}{Z^\nu(t)} M \,\middle|\, \mathcal{F}_t \right] \text{ and } M^\theta(t) := \mathbb{E}\left[\frac{Z^\theta(T)}{Z^\theta(t)} M \,\middle|\, \mathcal{F}_t \right]$$

for $t \in [0, T]$, where we take the right-continuous modification[3] for each process, we have $M^\nu(\cdot) \leq M^\theta(\cdot)$ almost surely. Furthermore, if both $Z^\nu(\cdot)$ and $Z^\theta(\cdot)$ are $\mathcal{F}^S(T)$-measurable, then $Z^\nu(T) \leq Z^\theta(T)$ almost surely.

Proof. Due to the right-continuity of $M^\nu(\cdot)$ and $M^\theta(\cdot)$ it suffices to show for all $t \in [0, T]$ that $M^\nu(t) \leq M^\theta(t)$ almost surely. We define $c(\cdot) := \nu(\cdot) - \theta(\cdot, S(\cdot))$. For the sequence of stopping

[2] In the bubbles literature, there has been an alternative definition, based upon the characterization of the pricing operator as a finitely additive measure. It can be shown that this characterization is equivalent to the one here; see Jarrow et al. (2010), Section 8 for the proof and literature which relies on this alternative characterization.

[3] See Karatzas and Shreve (1991), Theorem 1.3.13.

times

$$\tau_n := T \wedge \inf\left\{t \in [0,T] : \int_0^t c^2(s)ds \geq n\right\},$$

where $n \in \mathbb{N}$, we set $c^n(\cdot) := c(\cdot)\mathbf{1}_{\{\tau_n \geq \cdot\}}$ and observe that

$$\begin{aligned}\frac{Z^\nu(T)}{Z^\nu(t)} &= \frac{Z^c(T)}{Z^c(t)} \cdot \exp\left(-\int_t^T \theta^\mathsf{T}(u,S(u))(dW(u)+c(u)du) - \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right) \\ &= \lim_{n\to\infty} \frac{Z^{c^n}(T)}{Z^{c^n}(t)} \cdot \exp\left(-\int_t^T \theta^\mathsf{T}(u,S(u))(dW(u)+c^n(u)du) - \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right)\end{aligned}$$

with $Z^c(\cdot)$ and $Z^{c^n}(\cdot)$ defined as in (3.3). The limit holds almost surely since both $v(\cdot)$ and $\theta(\cdot,\cdot)$ are square-integrable, which again yields the square-integrability of $c(\cdot)$. Since $\int_0^T c^{n^2}(t)dt \leq n$, Novikov's Condition (see Karatzas and Shreve, 1991, Proposition 3.5.12) yields that $Z^{c^n}(\cdot)$ is a martingale. Now, Fatou's lemma, Girsanov's theorem and Bayes' rule (see Karatzas and Shreve, 1991, Chapter 3.5) yield

$$M^\nu(t) \leq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}^n}\left[\exp\left(-\int_t^T \theta^\mathsf{T}(u,S(u))dW^n(u) - \frac{1}{2}\int_t^T \|\theta(u,S(u))\|^2 du\right) M \,\middle|\, \mathcal{F}_t\right], \tag{3.4}$$

where $d\mathbb{Q}^n(\cdot) := Z^{c^n}(T)d\mathbb{P}(\cdot)$ is a probability measure, $\mathbb{E}^{\mathbb{Q}^n}$ its expectation operator, and $W^n(\cdot) := W(\cdot) + \int_0^\cdot c^n(u)du$ a K-dimensional $\mathbb{Q}^n$-Brownian motion. Since $\sigma(\cdot,S(\cdot))c^n(\cdot) \equiv 0$ we can replace $W(\cdot)$ by $W^n(\cdot)$ in (2.1). This yields that the process $S(\cdot)$ has the same dynamics under $\mathbb{Q}^n$ as under $\mathbb{P}$. Furthermore, both $\theta(\cdot,S(\cdot))$ and M have, as functionals of $S(\cdot)$, the same distribution under $\mathbb{Q}^n$ as under $\mathbb{P}$. Therefore, we can replace the expectation operator $\mathbb{E}^{\mathbb{Q}^n}$ by $\mathbb{E}$ and the Brownian motion $W^n(\cdot)$ by $W(\cdot)$ in (3.4) and obtain the first part of the statement. The last inequality of the statement follows from setting $M = \mathbf{1}_{\{Z^\nu(T) > Z^\theta(T)\}}$ and observing that M must equal zero almost surely. □

We remark that the inequality $M^\nu(\cdot) \leq M^\theta(\cdot)$ can be strict. For an example, choose $M = 1$ and a market with one stock and two Brownian motions, to wit, $d = 1$ and $K = 2$. We set $\mu(\cdot,\cdot) \equiv 0$, $\sigma(\cdot,\cdot) \equiv (1,0)$ and observe that $\theta(\cdot,S(\cdot)) \equiv (0,0)^\mathsf{T}$ is a Markovian MPR. Another MPR $\nu(\cdot) \equiv (\nu_1(\cdot),\nu_2(\cdot))^\mathsf{T}$ is defined via $\nu_1(\cdot) \equiv 0$, the stochastic differential equation $d\nu_2(t) = -\nu_2^2(t)dW_2(t)$ for all $t \in [0,T]$ and $\nu_2(0) = 1$. That is, $\nu_2(\cdot)$ is the reciprocal of a three-dimensional Bessel process starting at one. Since $Z^\nu(\cdot)$ also satisfies the stochastic differential equation $dZ^\nu(t) = -Z^\nu(t)\nu_2(t)dW_2(t)$ we have from Jacod and Shiryaev (2003), Theorem 1.4.61 that $Z^\nu(\cdot) \equiv \nu_2(\cdot)$, which is a strict local martingale (see Karatzas and Shreve, 1991, Exercise 3.3.36), and thus $M^\nu(0) = \mathbb{E}[Z^\nu(T)] < 1 = \mathbb{E}[Z^\theta(T)] = M^\theta(0)$.

Under the assumption that an ELMM exists, Jacka (1992), Theorem 12, Ansel and Stricker (1993), Theorem 3.2 or Delbaen and Schachermayer (1995c), Theorem 16 show that a contingent claim can be hedged if and only if the supremum over all expectations of the terminal value of the contingent claim under all ELMMs is a maximum. In our setup, we also observe that the supremum over all $M^{\tilde{\nu}}(0)$ in the last proposition is a maximum, attained by any Markovian MPR. Indeed, we will prove in Theorem 4.1 that, under weak analytic assumptions, claims of the form $M = p(S(T))$ can be hedged. The general theory lets us conjecture that all claims measurable with respect to $\mathcal{F}^S(T)$ can be hedged.

As pointed out by Ioannis Karatzas in a personal communication (2010), Proposition 1 might be related to the "Markovian selection results," as in Krylov (1973), Ethier and Kurtz (1986), Section 4.5, and Stroock and Varadhan (2006), Chapter 12. There, the existence of a Markovian solution for a martingale problem is studied. It is observed that a supremum over a set of expectations indexed by a family of distributions is attained and the maximizing distribution is a Markovian solution of the martingale problem. This potential connection needs to be worked out in a future research project.

From this point forward, we shall always assume the MPR to be Markovian. As we shall see, this choice will lead directly to the optimal trading strategy.

4 OPTIMAL STRATEGIES

In this section, we show that delta hedging provides the optimal trading strategy in terms of minimal required initial capital to replicate a given terminal payoff. Next, we prove a modified put-call parity. In order to ensure the existence of the delta hedge, we derive sufficient conditions for the differentiability of expectations indexed over the initial market configuration.

We will rely on the following notation. If Y is a nonnegative $\mathcal{F}(T)$-measurable random variable such that $\mathbb{E}[Y|\mathcal{F}(t)]$ is a function of t and $S(t)$ for all $t \in [0,T]$, we use the Markovian structure of $S(\cdot)$ to denote conditioning on the event $\{S(t) = s\}$ by $\mathbb{E}^{t,s}[Y]$. Outside of the expectation operator we denote by $(S^{t,s}(u))_{u\in[t,T]}$ a stock price process with the dynamics of (2.1) and $S(t) = s$, in particular, $S^{0,S(0)}(\cdot) \equiv S(\cdot)$. We observe that $Z^\theta(u)/Z^\theta(t)$ depends for $u \in (t,T]$ on $\mathcal{F}(t)$ only through $S(t)$ and we write similarly $(\tilde{Z}^{\theta,t,s}(u))_{u\in[t,T]}$ for $(Z^\theta(u)/Z^\theta(t))_{u\in[t,T]}$ with $\tilde{Z}^{\theta,t,s}(t) = 1$ on the event $\{S(t) = s\}$. When we want to stress the dependence of a process on the state $\omega \in \Omega$ we will write, for example, $S(t,\omega)$.

Let us denote by supp$(S(\cdot))$ the support of $S(\cdot)$, that is, the smallest closed set in $[0,T] \times \mathbb{R}^n$ such that

$$\mathbb{P}((t,S(t)) \in \text{supp}(S(\cdot)) \text{ for all } t \in [0,T]) = 1.$$

We call i-supp$(S(\cdot))$ the union of $(0,S(0))$ and the interior of supp$(S(\cdot))$ and assume that

$$\mathbb{P}((t,S(t)) \in \text{i-supp}(S(\cdot)) \text{ for all } t \in [0,T)) = 1.$$

This assumption is made to exclude degenerate cases, where $S(\cdot)$ can hit the boundary of its support with positive probability. We shall call any $(t,s) \in$ i-supp$(S(\cdot))$ a *point of support for* $S(\cdot)$ and we remark that each such point (t,s) satisfies $t < T$. For example, if $S(\cdot)$ is a one-dimensional geometric Brownian motion then the set of points of support for $S(\cdot)$ is exactly $(0,S(0)) \cup \{(t,s) \in (0,T) \times \mathbb{R}_+\}$.

We define for any measurable function $p : \mathbb{R}^d_+ \to [0,\infty)$ a candidate $h^p : [0,T] \times \mathbb{R}^d_+ \to [0,\infty)$ for the hedging price of the corresponding European option:

$$h^p(t,s) := \mathbb{E}^{t,s}\left[\tilde{Z}^{\theta,t,s}(T)p(S(T))\right]. \tag{4.1}$$

Since $S(\cdot)$ is Markovian, h^p is well-defined. Proposition 1 yields that h^p does not depend on the choice of the (Markovian) MPR $\theta(\cdot)$. Equation (4.1) has appeared as the "real-world pricing formula" in the BA; compare Platen and Heath (2006), Equation (9.1.30). Simple examples for payoffs could be the market portfolio ($\tilde{p}(s) = \sum_{i=1}^{d} s_i$), the money market ($p^0(s) = 1$), a stock ($p^1(s) = s_1$), or a call ($p^C(s) = (s_1 - L)^+$ for some $L \in \mathbb{R}$). We can now prove the first main result, which in particular provides a mechanism for pricing and hedging contingent claims under the BA. We denote by D_i, $D^2_{i,j}$ the partial derivatives with respect to the variable s.

Theorem 4.1 (Markovian representation for non path-dependent European claims)**.** *Assume that we have a contingent claim of the form $p(S(T)) \geq 0$ and that the function h^p of* (4.1) *is sufficiently differentiable or, more precisely, that for all points of support (t, s) for $S(\cdot)$ we have $h^p \in C^{1,2}(\mathcal{U}_{t,s})$ for some neighborhood $\mathcal{U}_{t,s}$ of (t, s). Then, with*

$$\eta_i^p(t, s) := D_i h^p(t, s)$$

for all $i = 1, \ldots, d$ and $(t, s) \in [0, T] \times \mathbb{R}^d_+$, and with $v^p := h^p(0, S(0))$, we get

$$V^{v^p,\eta^p}(t) = h^p(t, S(t))$$

for all $t \in [0, T]$. The strategy η^p is optimal in the sense that for any $\tilde{v} > 0$ and for any strategy $\tilde{\eta}$ whose associated wealth process is nonnegative and satisfies $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$ almost surely, we have $\tilde{v} \geq v^p$. Furthermore, h^p solves the PDE

$$\frac{\partial}{\partial t} h^p(t, s) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} s_i s_j a_{i,j}(t, s) D^2_{i,j} h^p(t, s) = 0 \tag{4.2}$$

at all points of support (t, s) for $S(\cdot)$.

Proof. Let us start by defining the martingale $N^p(\cdot)$ as

$$N^p(t) := \mathbb{E}[Z^\theta(T) p(S(T)) | \mathcal{F}(t)] = Z^\theta(t) h^p(t, S(t))$$

for all $t \in [0, T]$. Although h^p is not assumed to be in $C^{1,2}([0, T) \times \mathbb{R}^d)$ but only to be locally smooth, we can apply a localized version of Itô's formula (see for example Revuz and Yor, 1999, Section IV.3) to it. Then, the product rule of stochastic calculus can be used to obtain the dynamics of $N^p(\cdot)$. Since $N^p(\cdot)$ is a martingale, the corresponding dt term must disappear. This observation, in connection with (3.1) and the positivity of $Z^\theta(\cdot)$, yields PDE (4.2). Itô's formula, now applied to $h^p(\cdot, S(\cdot))$, and PDE (4.2) imply

$$dh^p(t, S(t)) = \sum_{i=1}^{d} D_i h^p(t, S(t)) dS_i(t) = dV^{v^p,\eta^p}(t)$$

for all $t \in [0, T]$. This yields directly $V^{v^p,\eta^p}(\cdot) \equiv h^p(\cdot, S(\cdot))$.

Next, we prove optimality. Assume we have some initial wealth $\tilde{v} > 0$ and some strategy $\tilde{\eta}$ with nonnegative associated wealth process such that $V^{\tilde{v},\tilde{\eta}}(T) \geq p(S(T))$ is satisfied almost surely. Then, $Z^\theta(\cdot) V^{\tilde{v},\tilde{\eta}}(\cdot)$ is bounded from below by zero, thus a supermartingale. This implies

$$\tilde{v} \geq \mathbb{E}[Z^\theta(T) V^{\tilde{v},\tilde{\eta}}(T)] \geq \mathbb{E}[Z^\theta(T) p(S(T))] = \mathbb{E}[Z^\theta(T) V^{v^p,\eta^p}(T)] = v^p,$$

which concludes the proof. $\square$

The last result generalizes Platen and Hulley (2008), Proposition 3, where the same statement has been shown for a one-dimensional, complete market with a time-transformed squared Bessel process of dimension four modeling the stock price process. There are usually several strategies to obtain the same payoff. For example, if the first stock has a bubble, that is, if $\mathbb{E}[Z^\theta(T)S_1(T)] < S_1(0)$, then one could either delta hedge with initial capital $\mathbb{E}[Z^\theta(T)S_1(T)]$ as the last theorem describes, or hold the stock with initial capital $S_1(0)$. The last result shows that the delta hedge is optimal in the sense of minimal required initial capital. Platen (2008) has suggested calling the fact that an optimal strategy exists the "Law of the Minimal Price" to contrast it to the classical "Law of the One Price," which appears if there is an equivalent martingale measure.

We would like to emphasize that we have not shown that η^p is unique. Indeed, since we have not excluded the case that two stock prices have identical dynamics this is not necessarily true. The next remark discusses the fact that we have not assumed the completeness of the market.

Remark 4.1 (Completeness of the market). One remarkable feature of the last theorem is that it does not require the market to be complete. In particular, at no point have we assumed invertibility or full rank of the volatility matrix $\sigma(\cdot,\cdot)$. In contrast to Fernholz and Karatzas (2010), we do not rely on the martingale representation theorem here but instead directly derive a representation for the conditional expectation process of the final wealth $p(S(T))$. The explanation for this phenomenon is that all relevant sources of risk for hedging are completely captured by the tradeable stocks. However, we remind the reader that we live here in a setting in which the mean rates of return and volatilities do not depend on an extra stochastic factor. In a "more incomplete" model, with jumps or additional risk factors in mean rates of return or volatilities, this result can no longer be expected to hold. Furthermore, there is no hope to be able to hedge all contingent claims of the Brownian motion $W(T)$. However, $W(T)$ appears in the model only as a nuisance parameter and it is of no economic interest to trade in it directly.

In the next remark we discuss PDE (4.2).

Remark 4.2 (Non-uniqueness of PDE (4.2)). Parabolic PDEs generally do not have unique solutions. The hedging price for the stock of Example 6.3 in (6.5), for instance, is one of many solutions of polynomial growth for the corresponding Black-Scholes type PDE with terminal condition $p(s) = s$ and boundary condition $f(t) = 0$. Another solution is of course $h(t,s) = s$. The reason for non-uniqueness in this case is the fact that the second-order coefficient has super-quadratic growth preventing standard theory cannot from being applied; see, for example, Karatzas and Shreve (1991), Section 5.7.B. However, one can show easily that, given that h^p is sufficiently differentiable, h^p can be characterized as the minimal nonnegative classical solution of PDE (4.2) with terminal condition $h^p(T,s) = p(s)$; compare the proof of Fernholz and Karatzas (2010), Theorem 1.

Fernholz et al. (2005), Example 9.2.2 illustrates that the classical put-call parity can fail. However, a modified version holds. An equivalent version for the situation of an ELMM with possible bubbles has already been found in Jarrow et al. (2007), Lemma 7.

Corollary 4.1 (Modified put-call parity). *For any $L \in \mathbb{R}$ we have the modified put-call parity for the call- and put-options $(S_1(T) - L)^+$ and $(L - S_1(T))^+$, respectively, with strike price L:*

$$\mathbb{E}^{t,s}\left[\tilde{Z}^{\theta,t,s}(T)(L - S_1(T))^+\right] + h^{p^1}(t,s) = \mathbb{E}^{t,s}\left[\tilde{Z}^{\theta,t,s}(T)(S_1(T) - L)^+\right] + Lh^{p^0}(t,s), \quad (4.3)$$

where $p^0(\cdot) \equiv 1$ denotes the payoff of one monetary unit and $p^1(s) = s_1$ the price of the first stock for all $s \in \mathbb{R}^d_+$.

Proof. The statement follows from the linearity of expectation. □

Due to Theorem 4.1, there exist, under weak differentiability assumptions, optimal strategies for the money market, the stock $S_1(T)$, the call and the put. Thus, the left-hand side of (4.3) corresponds to the sum of the hedging prices of a put and the stock, and the right-hand side corresponds to the sum of the hedging prices of a call and L monetary units. The difference between this and the classical put-call parity is that the current stock price and the strike L are replaced by their hedging prices. Bayraktar et al. (2010), Section 2.2 have recently observed another version. Instead of replacing the current stock price by its hedging price, they replace the European call price by the American call price and restore the put-call parity this way.

Next, we will provide sufficient conditions under which the function h^p is sufficiently smooth. We shall call a function $f : [0,T] \times \mathbb{R}^d_+ \to \mathbb{R}$ *locally Lipschitz and locally bounded* on $\mathbb{R}^d_+$ if for all $s \in \mathbb{R}^d_+$ the function $t \to f(t,s)$ is right-continuous with left limits and for all $M > 0$ there exists some $C(M) < \infty$ such that

$$\sup_{\substack{\frac{1}{M} \le \|y\|, \|z\| \le M \\ y \neq z}} \frac{|f(t,y) - f(t,z)|}{\|y - z\|} + \sup_{\frac{1}{M} \le \|y\| \le M} |f(t,y)| \le C(M)$$

for all $t \in [0,T]$. In particular, if f has continuous partial derivatives, it is locally Lipschitz and locally bounded. We require several assumptions in order to show the differentiability of h^p in Theorem 4.2 below.

(A1) The functions θ_k and $\sigma_{i,k}$ are for all $i = 1, \ldots, d$ and $k = 1, \ldots, K$ locally Lipschitz and locally bounded.

(A2) For all points of support (t,s) for $S(\cdot)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of (t,s) such that

$$\sum_{i=1}^{d} \sum_{j=1}^{d} a_{i,j}(u,y)\xi_i\xi_j \ge C\|\xi\|^2 \tag{4.4}$$

for all $\xi \in \mathbb{R}^d$ and $(u,y) \in \mathcal{U}$.

(A3) The payoff function p is chosen so that for all points of support (t,s) for $S(\cdot)$ there exist some $C > 0$ and some neighborhood $\mathcal{U}$ of (t,s) such that $h^p(u,y) \le C$ for all $(u,y) \in \mathcal{U}$.

If h^p is constant for $\tilde{d} \le d$ coordinates, say the last ones, Assumption (A2) can be weakened to requesting the uniform ellipticity only in the remaining $d - \tilde{d} - 1$ coordinates; that is, the sum in (4.4) goes only to $d - \tilde{d} - 1$ and $\xi \in \mathbb{R}^{d - \tilde{d} - 1}$. Assumption (A3) holds in particular if p is of linear growth; that is, if $p(s) \le C \sum_{i=1}^d s_i$ for some $C > 0$ and all $s \in \mathbb{R}^d_+$, since $\tilde{Z}^{\theta,t,s}(\cdot) S_i^{t,s}(\cdot)$ is a nonnegative supermartingale for all $i = 1, \ldots, d$. We emphasize that the conditions here are weaker than the ones by Fernholz and Karatzas (2010), Section 9 for the case of the market portfolio which can be represented as $p(s) = \sum_{i=1}^d s_i$. In particular, the stochastic integral component in $Z^\theta(\cdot)$ does not present any technical difficulty in our approach.

We proceed in two steps. In the first step we use the theory of stochastic flows to derive continuity of $S^{t,s}(T)$ and $\tilde{Z}^{\phi,t,s}(T)$ in t and s. This theory relies on Kolmogorov's lemma, see, for example, Protter (2003), Theorem IV.73, and studies continuity of stochastic processes as functions of their initial conditions. We refer the reader to Protter (2003), Chapter V for an introduction to and further references for stochastic flows. We will prove continuity of $S^{t,s}(\cdot)$ and $\tilde{Z}^{\phi,t,s}(\cdot)$ at once and introduce for that the $d+1$-dimensional process $X^{t,s,z}(\cdot) := (S^{t,s\mathsf{T}}(\cdot), z\tilde{Z}^{\phi,t,s}(\cdot))^{\mathsf{T}}$.

Lemma 4.1 (Stochastic flow)**.** *We fix a point $(t,s) \in [0,T] \times \mathbb{R}_+^d$ so that $X^{t,s,1}(\cdot)$ is strictly positive and an $\mathbb{R}_+^{d+1}$-valued process. Then under Assumption (A1) we have for all sequences $(t_k, s_k)_{k\in\mathbb{N}} \subset [0,T] \times \mathbb{R}_+^d$ with $\lim_{k\to\infty}(t_k, s_k) = (t,s)$ that*

$$\lim_{k\to\infty} \sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| = 0$$

almost surely, where we set $X^{t_k,s_k,1}(u) := (s_k^{\mathsf{T}}, 1)^{\mathsf{T}}$ for $u \leq t_k$. In particular, for $K(\omega)$ sufficiently large we have that $X^{t_k,s_k,1}(u,\omega)$ is strictly positive and $\mathbb{R}_+^{d+1}$-valued for all $k > K(\omega)$ and $u \in [t,T]$.

Proof. Since the class of locally Lipschitz and locally bounded functions is closed under summation and multiplication, Assumption (A1) yields that the drift and diffusion coefficients of $X^{u,y,z}(\cdot)$ are locally Lipschitz for all $(u,y,z) \in [0,T] \times \mathbb{R}_+^d \times \mathbb{R}_+$. We start by assuming $t_k \geq t$ for all $k \in \mathbb{N}$ and obtain

$$\begin{aligned}\sup_{u\in[t,T]} \|X^{t_k,s_k,1}(u) - X^{t,s,1}(u)\| \leq{}& \sup_{u\in[t,t_k]} \|(s_k^{\mathsf{T}}, 1)^{\mathsf{T}} - X^{t,s,1}(u)\| + \sup_{u\in[t_k,T]} \|X^{t_k,s_k,1}(u) - X^{t_k,s,1}(u)\| \\ &+ \sup_{u\in[t_k,T]} \|X^{t_k,s,1}(u) - X^{t_k,S^{t,s}(t_k),\tilde{Z}^{\phi,t,s}(t_k)}(u)\| \end{aligned} \tag{4.5}$$

for all $k \in \mathbb{N}$. The first term on the right-hand side of the last inequality goes to zero as k increases by the continuity of the sample paths of $X^{t,s,1}(\cdot)$. The arguments in the proof of Protter (2003), Theorem V.38 yield that

$$\lim_{k\to\infty} \sup_{u\in[\tilde{t},T]} \|X^{\tilde{t},y_k,z_k}(u) - X^{\tilde{t},s,1}(u)\| = 0$$

for all $\tilde{t} \in \{t, t_1, t_2, \ldots\}$ and any sequence $((y_k^{\mathsf{T}}, z_k)^{\mathsf{T}})_{k\in\mathbb{N}} \subset \mathbb{R}_+^{d+1}$ with $(y_k^{\mathsf{T}}, z_k)^{\mathsf{T}} \to (s^{\mathsf{T}}, 1)^{\mathsf{T}}$ as $k \to \infty$ almost surely. An analysis of the arguments in Protter (2003), Theorems V.37 and IV.73 yields that the convergence is uniformly in $\tilde{t} \in \{t, t_1, t_2, \ldots\}$, see also Ruf (2011), Lemma 1. We now choose for $(y_k^{\mathsf{T}}, z_k)^{\mathsf{T}}$ the sequences $(s_k^{\mathsf{T}}, 1)^{\mathsf{T}}$ and $(S^{t,s\mathsf{T}}(t_k,\omega), \tilde{Z}^{\phi,t,s}(t_k,\omega))^{\mathsf{T}}$ for all $\omega \in \Omega$. This proves the statement if $t_k \geq t$ for all $k \in \mathbb{N}$. In the case of the reversed inequality $t_k \leq t$, a small modification of the inequality in (4.5) yields the lemma. □

In the second step, we use techniques from the theory of PDEs to conclude the necessary smoothness of h^p. The following result has been used by Ekström, Janson and Tysk. We present it here on its own to underscore the analytic component of our argument.

Lemma 4.2 (Schauder estimates and smoothness)**.** *Fix a point $(t,s) \in [0,T) \times \mathbb{R}_+^d$ and a neighborhood $\mathcal{U}$ of (t,s). Suppose Assumption (A1) holds in conjunction with Inequality* (4.4) *for all $\xi \in \mathbb{R}^d$ and $(u,y) \in \mathcal{U}$ and some $C > 0$. Let $(f_k)_{k\in\mathbb{N}}$ denote a sequence of solutions of PDE* (4.2) *on $\mathcal{U}$, uniformly bounded under the supremum norm on $\mathcal{U}$. If $\lim_{k\to\infty} f_k(t,s) = f(t,s)$ on $\mathcal{U}$ for some function $f : \mathcal{U} \to \mathbb{R}$, then f solves PDE* (4.2) *on some neighborhood $\tilde{\mathcal{U}}$ of (t,s). In particular, $f \in C^{1,2}(\tilde{\mathcal{U}})$.*

Proof. We refer the reader to the arguments and references provided in Janson and Tysk (2006), Section 2 and Ekström and Tysk (2009), Theorem 3.2. The central idea is to use the interior Schauder estimates by Knerr (1980) along with Arzelà-Ascoli type of arguments to prove the existence of first- and second-order derivatives of f. $\square$

We can now prove the smoothness of the hedging price h^p.

Theorem 4.2. *Under Assumptions (A1)-(A3) there exists for all points of support* (t, s) *for* $S(\cdot)$ *some neighborhood* $\mathcal{U}$ *of* (t, s) *such that the function* h^p *defined in* (4.1) *is in* $C^{1,2}(\mathcal{U})$.

Proof. We define $\tilde{p} : \mathbb{R}_+^{d+1} \to \mathbb{R}_+$ by $\tilde{p}(s_1, \ldots, s_d, z) := zp(s_1, \ldots, s_d)$ and $\tilde{p}^M : \mathbb{R}_+^{d+1} \to \mathbb{R}_+$ by $\tilde{p}^M(\cdot) := \tilde{p}(\cdot)\mathbf{1}_{\{\tilde{p}(\cdot)\leq M\}}$ for some $M > 0$ and approximate $\tilde{p}^M$ by a sequence of continuous functions $\tilde{p}^{M,m}$ (compare for example Evans, 1998, Appendix C.4) such that $\lim_{m\to\infty} \tilde{p}^{M,m} = \tilde{p}^M$ pointwise and $\tilde{p}^{M,m} \leq 2M$ for all $m \in \mathbb{N}$. The corresponding expectations are defined as

$$\tilde{h}^{p,M}(u, y) := \mathbb{E}^{u,y}[\tilde{p}^M(S_1(T), \ldots, S_d(T), \tilde{Z}^{\theta,u,y}(T))]$$

for all $(u, y) \in \tilde{\mathcal{U}}$ for some neighborhood $\tilde{\mathcal{U}}$ of (t, s) and equivalently $\tilde{h}^{p,M,m}$.

We start by proving continuity of $\tilde{h}^{p,M,m}$ for large m. For any sequence $(t_k, s_k)_{k\in\mathbb{N}} \subset [0, T] \times \mathbb{R}_+^d$ with $\lim_{k\to\infty}(t_k, s_k) = (t, s)$, Lemma 4.1, in connection with Assumption (A1), yields

$$\lim_{k\to\infty} \tilde{p}^{M,m}(S^{t_k,s_k}(T), \tilde{Z}^{\theta,t_k,s_k}(T)) = \tilde{p}^{M,m}(S^{t,s}(T), \tilde{Z}^{\theta,t,s}(T)).$$

The continuity of $\tilde{h}^{p,M,m}$ follows then from the bounded convergence theorem.

Now, Janson and Tysk (2006), Lemma 2.6, in connection with Assumption (A2), guarantees that $\tilde{h}^{p,M,m}$ is a solution of PDE (4.2). Lemma 4.2 then yields that firstly, $\tilde{h}^{p,M}$ and secondly, in connection with Assumption (A3), h^p also solve PDE (4.2) on some neighborhood $\mathcal{U}$ of (t, s). In particular, h^p is in $C^{1,2}(\mathcal{U})$. $\square$

The last theorem is a generalization of the results in Ekström and Tysk (2009) to several dimensions and to non-continuous payoff functions p. Friedman (1976), Chapters 6 and 15 and Janson and Tysk (2006) have related results, but they impose linear growth conditions on $a(\cdot, \cdot)$ so that PDE (4.2) has a unique solution of polynomial growth. We are especially interested in the situation in which multiple solutions may exist. Heath and Schweizer (2000) present results in the case when the process corresponding to PDE (4.2) does not leave the positive orthant. As Fernholz and Karatzas (2010) observe, this condition does not necessarily hold if there is no ELMM. In the case of $Z^\theta(\cdot)$ being a martingale, our assumptions are only weakly more general than the ones in Heath and Schweizer (2000) by not requiring $a(\cdot, \cdot)$ to be continuous in the time dimension. However, in all these research articles the authors show that the function h^p indeed solves PDE (4.2) not only locally but globally and satisfies the corresponding boundary conditions. We have here abstained from imposing the stronger assumptions these papers rely on and concentrate on the local properties of h^p. For our application it is sufficient to observe that $h^p(t, S(t))$ converges to $p(S(T))$ as t goes to T; compare the proof of Theorem 4.1.

The next section provides an interpretation of our approach to prove the differentiability of h^p; all problems on the spatial boundary, arising for example from a discontinuity of $a(\cdot, \cdot)$ on the boundary of the positive orthant, have been "conditioned away," so that $S(\cdot)$ can get close to but never actually attains the boundary.

5 CHANGE OF MEASURE

In order to compute optimal strategies we need to compute the "deltas" of expectations. To simplify the computations we suggest in this section a change of measure under which the dynamics of the stock price process simplify.

Delbaen and Schachermayer (1995b), Theorem 1.4 show that NA implies the existence of a local martingale measure absolutely continuous with respect to $\mathbb{P}$. On the other side, a consequence of this section is the existence of a local martingale measure under NUPBR, such that $\mathbb{P}$ is absolutely continuous with respect to it. Indeed, NA and NUPBR together yield NFLVR (compare Delbaen and Schachermayer, 1994; Karatzas and Kardaras, 2007, Proposition 3.2), which again yields an ELMM corresponding exactly to the one discussed in this section. Another point of view, which we do not take here, is the recent insight by Kardaras (2010) on the equivalence of NUPBR and the existence of a finitely additive probability measure which is, in some sense, weakly equivalent to $\mathbb{P}$ and under which $S(\cdot)$ has some notion of weak local martingale property.

Our approach via a "generalized change of measure" is in the spirit of the work by Föllmer (1972), Meyer (1972), Delbaen and Schachermayer (1995a), Section 2, and Fernholz and Karatzas (2010), Section 7. They show that for the strictly positive $\mathbb{P}$-local martingale $Z^\theta(\cdot)$ a probability measure $\mathbb{Q}$ exists such that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ and $d\mathbb{P}/d\mathbb{Q} = 1/Z^\theta(T \wedge \tau^\theta)$, where τ^θ is the first hitting time of zero by the process $1/Z^\theta(\cdot)$. Their analysis has been built upon by several authors, for example by Pal and Protter (2010), Section 2. We complement this research direction by determining the dynamics of the $\mathbb{P}$-Brownian motion $W(\cdot)$ under the new measure $\mathbb{Q}$. These dynamics do not follow directly from an application of a Girsanov-type argument since $\mathbb{Q}$ need not be absolutely continuous with respect to $\mathbb{P}$. Similar results for the dynamics have been obtained in Sin (1998), Lemma 4.2 and Delbaen and Shirakawa (2002), Section 2. However, they rely on additional assumptions on the existence of solutions for some stochastic differential equations. Wong and Heyde (2004) prove the existence of a measure $\tilde{\mathbb{Q}}$ satisfying $\mathbb{E}^{\mathbb{P}}[Z^\theta(T)] = \tilde{\mathbb{Q}}(\tau^\theta > T)$, where $W(\cdot)$ has the same $\tilde{\mathbb{Q}}$-dynamics as we derive, but $\mathbb{P}$ is not necessarily absolutely continuous with respect to $\tilde{\mathbb{Q}}$.

For the results in this section, we make the technical assumption that the probability space Ω is the space of right-continuous paths $\omega : [0,T] \to \mathbb{R}^m \cup \{\Delta\}$ for some $m \in \mathbb{N}$ with left limits at $t \in [0,T]$ if $\omega(t) \neq \Delta$ and with an absorbing "cemetery" point Δ. By that we mean that $\omega(t) = \Delta$ for some $t \in [0,T]$ implies $\omega(u) = \Delta$ for all $u \in [t,T]$ and for all $\omega \in \Omega$. This point Δ will represent explosions of $Z^\theta(\cdot)$, which do not occur under $\mathbb{P}$, but may occur under a new probability measure $\mathbb{Q}$ constructed below. We further assume that the filtration $\mathbb{F}$ is the right-continuous modification of the filtration generated by the paths ω or, more precisely, by the projections $\xi_t(\omega) := \omega(t)$. Concerning the original probability measure we assume that $\mathbb{P}(\omega : \omega(T) = \Delta) = 0$ and that for all $t \in [0,T]$, ∞ is an absorbing state for $Z^\theta(\cdot)$; that is, $Z^\theta(t) = \infty$ implies $Z^\theta(u) = \infty$ for all $u \in [t,T]$. This assumption specifies $Z^\theta(\cdot)$ only on a set of measure zero and is made for notational convenience.

We emphasize that we have not assumed completeness of the filtration $\mathbb{F}$. Indeed, we shall construct a new probability measure $\mathbb{Q}$ which is not necessarily equivalent to the original measure $\mathbb{P}$ and can assign positive probability to nullsets of $\mathbb{P}$. If we had assumed completeness of $\mathbb{F}$, we could not guarantee that $\mathbb{Q}$ could be consistently defined on all subsets of these nullsets, which had been included in $\mathbb{F}$ during the completion process. The fact that we need the cemetery point Δ and cannot restrict ourselves to the original canonical space is also not surprising. The point Δ

represents events which have under $\mathbb{P}$ probability zero, but under $\mathbb{Q}$ have positive probability.

All these assumptions are needed to prove the existence of a measure $\mathbb{Q}$ with $d\mathbb{P}/d\mathbb{Q} = 1/Z^\theta(T \wedge \tau^\theta)$. After having ensured its existence, one then can take the route suggested by Delbaen and Schachermayer (1995a), Theorem 5 and start from any probability space satisfying the usual conditions, construct a canonical probability space satisfying the technical assumptions mentioned above, doing all necessary computations on this space, and then going back to the original space.

For now, the goal is to construct a measure $\mathbb{Q}$ under which the computation of h^p simplifies. For that, we define the sequence of stopping times

$$\tau_i^\theta := \inf\{t \in [0,T] : Z^\theta(t) \geq i\}$$

with $\inf \emptyset := \infty$ and the sequence of σ-algebras $\mathcal{F}^i := \mathcal{F}(\tau_i^\theta \wedge T)$ for all $i \in \mathbb{N}$. We observe that the definition of $\mathcal{F}^i$ is independent of the probability measure and define the stopping time $\tau^\theta := \lim_{i\to\infty} \tau_i^\theta$ with corresponding σ-algebra $\mathcal{F}^{\infty,\theta} := \mathcal{F}(\tau^\theta \wedge T)$ generated by $\cup_{i=1}^\infty \mathcal{F}^{i,\theta}$.

Within this framework, Meyer (1972) and Föllmer (1972), Example 6.2.2 rely on an extension theorem (compare Parthasarathy, 1967, Chapter 5) to show the existence of a measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}(T))$ satisfying

$$\mathbb{Q}(A) = \mathbb{E}^\mathbb{P}\left[Z^\theta(\tau_i^\theta \wedge T)\mathbf{1}_A\right] \tag{5.1}$$

for all $A \in \mathcal{F}^{i,\theta}$, where we now write $\mathbb{E}^\mathbb{P}$ for the expectation under the original measure. We summarize these insights in the following theorem, which also generalizes the well-known Bayes' rule for classical changes of measures (compare Karatzas and Shreve, 1991, Lemma 3.5.3).

Theorem 5.1 (Generalized change of measure, Bayes' rule)**.** *There exists a measure $\mathbb{Q}$ such that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ and such that for all $\mathcal{F}(T)$-measurable random variables $Y \geq 0$ we have*

$$\mathbb{E}^\mathbb{Q}\left[Y\mathbf{1}_{\left\{1/Z^\theta(T)>0\right\}}\middle|\mathcal{F}(t)\right] = \mathbb{E}^\mathbb{P}\left[Z^\theta(T)\,Y|\mathcal{F}(t)\right]\frac{1}{Z^\theta(t)}\mathbf{1}_{\left\{1/Z^\theta(t)>0\right\}} \tag{5.2}$$

$\mathbb{Q}$-almost surely (and thus, $\mathbb{P}$-almost surely) for all $t \in [0,T]$, where $\mathbb{E}^\mathbb{Q}$ denotes the expectation with respect to the new measure $\mathbb{Q}$. Under this measure $\mathbb{Q}$, the process $\widetilde{W}(\cdot) = \left(\widetilde{W}_1(\cdot), \ldots \widetilde{W}_K(\cdot)\right)^\mathsf{T}$ with

$$\widetilde{W}_k(t\wedge\tau^\theta) := W_k(t\wedge\tau^\theta) + \int_0^{t\wedge\tau^\theta}\theta_k(u,S(u))du \tag{5.3}$$

for all $k = 1,\ldots,K$ and $t \in [0,T]$ is a K-dimensional Brownian motion stopped at time τ^θ.

Proof. The existence of a measure $\mathbb{Q}$ satisfying (5.1) follows as in the discussion above. We fix an arbitrary set $B \in \mathcal{F}(t)$. It is sufficient to show the statement for $Y = \mathbf{1}_A$ where $A \in \mathcal{F}(T)$. We have

$$A = \left(A \cap \left\{\tau^\theta \leq T\right\}\right) \cup \bigcup_{i=1}^\infty \left(A \cap \left\{\tau_{i-1}^\theta < T \leq \tau_i^\theta\right\}\right).$$

From the fact that $\tau^\theta \leq T$ holds if and only if $1/Z^\theta(T) = 0$ holds, from the identity in (5.1), and from the observation that $\mathbb{P}\left(\tau^\theta \leq T\right) = 0$, we obtain

$$\mathbb{Q}\left(A \cap \left\{\frac{1}{Z^\theta(T)} > 0\right\} \cap B\right) = \sum_{i=1}^\infty \mathbb{Q}\left(A \cap \left\{\tau_{i-1}^\theta < T \leq \tau_i^\theta\right\} \cap B\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty} \mathbb{E}^{\mathbb{P}} \left[Z^{\theta}(\tau_i^{\theta} \wedge T) \mathbf{1}_{A \cap \left\{ \tau_{i-1}^{\theta} < T \leq \tau_i^{\theta} \right\} \cap B} \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[Z^{\theta}(T) \mathbf{1}_{A \cap B} \right] \\
&= \mathbb{E}^{\mathbb{P}} \left[Z^{\theta}(t) \mathbb{E}^{\mathbb{P}} \left[Z^{\theta}(T) \mathbf{1}_A \middle| \mathcal{F}(t) \right] \frac{1}{Z^{\theta}(t)} \mathbf{1}_B \right] \\
&= \mathbb{E}^{\mathbb{Q}} \left[\mathbb{E}^{\mathbb{P}} \left[Z^{\theta}(T) \mathbf{1}_A \middle| \mathcal{F}(t) \right] \frac{1}{Z^{\theta}(t)} \mathbf{1}_{\left\{ 1/Z^{\theta}(t) > 0 \right\}} \mathbf{1}_B \right].
\end{aligned}
$$

Here, the last equality follows as the first ones with $\mathbf{1}_A$ replaced by the random variable inside the last expression and T replaced by t. This yields (5.2). The fact that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ follows from setting $t = 0$ in (5.2). From Girsanov's theorem (compare Revuz and Yor, 1999, Theorem 8.1.4) we obtain that on $\mathcal{F}^{i,\theta}$ the process $\widetilde{W}(\cdot)$ is under $\mathbb{Q}$ a K-dimensional Brownian motion stopped at $\tau_i^{\theta} \wedge T$. Since $\cup_{i=1}^{\infty} \mathcal{F}^{i,\theta}$ generates $\mathcal{F}^{\infty,\theta}$ and forms a π-system, we get the dynamics of (5.3). □

Thus, an ELMM exists if and only if $\mathbb{Q}(1/Z^{\theta}(T) > 0) = 1$. A further consequence of Theorem 5.1 is the fact that the dynamics of the stock price process and the reciprocal of the SDF simplify under $\mathbb{Q}$ as the next corollary shows.

Corollary 5.1 (Evolution of important processes under $\mathbb{Q}$)**.** *The stock price process $S(\cdot)$ and the reciprocal $1/Z^{\theta}(\cdot)$ of the SDF evolve until the stopping time τ^{θ} under $\mathbb{Q}$ according to*

$$
\begin{aligned}
dS_i(t) &= S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) d\widetilde{W}_k(t), \\
d\left(\frac{1}{Z^{\theta}(t)} \right) &= \frac{1}{Z^{\theta}(t)} \sum_{k=1}^{d} \theta_k(t, S(t)) d\widetilde{W}_k(t)
\end{aligned}
$$

for all $i = 1, \ldots, d$ and $t \in [0, T]$. Furthermore, for any process $N(\cdot)$, $N(\cdot)\mathbf{1}_{\{1/Z^{\theta}(\cdot)>0\}}$ is a $\mathbb{Q}$-martingale if and only if $N(\cdot)Z^{\theta}(\cdot)$ is a $\mathbb{P}$-martingale. In particular, the process $1/Z^{\theta}(\cdot)$ is a $\mathbb{Q}$-martingale.

Proof. The dynamics are a direct consequence of the representation of $\widetilde{W}(\cdot)$ in (5.3) and the definition of the MPR. The other statements follow from choosing $Y = N(T)$ and $Y = 1/Z^{\theta}(T)$ in (5.2). □

The results of the last corollary play an essential role when we do computations, since the first hitting time of the reciprocal of the SDF can in most cases be easily represented as a first hitting time of the stock price. This now usually follows some more tractable dynamics, as we shall see in Section 6. For the case of strict local martingales the equivalence of the last corollary is generally not true. Take as an example $N(\cdot) \equiv 1$ and $Z^{\theta}(\cdot)$ a strict local martingale under $\mathbb{P}$. Then, $Z^{\theta}(\cdot)N(\cdot) \equiv Z^{\theta}(\cdot)$ is a local $\mathbb{P}$-martingale but $N(\cdot)\mathbf{1}_{\{1/Z^{\theta}(\cdot)>0\}} \equiv \mathbf{1}_{\{1/Z^{\theta}(\cdot)>0\}}$ is clearly not a local $\mathbb{Q}$-martingale. The reason for this lack of symmetry is that a sequence of stopping times which converges $\mathbb{P}$-almost surely to T need not necessarily converge $\mathbb{Q}$-almost surely to T.

6 EXAMPLES

In this section, we discuss several examples for markets which imply arbitrage opportunities. Examples 6.1 and 6.2 treat the case of a three-dimensional Bessel process with drift for various payoffs. Example 6.3 concentrates on the reciprocal of the three-dimensional Bessel, a standard example in the bubbles literature.

Example 6.1 (Three-dimensional Bessel process with drift - money market). One of the best known examples for markets without an ELMM is the three-dimensional Bessel process, as discussed in Karatzas and Shreve (1991), Section 3.3.C. We study here a class of models which contain the Bessel process as special case and generalize the example for arbitrage of A.V. Skorohod in Karatzas and Shreve (1998), Section 1.4. For that, we begin with defining an auxiliary stochastic process $X(\cdot)$ as a Bessel process with drift $-c$, that is,

$$dX(t) = \left(\frac{1}{X(t)} - c\right) dt + dW(t) \tag{6.1}$$

for all $t \in [0,T]$ with $W(\cdot)$ denoting a Brownian motion on its natural filtration $\mathbb{F} = \mathbb{F}^W$ and $c \in [0,\infty)$ a constant. The process $X(\cdot)$ is strictly positive, since it is a Bessel process, thus strictly positive under the equivalent measure where $\{W(t) - ct\}_{0\leq t\leq T}$ is a Brownian motion. The stock price process is now defined via the stochastic differential equation

$$dS(t) = \frac{1}{X(t)} dt + dW(t) \tag{6.2}$$

for all $t \in [0,T]$. Both processes $X(\cdot)$ and $S(\cdot)$ are assumed to start at the same point $S(0) > 0$. From (6.1) and (6.2) we obtain directly $S(t) = X(t) + ct > 0$ for all $t \in [0,T]$. If $c = 0$ then $S(\cdot) \equiv X(\cdot)$ and the stock price process is a Bessel process. Of course, the MPR is exactly $\theta(t,s) = 1/(s-ct)$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$ with $s > ct$. Thus, the reciprocal $1/Z^\theta(\cdot)$ of the SDF hits zero exactly when $S(t)$ hits ct. This follows directly from the $\mathbb{Q}$-dynamics of $1/Z^\theta(\cdot)$ derived in Corollary 5.1 and a strong law of large numbers as in Kardaras (2008), Lemma A.2.

Let us start by looking at a general, for the moment not-specified payoff function p. For all $(t,s) \in [0,T] \times \mathbb{R}_+$ with $s > ct$, by relying on Theorem 5.1, using the density of a Brownian motion absorbed at zero (compare Karatzas and Shreve, 1991, Problem 2.8.6) and some simple computations, we obtain

$$\begin{aligned}
h^p(t,s) =& \mathbb{E}^{t,s}\left[\tilde{Z}^{\theta,t,s}(T) p(S(T))\right] \\
=& \mathbb{E}^{\mathbb{Q}}\left[p(S(T))\mathbf{1}_{\{\min_{t\leq u\leq T}\{S(u)-cu\}>0\}}\middle|\,\mathcal{F}(t)\right]\Big|_{S(t)=s} \\
=& \int_{\frac{cT-s}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) p(z\sqrt{T-t}+s)dz \\
&- \exp(2cs - 2c^2t) \int_{\frac{cT-2ct+s}{\sqrt{T-t}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) p(z\sqrt{T-t}-s+2ct)dz.
\end{aligned} \tag{6.3}$$

Let us consider the investment in the money market only, to wit, $p(s) \equiv p^0(s) \equiv 1$ for all $s > 0$. The expression in (6.3) yields the hedging price of one monetary unit

$$h^{p^0}(t,s) = \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - \exp(2cs - 2c^2t)\Phi\left(\frac{-s-cT+2ct}{\sqrt{T-t}}\right), \tag{6.4}$$

where Φ denotes the cumulative standard normal distribution function. It can be easily checked that h^{p^0} solves PDE (4.2) for all $(t,s) \in [0,T] \times \mathbb{R}_+$ with $s > ct$. Thus, by Theorem 4.1 the optimal hedging strategy η^0 of one monetary unit is

$$\eta^0(t,s) = \frac{2}{\sqrt{T-t}}\phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - 2c\exp(2cs-2c^2t)\Phi\left(\frac{-s-cT+2ct}{\sqrt{T-t}}\right),$$

where ϕ denotes the standard normal density.

It is well-known that a Bessel process allows for arbitrage. Compare for example Karatzas and Kardaras (2007), Example 3.6 for an ad-hoc strategy which corresponds to a hedging price of $\Phi(1)$ for a monetary unit if $c = 0$ and $S(0) = T = 1$. We have improved here the existing strategies and found the optimal one, which corresponds in this setup to a hedging price of $h^{p^0}(0,1) = 2\Phi(1) - 1 < \Phi(1)$.

Remark 6.1 (Multiple solutions for PDE (4.2)). We observe that the hedging price h^{p^0} in (6.4) depends on the drift c. Also, h^{p^0} is sufficiently differentiable, thus by Remark 4.2 uniquely characterized as the minimal nonnegative solution of PDE (4.2), which does not depend on the drift c. The uniqueness of h^{p^0} by Remark 4.2 and the dependence of h^{p^0} on c do not contradict each other, since the nonnegativity of h^{p^0} has only to hold at the points of support for $S(\cdot)$. For a given time $t \in [0,T]$, these are only the points $s > ct$. Thus, as c increases, the nonnegativity condition weakens since it has to hold for fewer points, and thus h^p can become smaller and smaller. Indeed, plugging in (6.4) the point $s = ct$ yields $h^{p^0}(t,ct) = 0$. In summary, while the PDE itself does only depend on the (more easily observable) volatility structure of the stock price dynamics, the mean rate of return determines where the PDE has to hold.

In the next example we price and hedge a European call within the same class of models as in the last example.

Example 6.2 (Three-dimensional Bessel process with drift - stock and European call). Plugging in (6.3) the payoff $p(s) = p^C(s) = (y-L)^+$ for some $L \geq 0$ and writing $\tilde{L} := \max\{cT, L\}$, a simple computation yields

$$\begin{aligned} h^{p^C}(t,s) =& \sqrt{\frac{T-t}{2\pi}}\exp\left(-\frac{(s-\tilde{L})^2}{2(T-t)}\right) + (s-L)\Phi\left(\frac{s-\tilde{L}}{\sqrt{T-t}}\right) - \exp(2cs-2c^2t) \\ & \cdot \left(\sqrt{\frac{T-t}{2\pi}}\exp\left(-\frac{(\tilde{L}-2ct+s)^2}{2(T-t)}\right) + (2ct-s-L)\Phi\left(\frac{-\tilde{L}+2ct-s}{\sqrt{T-t}}\right)\right). \end{aligned}$$

If $L \leq cT$, in particular if $L = 0$, the last expression simplifies to

$$h^{p^C}(t,s) = s\Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) + \exp(2cs-2c^2t)\Phi\left(\frac{2ct-s-cT}{\sqrt{T-t}}\right)(s-2ct) - Lh^{p^0}(t,s),$$

where h^{p^0} denotes the hedging price of one monetary unit given in (6.4). It is simply the difference between the hedging price of the stock and L monetary units since if $L \leq cT$, the call is always exercised. Using $L = 0$ we get the value of the stock. We could now proceed by computing the derivative of h^{p^C} in s to get the hedge. Furthermore, the modified put-call parity of Corollary 4.1 provides us directly with the hedging price for a put.

If $L = c = 0$, we write $p^1 \equiv p^L$ and the last equality yields $h^{p^1}(t,s) = s$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$ and holding the stock is optimal. There are two other ways to see this result right away. Simple computations show directly that $\tilde{Z}^{\theta,t,s}(T) = s/S(T)$ if $c = 0$, thus $h^{p^1}(t,s) = s$ for all $(t,s) \in [0,T] \times \mathbb{R}_+$. Alternatively, using the representation of $h^{p^1}(t,s)$ implied by (5.2) we see that the hedging price is just the expectation of a Brownian motion stopped at zero, thus the expectation of a martingale started at s.

Two notable observations can be made. First, in this model both the money market and the stock simultaneously have a hedging price cheaper than their current price, as long as $c > 0$. Second, in contrast to classical theory, the mean rate of return under the "real-world" measure does matter in determining the hedging price of calls (or other derivatives).

Pal and Protter (2010) compute call prices for the reciprocal Bessel process model. We discuss next how the results of the last examples relate to this model.

Example 6.3 (Reciprocal of the three-dimensional Bessel process). Let the stock price $\tilde{S}(\cdot)$ have the dynamics

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

for all $t \in [0,T]$ with $W(\cdot)$ denoting a Brownian motion on its natural filtration $\mathbb{F} = \mathbb{F}^W$. The process $\tilde{S}(\cdot)$ is exactly the reciprocal of the process $S(\cdot)$ of Examples 6.1 and 6.2 with $c = 0$, thus strictly positive. We observe that $\mathbb{P}$ is already a martingale measure. However, if one wants to hold the stock at time T, one should not buy the stock at time zero, but use the strategy η^1 below for a hedging price smaller than $\tilde{S}(0)$ along with the suboptimal strategy $\eta(\cdot,\cdot) \equiv 1$. That is, the stock has a bubble.

We have already observed that $\tilde{S}(T) = 1/S(T)$, which is exactly the SDF in Example 6.1 for $c = 0$ multiplied by $\tilde{S}(t)$. Thus, as in (6.4) with $c = 0$, the hedging price for the stock is

$$h^{p^1}(t,s) = 2s\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - s < s \tag{6.5}$$

along with the optimal strategy

$$\eta^1(t,s) = 2\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - 1 - \frac{2}{s\sqrt{T-t}}\phi\left(\frac{1}{s\sqrt{T-t}}\right)$$

for all $(t,s) \in [0,T) \times \mathbb{R}_+$. For pricing calls, we observe

$$\left(\tilde{S}(T) - L\right)^+ = L\tilde{S}(T)\left(\frac{1}{L} - \frac{1}{\tilde{S}(T)}\right)^+ = \frac{L}{S(t)} \cdot \frac{S(t)}{S(T)}\left(\frac{1}{L} - S(T)\right)^+$$

for $L > 0$. Thus, the price at time t of a call with strike L in the reciprocal Bessel model is the price of $L\tilde{S}(t)$ puts with strike $1/L$ in the Bessel model and can be computed from Example 6.2 and Corollary 4.1. For $S(0) = 1$, simple computations will lead directly to Equation (6) of Pal and Protter (2010). The optimal strategy could now be derived with Theorem 4.1.

7 CONCLUSION

It has been proven that, under weak technical assumptions, there is no equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge. To ensure its existence, weak sufficient conditions have been introduced which guarantee the differentiability of an expectation parameterized over time and over the original market configuration. The dynamics of stochastic processes simplify after a non-equivalent change of measure and a generalized Bayes' rule has been derived. With this newly developed machinery, some optimal trading strategies have been computed addressing standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

References

Ansel, J.-P. and Stricker, C. (1993). Couverture des actifs contingents. *Annales de l'Institut Henri Poincaré*, 30(2):303–315.

Bayraktar, E., Kardaras, C., and Xing, H. (2010). Strict local martingale deflators and pricing American call-type options. *Finance and Stochastics*, forthcoming.

Cox, A. and Hobson, D. (2005). Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492.

Delbaen, F. and Schachermayer, W. (1994). A general version of the Fundamental Theorem of Asset Pricing. *Mathematische Annalen*, 300(3):463–520.

Delbaen, F. and Schachermayer, W. (1995a). Arbitrage possibilities in Bessel processes and their relations to local martingales. *Probability Theory and Related Fields*, 102(3):357–366.

Delbaen, F. and Schachermayer, W. (1995b). The existence of absolutely continuous local martingale measures. *Annals of Applied Probability*, 5(4):926–945.

Delbaen, F. and Schachermayer, W. (1995c). The no-arbitrage property under a change of numéraire. *Stochastics and Stochastic Reports*, 53:213–226.

Delbaen, F. and Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer.

Delbaen, F. and Shirakawa, H. (2002). No arbitrage condition for positive diffusion price processes. *Asia-Pacific Financial Markets*, 9:159–168.

Ekström, E. and Tysk, J. (2009). Bubbles, convexity and the Black-Scholes equation. *Annals of Applied Probability*, 19(4):1369–1384.

Ethier, S. N. and Kurtz, T. G. (1986). *Markov Processes: Characterization and Convergence*. John Wiley & Sons.

Evans, L. C. (1998). *Partial Differential Equations*. American Mathematical Society.

Fernholz, D. and Karatzas, I. (2010). On optimal arbitrage. *Annals of Applied Probability*, 20(4):1179–1204.

Fernholz, E. R. (2002). *Stochastic Portfolio Theory*. Springer.

Fernholz, E. R. and Karatzas, I. (2009). Stochastic Portfolio Theory: a survey. In Bensoussan, A., editor, *Handbook of Numerical Analysis*, volume Mathematical Modeling and Numerical Methods in Finance. Elsevier.

Fernholz, E. R., Karatzas, I., and Kardaras, C. (2005). Diversity and relative arbitrage in equity markets. *Finance and Stochastics*, 9(1):1–27.

Föllmer, H. (1972). The exit measure of a supermartingale. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 21:154–166.

Friedman, A. (1976). *Stochastic Differential Equations and Applications. Vols 1 and 2*. Academic Press.

Heath, D. and Platen, E. (2002a). Consistent pricing and hedging for a modified constant elasticity of variance model. *Quantitative Finance*, 2(6):459–467.

Heath, D. and Platen, E. (2002b). Perfect hedging of index derivatives under a minimal market model. *International Journal of Theoretical and Applied Finance*, 5(7):757–774.

Heath, D. and Schweizer, M. (2000). Martingales versus PDEs in finance: an equivalence result with examples. *Journal of Applied Probability*, 37:947 – 957.

Heston, S., Loewenstein, M., and Willard, G. (2007). Options and bubbles. *Review of Financial Studies*, 20.

Jacka, S. (1992). A martingale representation result and an application to incomplete financial markets. *Mathematical Finance*, 2:239–250.

Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer, 2nd edition.

Janson, S. and Tysk, J. (2006). Feynman-Kac formulas for Black-Scholes-type operators. *Bulletin of the London Mathematical Society*, 38(2):269–282.

Jarrow, R., Protter, P., and Shimbo, K. (2007). Asset price bubbles in complete markets. In Fu, M. C., Jarrow, R. A., Yen, J.-Y. J., and Elliott, R. J., editors, *Advances in Mathematical Finance*, volume in honor of Dilip Madan, pages 97–121. Birkhäuser.

Jarrow, R., Protter, P., and Shimbo, K. (2010). Asset price bubbles in incomplete markets. *Mathematical Finance*, 20(2):145–185.

Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 11(4):447–493.

Karatzas, I., Lehoczky, J., Shreve, S. E., and Xu, G. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29.

Karatzas, I. and Shreve, S. E. (1991). *Brownian Motion and Stochastic Calculus*. Springer, 2nd edition.

Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance*. Springer.

Kardaras, C. (2008). Balance, growth and diversity of financial markets. *Annals of Finance*, 4(3):369–397.

Kardaras, C. (2010). Finitely additive probabilities and the Fundamental Theorem of Asset Pricing. In Chiarella, C. and Novikov, A., editors, *Contemporary Mathematical Finance*. Springer.

Knerr, B. F. (1980). Parabolic interior Schauder estimates by the maximum principle. *Archive for Rational Mechanics and Analysis*, 75(1):51–85.

Krylov, N. V. (1973). On the selection of a Markov process from a system of processes and the construction of quasi-diffusion processes. *Mathematics of the USSR-Izvestiya*, 7:691–709.

Loewenstein, M. and Willard, G. A. (2000a). Local martingales, arbitrage, and viability. Free snacks and cheap thrills. *Economic Theory*, 16(1):135–161.

Loewenstein, M. and Willard, G. A. (2000b). Rational equilibrium asset-pricing bubbles in continuous trading models. *Journal of Economic Theory*, 91(1):17–58.

Merton, R. C. (1973). Theory of rational option pricing. *Bell Journal of Economics*, 4(1):141–183.

Meyer, P. (1972). La mesure de H. Föllmer en théorie de surmartingales. In *Séminaire de Probabilités, VI*. Springer.

Pal, S. and Protter, P. E. (2010). Analysis of continuous strict local martingales via h-transforms. *Stochastic Processes and their Applications*, 120(8):1424–1443.

Parthasarathy, K. (1967). *Probability Measures on Metric Spaces*. Academic Press.

Platen, E. (2002). Arbitrage in continuous complete markets. *Advances in Applied Probability*, 34(3):540–558.

Platen, E. (2008). The law of minimum price.

Platen, E. and Heath, D. (2006). *A Benchmark Approach to Quantitative Finance*. Springer.

Platen, E. and Hulley, H. (2008). Hedging for the long run.

Protter, P. E. (2003). *Stochastic Integration and Differential Equations*. Springer, 2nd edition.

Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*. Springer, 3rd edition.

Ruf, J. (2011). *Optimal Trading Strategies Under Arbitrage*. PhD thesis, Columbia University, New York, USA.

Schweizer, M. (1992). Martingale densities for general asset prices. *Journal of Mathematical Economics*, 21:363–378.

Sin, C. (1998). Complications with stochastic volatility models. *Advances in Applied Probability*, 30(1):256–268.

Stroock, D. W. and Varadhan, S. R. S. (2006). *Multidimensional Diffusion Processes*. Springer, Berlin. Reprint of the 1997 edition.

Wong, B. and Heyde, C. C. (2004). On the martingale property of stochastic exponentials. *Journal of Applied Probability*, 41(3):654–664.