Structural Default Models with Jumps

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Contents

Introduction

Abstract

In this thesis, we introduce a structural default model with an arbitrary jump-size distribution which is used to price corporate bonds. We model the value of a company and a default threshold using two jump-diffusion processes. A default of the company is triggered when the process representing the company's value first crosses the default threshold. Including jumps implies credit spreads which do not vanish as maturity decreases, as we prove. In case of a default, the bond holder receives a possibly stochastic fraction of the promised payments, the so called recovery rate. We present and compare three methods to price a corporate bond within this setup: Zhou's algorithm, the Brownian-bridge pricing technique and an approach using the Laplace transforms of the default probabilities. The first two methods allow all jump-size distributions which can be simulated numerically. Moreover, they allow a stochastic recovery rate. The third method is based on two-sided exponentially distributed jump sizes and a constant recovery rate.

Economic motivation and modelling background

Bonds promise their investors fixed payments. However, the bond holder cannot be sure to receive all payments, since the bond-issuing institution may default. If this occurs, the bond holder receives only a fraction of his promised payments. Therefore, it is not trivial to determine the fair price of a bond which includes a default risk. A pricing model is needed.[1]

Two classes of models are mainly discussed to price corporate bonds: intensity-based and structural models. In intensity-based models, a default is triggered

[1] Of course, bond prices can be observed on the capital markets. However, being able to determine bond prices allows us to calibrate the model parameters, which then can be used to price more complex credit derivatives.

by the first jump of a Poisson process with stochastic intensity. In structural models, a default is triggered by the event that a stochastic process representing the value of the company crosses a default barrier. We concentrate on structural models and compute the price of a defaultable bond on the basis of the default probabilities, which are implied by the model.

The firm-value process and the default threshold in our thesis are represented by jump-diffusion processes. There are several reasons for allowing jumps and not restricting ourselves to a more tractable pure diffusion setup. Firstly, the value of a company does not evolve continuously. Special events, such as a bulk order, winning a lawsuit or a computer crash suddenly increase or decrease the value of a company. Secondly, we observe credit spreads on the capital markets which do not vanish as maturity decreases to zero. However, pure diffusion models are not able to simulate sudden defaults, since the value process approaches the default barrier continuously. Hence, credit spreads that are not close to zero for short maturities cannot be obtained by pure diffusion models. In contrast, including the possibility of large negative jumps produces credit spreads that are strictly positive even for short maturities. Thirdly, jumps allow us to endogenously include a stochastic recovery rate. This is the payment that a bond investor obtains if the bond-issuing company defaults. In case of a default, in the pure diffusion setup, the value of a company equals the value of the default barrier, which is often assumed to be constant. This is, however, not true in the jump-diffusion setup. If the company defaults due to a jump of the underlying firm-value process, the value of the company at time of default is stochastic. This randomness allows us to model the recovery rate based on the company's value at time of default.

However, no analytical solution of the bond-pricing problem is known when the value of a company is modelled using a jump-diffusion process. Therefore, we have to perform simulations to obtain prices. We concentrate on three methods. Zhou's algorithm discretizes the time to maturity and checks at finitely many time points whether a default occurred. The Brownian-bridge pricing technique simulates the jumps and calculates the probability of a default based on the information about the jumps. In the Laplace-transform approach, we make assumptions about the jump-size distribution and calculate the bond price by means of the Laplace transform of the default probabilities. The three algorithms require different prerequisites. Moreover, they have different implications concerning the quality of the bond price and running time, which we discuss.

Structure of this thesis

In the first chapter, we present most of the mathematical tools which are used in the following chapters. The first section serves as a concise introduction to the basics of probability theory and our notation. The next section gives an overview of Lévy processes. We define a Wiener process and a compound Poisson process as the components of a jump-diffusion process. Then, we discuss important properties of these processes, such as the distribution of the running minimum of a Brownian motion. Furthermore, to understand jump-diffusion processes in the context of Lévy processes, important characteristics and properties of Lévy processes, such as the Lévy-Khinchin representation, are briefly presented. In the last section of this chapter, jump-diffusion processes are defined, their first moments are calculated, and it is shown that the difference of two jump-diffusion processes remains a jump-diffusion process. This statement allows a simplification of the latter introduced default model.

In the first section of the second chapter, we discuss bonds in general and risks associated with them. In the second section, we introduce a structural default model with a stochastic default barrier, which is modelled as a jump-diffusion process, and we show that this model can be simplified to a model with a constant default barrier. In the third section, we give an overview of approaches other than the structural one and compare those. In the fourth section, we prove that the credit spreads do not vanish as maturity decreases to zero.

The third chapter contains three different methods to price a corporate zero-coupon bond and a comparison of these methods. In the first section, Zhou's algorithm is presented and a justifying theorem is proven for all jump-size distributions. In the second section, Brownian bridges are defined and a Brownian-bridge pricing technique which allows us to assume stochastic recovery rates is introduced. Additionally, an approximation for an integral used in the Brownian-bridge pricing technique is discussed. In the third section, a Laplace-transform approach is briefly introduced. Finally, in the fourth section, the results and running times of the different algorithms are compared and the bias generated by Zhou's algorithm is explained.

In Appendix A, we introduce the Laplace transform, some of its basic properties and a Laplace inversion which we use to obtain the approximation of an integral used in the Brownian-bridge pricing technique. Appendix B contains some details about the implementations we have programmed to perform the simulations.

Chapter 1

Jump-diffusion processes

Later on, we model the value of a company by an exponential of a jump-diffusion process, which is a stochastic process belonging to the class of Lévy processes. The main purpose of this chapter is to present definitions and important mathematical results about jump-diffusion processes, necessary to understand the models and valuation formulas which we discuss later. The structure of this chapter is as follows: Firstly, we concisely introduce some standard notations and definitions to be used throughout this thesis. Secondly, we define Lévy processes and discuss some of their basic properties. Thirdly, we introduce jump-diffusion processes as the most important subclass of Lévy processes within this thesis. Most of the time, we follow the notation of Cont and Tankov, presented in [CT]. For a further treatment of Lévy processes see [Be] and [Sa].

1.1 Preliminaries

All definitions and statements made in this section also hold for finite time horizons. We use $\mathbb{R}$ (resp. $\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}^-$) for the set of all (resp. strictly positive, positive including zero, strictly negative) real numbers. The natural numbers without (resp. with) zero are denoted by $\mathbb{N}$ (resp. $\mathbb{N}_0$). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, that is a triple of a non-empty set Ω, a σ-Algebra $\mathcal{F}$ on Ω and a σ-additive function $\mathbb{P} : \mathcal{F} \to [0;1]$, and let $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t < \infty}$ be a filtration (which is an increasing family of σ-algebras, that is, $F_s \subset F_t$ for $s < t$) with $\mathcal{F} \supset \mathcal{F}_\infty := \sigma(\bigcup_{s \geq 0} \mathcal{F}_s)$.

Definition 1.1 ($\mathbb{P}$-complete, right-continuous)
A filtration $\mathbb{F}$ is called $\mathbb{P}$-complete and right-continuous if

1. *every* $\mathbb{P}$*-null set in* $\mathcal{F}_\infty$ *belongs to* $\mathcal{F}_0$ *(and so to all* $\mathcal{F}_t$ *) and*
2. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ *for all* $t > 0$ *.*

From now on, we deal with a filtered prob space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, equipped with a right-continuous, $\mathbb{P}$-complete filtration $\mathbb{F}$. We use the abbreviation "a.s." if a statement holds $\mathbb{P}$-almost surely, that is, there exists a set N of $\mathbb{P}$-measure zero such that the statement holds for all $\omega \in \Omega \setminus N$. Furthermore, all results with random objects hold almost surely in general. If this is obvious from the context it is occasionally not pointed out explicitly. The Borel σ-algebra, which is the σ-algebra generated by all open sets, is denoted by $\mathcal{B}(\mathbb{R})$. A measurable function from Ω into $\mathbb{R}$ equipped with the Borel σ-algebra is called a random variable. All random variables in this thesis take values in $\mathbb{R}$. If the random variable Z has the distribution $\mathbb{P}_Z$ with $\int_A \mathbb{P}_Z(dz) = \mathbb{P}_Z(A) := \mathbb{P}(Z^{-1}(A))$ for all $A \in \mathcal{B}(\mathbb{R})$, then we call $F_Z(x) := \int_{-\infty}^{x} \mathbb{P}_Z(du) = \mathbb{P}_Z((-\infty; x])$ the (cumulative) distribution function of Z. We also use the notation $F_Z(x-) := \lim_{s\uparrow x} F_Z(s)$. We denote by Φ the cumulative normal distribution:

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy.$$

The expectation of a random variable Z, with respect to the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}(Z)$, its variance by $\operatorname{Var}(Z)$, and the covariance of the random variables Z and $\tilde{Z}$ by $\operatorname{Cov}(Z, \tilde{Z})$.

Definition 1.2 (Characteristic function)
The characteristic function of a random variable Z *is the function* $\Phi_Z : \mathbb{R} \to \mathbb{C}$ *defined by*

$$\Phi_Z(z) := \mathbb{E}(\exp(izZ)) = \int_{\mathbb{R}} e^{izu} d\mathbb{P}_Z(du), \quad \forall z \in \mathbb{R}.$$

A stochastic process $X = \{X_t\}_{0 \le t < \infty}$, that is a family of random variables indexed over time, is said to be adapted if X_t is $\mathcal{F}_t$-measurable for all t. We call a stochastic process Y a modification of X if $X_t \overset{\text{a.s.}}{=} Y_t$ for all t.

Definition 1.3 (Càdlàg property)
An adapted stochastic process X *is called càdlàg*[2] *if* X *has almost surely sample paths which are right-continuous with left limits. We use the notations* $X_{t-} := \lim_{s\uparrow t} X_s$ *and* $\Delta X_t := X_t - X_{t-}$ *.*

[2] Continue à droite, limites à gauche in French. Sometimes it is also called RCLL or Skorokhod. A well-known example for càdlàg functions are cumulative distribution functions.

The time of default is modelled by a stopping time.

Definition 1.4 (Stopping time)
A random variable $\tau : \Omega \to [0;\infty]$ *is a stopping time if the event* $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ *for all* $t \in [0;\infty)$ *.*

We use mainly stopping times of the following form:

Lemma 1.1 (First-passage times are stopping times)
Let X *denote a càdlàg stochastic process and* $d \in \mathbb{R}$ *a real number. Then the first time* τ_d *, when* X *passes the threshold* d *,*

$$\tau_d := \inf\{t \geq 0 : X_t \leq d\}$$

with $\inf \emptyset := \infty$ *, is a stopping time.*

Proof: For the proof we refer to [RW], Lemma II.73.10, page 182 and Theorem II.76.1, page 186.[3] $\diamondsuit$

For a further introduction to probability theory and as a reference, we refer to [Fe], [Bi], and [RW].

1.2 Lévy processes

In this section, we give the definition of a Lévy process and briefly define a Wiener process as the most important and well-known example of a Lévy process. In the last paragraph of this section, we discuss some general properties, such as the Lévy-Khinchin representation.

1.2.1 Definition and examples

In this paragraph, we define a Lévy process and present two main examples of Lévy processes which are essential components of jump-diffusion processes: Wiener process and compound Poisson process.

[3] This is an example where we need the right-continuity of the underlying filtration $\mathbb{F}$.

Definition 1.5 (Lévy process)
An adapted càdlàg[4] *stochastic process* $X = \{X_t\}_{0 \le t < \infty}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}^d$, *such that* $X_0 = 0$, *is called a Lévy process if it possesses the following properties:*

1. *Independent increments: For every increasing sequence of times* $t_0, \ldots, t_n$, *the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*
2. *Stationary increments: The law of* $X_{t+h} - X_t$ *does not depend on* t.
3. *Stochastic continuity:* $\forall \epsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \ge \epsilon) = 0$.

We restrict ourselves to the one-dimensional case, that is, $d = 1$. By having independent and stationary increments, Lévy processes are the continuous equivalent to random walks in discrete time. The sample paths of a Lévy process are not continuous in general. The Lévy processes in the second example of this paragraph even have discontinuous sample paths with probability one at an infinite time horizon.[5] However, the most famous example of a Lévy process, the Wiener process, has almost surely continuous sample paths.

Definition 1.6 (Wiener process)
An adapted stochastic process $W = \{W_t\}_{0 \le t < \infty}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with values in* $\mathbb{R}$, *such that* $W_0 = 0$, *is called a Wiener process if*

1. W *has independent increments,*
2. $W_{t+h} - W_t \sim \mathcal{N}(0, h), \forall h > 0, t \ge 0$, *where* $\mathcal{N}(0, h)$ *denotes the Normal distribution with mean* 0 *and variance* h, *that is,* W *is Gaussian, and*
3. W *has almost surely continuous sample paths, that is,* $t \longrightarrow W_t$ *is continuous with probability one.*

Wiener processes and Brownian motions (Wiener processes with drift and modified standard deviation) are widely used and build the foundation for many mathematical models in financial applications. For a thorough introduction to the theory of Brownian motions we recommend [KS]. An overview of many formulas connected to Brownian motions is given in [BoS].

[4] As pointed out in [CT], Footnote 1, page 68, càdlàg can be postulated, since every Lévy process defined without this property has a unique modification which satisfies the càdlàg conditions.

[5] There exist also Lévy processes, for instance the so called infinite-activity Lévy processes, which have almost surely discontinuous sample paths on every finite time horizon. We do not deal with them in this thesis. For further details see for example [CT], Chapter 3.4.

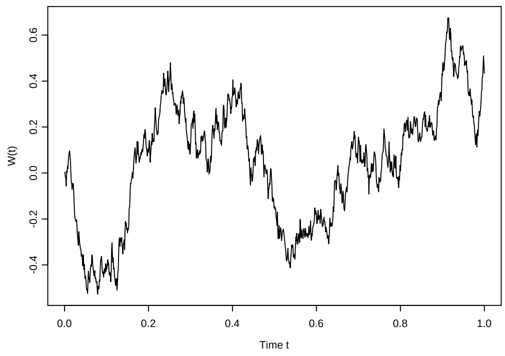

Figure 1.1: A sample path of a Wiener process W.

Lemma 1.2 (Wiener process is Lévy process)
A Wiener process is a Lévy process.

Proof: The statement follows directly from the definition. $\diamondsuit$

We use the next result about the running minimum of a Brownian motion several times.

Lemma 1.3 (Minimum of a Brownian motion)
The running minimum of a Brownian motion is inverse Gaussian distributed. More precisely, let W denote a Wiener process and $B_t = x+\gamma t+\sigma W_t$ represent a Brownian motion with drift γ and volatility σ starting in $B_0 = x$. The probability of B being below the threshold $b \in \mathbb{R}$ sometimes until time t is given by

$$\begin{aligned}
\tilde{\Phi}_b^{BM}(x,t) &:= \mathbb{P}\left(\min_{0\le s\le t} B_s \le b\right) \\
&= \mathbf{1}_{\{x\le b\}} + \mathbf{1}_{\{x>b\}}\left(\Phi\left(\frac{b-x-\gamma t}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\gamma(b-x)}{\sigma^2}\right)\Phi\left(\frac{b-x+\gamma t}{\sigma\sqrt{t}}\right)\right) \\
&= \mathbf{1}_{\{x\le b\}} + \mathbf{1}_{\{x>b\}}\int_0^t \frac{x-b}{\sqrt{2\pi}\sigma s^{3/2}}\exp\left(-\frac{(x-b+\gamma s)^2}{2s\sigma^2}\right) ds.
\end{aligned}$$

Proof: The first equality follows from [KS], Equation (3.41), page 265 by observing that $-W$ is also a Wiener process and setting $\gamma \to -\gamma/\sigma$ and $\beta \to (x-b)/\sigma$. The second equation is obtained by taking the derivative in t. $\diamondsuit$

To be able to define a compound Poisson process, we first introduce the Poisson process. Therefore, we remind the reader that an exponentially distributed random variable T with intensity λ is absolutely continuous with distribution $\mathbb{P}_T(dx) = \lambda \exp(-\lambda x) \cdot \mathbf{1}_{\{x>0\}} dx$.

Definition 1.7 (Poisson process)
Let $\{\tau_i\}_{i \geq 1}$ *denote a sequence of independent exponentially distributed random variables with parameter* λ *and* $T_n = \sum_{i=1}^n \tau_i$. *The process* $N = \{N_t\}_{0 \leq t < \infty}$ *defined by*

$$N_t := \sum_{n \geq 1} \mathbf{1}_{\{t \geq T_n\}}$$

is called a Poisson process with intensity λ.

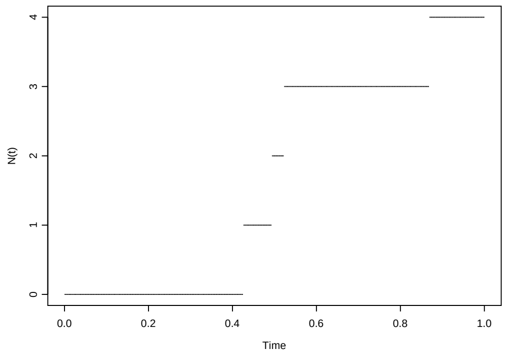

Figure 1.2: A sample path of a Poisson process N with intensity $\lambda = 3$.

If $\{\tau_i\}_{i \geq 1}$ model the waiting times within special occurrences (for instance some purchase orders for a company X), then N_t can be interpreted as the sum of these occurrences up to time t.

Lemma 1.4 (Distribution of Poisson process)
The process N *of Definition 1.7 follows a Poisson distribution, that is*

$$\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad \forall n \in \mathbb{N}_0. \tag{1.1}$$

Proof: This result can be shown by simple calculations. See [Bi], page 299. $\diamondsuit$

We later simulate stochastic processes with a Poisson-process component by conditioning on the number of jumps with the above lemma and on the location of the jumps. Therefore, we need to know the distribution of $(T_1, T_2, \ldots, T_k)$ conditioned on the events $\{\omega \in \Omega : N_t(\omega) = k\}$ for all $k \in \mathbb{N}$. The next lemma provides this information.

Lemma 1.5 (Distribution of jump times)
Under the condition $N_t = k$ *, the jump times* $(T_1, T_2, \dots, T_k)$ *of Definition 1.7 are distributed as order statistics, that is,*

$$\mathbb{P}_{(T_1,T_2,\dots,T_k|N_t=k)}(d(t_1,t_2,\dots,t_k)) = \mathbf{1}_{\{0<t_1<\dots<t_k\le t\}}\frac{k!}{t^k}d(t_1,t_2,\dots,t_k). \qquad (1.2)$$

Proof: Let $A_i := (-\infty; s_i]$ with $s_i \in \mathbb{R}$, $i = 1, \dots, k$. By the definition of conditional probability,

$$\mathbb{P}(T_1 \in A_1, \dots, T_k \in A_k | N_t = k) = \frac{\mathbb{P}(T_1 \in A_1, \dots, T_k \in A_k, N_t = k)}{P(N_t = k)}.$$

Since T_i is the sum of i independent, exponentially distributed random variables $\tau_1, \dots, \tau_i$,

$$\begin{aligned}
&\mathbb{P}(T_1 \in A_1, \dots, T_k \in A_k, N_t = k)\\
&= \mathbb{P}(T_1 \in A_1, \dots, T_{k-1} \in A_{k-1}, T_k \in [0; \min\{s_k, t\}], T_{t+1} > t)\\
&= \mathbb{P}(\tau_1 \in [0; s_1], \tau_2 \in [0; s_2 - \tau_1], \dots, \tau_k \in [0; \min\{s_k, t\} - \tau_{k-1}], \tau_{k+1} \in (t - \tau_k; \infty))\\
&= \int_{A_1\times\dots\times A_k} \mathbf{1}_{\{0<t_1<\dots<t_k\le t\}}\lambda^k e^{-\lambda t_1} \cdot \ldots \cdot e^{-\lambda(t_k - t_{k-1})} \cdot e^{-\lambda(t-t_k)} d(t_1, t_2, \dots, t_k)\\
&= \int_{A_1\times\dots\times A_k} \mathbf{1}_{\{0<t_1<\dots<t_k\le t\}}\lambda^k e^{-\lambda t} d(t_1, t_2, \dots, t_k).
\end{aligned}$$

From Equation (1.1) in Lemma 1.4, we obtain

$$\mathbb{P}(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

Combining these representations and using $\mathcal{B}(\mathbb{R}) = \sigma(\{(-\infty; x], x \in \mathbb{R}\})$ completes the proof. $\diamondsuit$

Having defined a Poisson process, it is possible to define a more general class of stochastic processes:

Definition 1.8 (Compound Poisson process)
A compound Poisson process with intensity $\lambda > 0$ *and jump-size distribution* $\mathbb{P}_Y$ *is an adapted stochastic process* $M = \{M_t\}_{0\le t<\infty}$ *, defined by*

$$M_t := \sum_{i=1}^{N_t} Y_i,$$

where the jump sizes Y_i *are independent and identically distributed with distribution* $\mathbb{P}_Y$ *, and* $N = \{N_t\}_{0\le t<\infty}$ *is a Poisson process with intensity* λ *, independent of* $Y = \{Y_i\}_{i\ge1}$ *.*

Remark 1.1 (Connection with Poisson process)
By choosing $\mathbb{P}_Y(\{1\}) = 1$ and thus having set $Y_i \equiv 1$ for all i, we see that a Poisson process is a special case of a compound Poisson process.

Lemma 1.6 (Compound Poisson process is Lévy process)
A compound Poisson process is a Lévy process.

Proof: By the definition of N in Definition 1.7 we see that a compound Poisson process is càdlàg. Independence and stationarity of increments and stochastic continuity can be simply validated. For detailed calculations we refer to [CT], Proposition 3.3, page 71. $\diamondsuit$

The Lévy-Khinchin formula, which is introduced later[6], yields an explicit representation of the characteristic function of a Lévy process at every time t. In the case of a compound Poisson process we can directly obtain this result. The proof is based on the commonly used technique to condition on the number of jumps.

Lemma 1.7 (Characteristic function of a compound Poisson process)
Let M denote a compound Poisson process with intensity λ and jump-size distribution $\mathbb{P}_Y$. Its characteristic function Φ_{M_t} at time t admits the following representation:

$$\Phi_{M_t}(z) = \mathbb{E}(\exp(izM_t)) = \exp\left(t\lambda \int_{\mathbb{R}} \left(e^{iuz} - 1\right) \mathbb{P}_Y(du)\right), \quad \forall z \in \mathbb{R}. \tag{1.3}$$

Proof: Let $M_t = \sum_{i=1}^{N_t} Y_i$ be as in Definition 1.8. Let $\hat{f}(z) = \Phi_{Y_1}(z)$ denote the characteristic function of Y_1. Since $\{Y_i\}_{i\geq 1}$ are independent and identically distributed, we find

$$\begin{aligned}
\Phi_{M_t}(z) &= \mathbb{E}\left(\mathbb{E}(\exp(izM_t)|N_t)\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(\exp\left(iz\sum_{i=1}^{N_t} Y_i\right)\middle| N_t\right)\right) \\
&= \mathbb{E}\left(\left(\hat{f}(z)\right)^{N_t}\right) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n \left(\hat{f}(z)\right)^n}{n!} \\
&= e^{-\lambda t} e^{\lambda t \hat{f}(z)}
\end{aligned}$$

[6] See Theorem 1.2.

$$= \exp\left(t\lambda\left(\hat{f}(z) - 1\right)\right)$$
$$= \exp\left(t\lambda \int_{\mathbb{R}} \left(e^{iuz} - 1\right) \mathbb{P}_Y(du)\right).$$

The last equation holds, since $\int_{\mathbb{R}} \mathbb{P}_Y(du) = 1$. $\diamondsuit$

1.2.2 General properties of Lévy processes

In this paragraph, we discuss general properties of Lévy processes and important results of Lévy-process theory. They not only serve as useful tools when dealing with Lévy processes but also show how jump-diffusion processes are characterized within the class of Lévy processes. Furthermore, they are helpful tools for understanding proofs connected to theorems, such as those in Section 3.3.

The jump structure of a Lévy process is described by the jump measure and the Lévy measure. It is well defined, since càdlàg functions have at most countably many jumps.[7]

Definition 1.9 (Jump measure)
Let X denote a Lévy process. The random measure[8] J_X defined by

$$J_X(A \times B) := \sum_{t \geq 0 : \Delta X_t > 0} \mathbf{1}_{t \in A} \cdot \mathbf{1}_{\{\Delta X_t \in B\}}, \quad \forall A \times B \in \mathcal{B}(\mathbb{R}_0^+ \times \mathbb{R})$$

is called the jump measure of X.

Definition 1.10 (Lévy measure)
Let X denote a Lévy process. The measure ν defined by

$$\nu(B) := \mathbb{E}(J_X([0;1] \times B)) = \mathbb{E}\left(\sum_{t \geq 0 : \Delta X_t > 0} \mathbf{1}_{t \in [0;1]} \cdot \mathbf{1}_{\{\Delta X_t \in B\}}\right), \quad \forall B \in \mathcal{B}(\mathbb{R})$$

is called the Lévy measure of X.

[7] For a proof, see [R], Appendix A.1.

[8] A random measure is a measure which additionally depends on the state of the world $\omega \in \Omega$, respectively here the realization of X. Mathematically, it is a function $J_X : \Omega \times \mathcal{B}([0;\infty[\times\mathbb{R}) \to \mathbb{R}^+$, $(\omega, C) \mapsto J_{X(\omega)}(C)$. For a discussion of random measures and especially Poisson-random measures, see [CT], Chapter 2.6.

Since a Lévy process is càdlàg, the number $\nu(A)$ is the expected number of jumps, per unit of time, whose sizes belong to A. Now we are able to give a first important result in Lévy-process theory: the Lévy-Itô decomposition, which describes the structure of the sample paths and allows us to characterize every real-valued[9] Lévy process with the so called Lévy triplet.

Theorem 1.1 (Lévy-Itô decomposition)
Let X denote a Lévy process, ν its Lévy measure, and J_X its jump measure. Then, there exist $\gamma_1 \in \mathbb{R}$, $\sigma \in \mathbb{R}_0^+$, and a Wiener process W such that

$$\begin{aligned} X_t &= \gamma_1 t + \sigma W_t + X_t^1 + \lim_{\epsilon \downarrow 0} \tilde{X}_t^\epsilon, \quad \textit{where} \\ X_t^1 &:= \int\limits_{|x|>1,\, s\in[0;t]} x J_X(ds \times dx) \qquad \textit{and} \\ \tilde{X}_t^\epsilon &:= \int\limits_{\epsilon \le |x| \le 1,\, s\in[0;t]} x(J_X(ds \times dx) - \nu(dx)ds). \end{aligned}$$

W, X^1 and $\lim_{\epsilon \downarrow 0} \tilde{X}^\epsilon$ are mutually independent.

Proof: We refer to [Sa], Theorem 19.2, page 120 and Chapter 20, page 125, where a proof is given. ◇

This representation is complicated, since on the one side, not all Lévy processes are compound Poisson processes and hence, the Lévy measure ν can have a singularity at zero.[10] On the other side, $\int_{|x|>1} |x| \nu(dx)$ does not have to be finite. We truncate the jump size in Theorem 1.1 at one, but we also could truncate it at any other size $\epsilon > 0$. From the Lévy-Itô decomposition, we realize that all distributional properties of a Lévy process are represented by γ_1, σ^2, and the Lévy measure ν[11]:

Definition 1.11 (Lévy triplet, center of the process)
The triplet $(\sigma^2, \nu, \gamma_1)_1$, where σ, ν, and γ_1 are as in Theorem 1.1 is called the Lévy triplet[12] *(with respect to the truncation function $\mathbf{1}_{\{x \le 1\}}$) of X. If the*

[9] Of course, there exists also a characterization of a vector-valued Lévy process. The standard-deviation parameter is replaced by a covariance matrix. However, we concentrate on the one-dimensional case in this thesis.

[10] From the càdlàg property it follows that zero is the only critical point.

[11] No distributional information gets lost at the transition from J_X to ν due to the independent and stationary increments of a Lévy process.

[12] The Lévy triplet is also called characteristic triplet.

condition $\int_{|x|>1} |x|\nu(dx) < \infty$ *is satisfied, then we call* $\gamma_c = \gamma + \int_{|x|>1} |x|\nu(dx)$ *the center of the process* X *and* $(\sigma^2, \nu, \gamma_c)_c$ *the Lévy triplet (without truncation) of* X *.*

The term center of the process is justified by the fact that a Lévy process X can be shown[13] to satisfy $I\!E(X_1) = \gamma_c$ if the integrability condition holds. The condition is satisfied if the Lévy process is a compound Poisson process with integrable jump sizes, that is, $I\!E(|Y_1|) < \infty$.

Another important result of Lévy-process theory is the possibility to express the characteristic function of a Lévy process at every time t in terms of its Lévy triplet:

Theorem 1.2 (Lévy-Khinchin representation)
Let X *denote a Lévy process with characteristic triplet* $(\sigma^2, \nu, \gamma)_1$ *. Then its characteristic function at time* t *satisfies:*

$$\begin{aligned} \Phi_{X_t}(z) &= e^{t\psi(z)}, \quad z \in \mathbb{R} \quad \textit{with} && (1.4)\\ \psi(z) &:= -\frac{1}{2}\sigma^2 z^2 + i\gamma z + \int\limits_{\mathbb{R}} \left(e^{iuz} - 1 - iuz\mathbf{1}_{\{|u|\le 1\}}\right) \nu(du). && (1.5) \end{aligned}$$

If $\int_{|u|>1} |x|\nu(du) < \infty$ *holds, Equation (1.5) can be replaced with the simpler expression*

$$\psi(z) := -\frac{1}{2}\sigma^2 z^2 + i\gamma_c z + \int\limits_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu(du), \qquad (1.6)$$

where $\gamma_c = \gamma + \int_{|u|>1} |u|\nu(du)$ *is as in Definition 1.11.*

Proof: We only show Equation (1.4). For Equation (1.5) we refer to [CT], Theorem 3.1, page 83, where the statement is proven. Equation (1.6) is a direct consequence of Equation (1.5). The stochastic continuity[14] of $t \mapsto X_t$ implies continuity in distribution and thus, continuity of $t \mapsto \Phi_{X_t}(z)$ for all $z \in \mathbb{R}$. We obtain for all $z \in \mathbb{R}$ due to the independent and stationary increments:

$$\Phi_{X_{t+s}}(z) = \Phi_{X_t + (X_{t+s} - X_t)}(z) = \Phi_{X_t}(z)\Phi_{X_{t+s}-X_t}(z) = \Phi_{X_t}(z)\Phi_{X_s}(z).$$

The exponential function is the only continuous function which satisfies this multiplicative property. $\diamondsuit$

[13] For instance, by differentiating Equation (1.4) in the below presented Theorem 1.2.
[14] See Definition 1.5, property 3.

Lemma 1.8 (Wiener process and compound Poisson process)

1. *A Wiener process has the Lévy triplet*[15]

$$(1,0,0)_1 \quad \textit{and} \quad (1,0,0)_c,$$

respectively.

2. *The Lévy triplet of a compound Poisson process with intensity* λ *and jump-size distribution* $\mathbb{P}_Y$ [16] *is*

$$\left(0, \lambda\mathbb{P}_Y(du), \lambda\int_{\mathbb{R}} u\mathbf{1}_{\{|u|\le 1\}}\mathbb{P}_Y(du)\right)_1.$$

If the jump sizes are integrable, that is, $\int_{\mathbb{R}} |u|\mathbb{P}_Y(du)$ *exists, then we do not need to truncate and have the Lévy triplet*

$$\left(0, \lambda\mathbb{P}_Y(du), \lambda\int_{\mathbb{R}} u\mathbb{P}_Y(du)\right)_c.$$

Proof: The parameters can be directly derived by comparing the well-known characteristic function $\Phi_{W_t}(z) = \exp(-0.5tz^2)$ of a Wiener process at time t [17] and Equation (1.3) in Lemma 1.7, respectively, to Equations (1.4) and (1.5) in the Levy-Khinchin representation. ♢

1.3 Jump-diffusion processes

In this section, we discuss the class of jump-diffusion processes, which is the most important class of Lévy processes in this thesis.

Definition 1.12 (Jump-diffusion process)
The process $X = \{X_t\}_{0\le t<\infty}$ *defined by*

$$X_t := \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

[15] This is one of numerous examples which show that in general Lévy measures are not probability measures.

[16] For some important jump-size distributions in financial applications, the explicit representations of the Lévy measures can be found in [CW], Table 1.

[17] Obtained by using the technique of completing the square.

where W is a Wiener process, independent from N and $\{Y_i\}_{i\geq 1}$ which are defined as in Definition 1.8, is called a jump-diffusion process with drift γ, volatility of diffusion σ, intensity λ, and jump-size distribution $\mathbb{P}_Y$. If there are no jumps, that is, $\mathbb{P}_Y(\{0\}) = 1$, then we also call the stochastic process pure diffusion process or Brownian motion.[18]

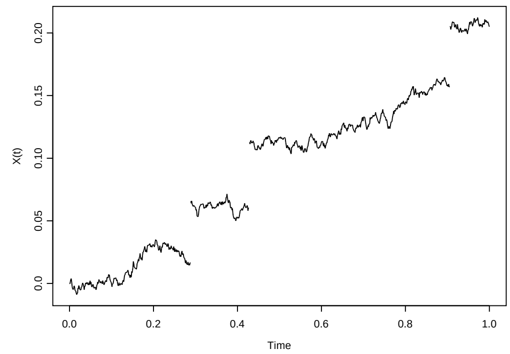

Figure 1.3: A sample path of a jump-diffusion process X with drift $\gamma = 0.025$, volatility of diffusion $\sigma = 0.05$, intensity $\lambda = 3$, and jumps of constant size 0.05, that is, $\mathbb{P}_Y(\{0.05\}) = 1$.

Remark 1.2 (Jump-diffusion process is Lévy process)
Since obviously $Z_t = \gamma t$ is a Lévy process and sums of independent Lévy processes are again Lévy processes, so is a jump-diffusion process. We know the Lévy-Khinchin representation of all three summands, can multiply their characteristic functions due to the independence of the underlying random variables and can compute the Lévy triplet for X

$$\left(\sigma^2, \lambda\mathbb{P}_Y(du), \gamma + \lambda\int_{\mathbb{R}} u\mathbf{1}_{\{|u|\leq 1\}}\mathbb{P}_Y(du)\right)_1$$

and if the integrability condition is satisfied

$$\left(\sigma^2, \lambda\mathbb{P}_Y(du), \gamma + \lambda\int_{\mathbb{R}} u\mathbb{P}_Y(du)\right)_c,$$

respectively.

We need the following lemma later.

[18] Compare to the notes after Definition 1.6.

Lemma 1.9 (Difference of two jump-diffusion processes)
Let

$$\begin{aligned} X_t &= \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \quad \textit{and} \\ \tilde{X}_t &= \tilde{\gamma} t + \tilde{\sigma} \tilde{W}_t + \sum_{i=1}^{\tilde{N}_t} \tilde{Y}_i \end{aligned}$$

denote two jump-diffusion processes, such that the covariance satisfies $Cov(W_t, \tilde{W}_t) = \rho\sigma\tilde{\sigma}t$ *, where the correlation* $\rho \in [-1;1]$ *is constant over time, and all other appearing random variables are mutually independent. The intensity of* N *(resp.* $\tilde{N}$ *) is denoted by* λ *(resp.* $\tilde{\lambda}$ *), the jump-size distribution of* Y *(resp.* $\tilde{Y}$ *) by* $\mathbb{P}_Y$ *(resp.* $\mathbb{P}_{\tilde{Y}}$ *). Then* $\hat{X} = X - \tilde{X}$ *is also a jump-diffusion process and takes the form*

$$\hat{X}_t \stackrel{\mathrm{d}}{=} \hat{\gamma} t + \hat{\sigma} \hat{W}_t + \sum_{i=1}^{\hat{N}_t} \hat{Y}_i,$$

where $\hat{\gamma} = \gamma - \tilde{\gamma}$ *,* $\hat{\sigma} = \sqrt{\sigma^2 + \tilde{\sigma}^2 - 2\sigma\tilde{\sigma}\rho}$ *,* $\hat{W}$ *is a Wiener process,* $\hat{N}$ *a Poisson process with intensity* $\hat{\lambda} = \lambda + \tilde{\lambda}$ *, and* $\hat{Y}$ *has the distribution* $\mathbb{P}_{\hat{Y}}(du) = \lambda/(\lambda + \tilde{\lambda})\mathbb{P}_Y(du) + \tilde{\lambda}/(\lambda+\tilde{\lambda})\mathbb{P}_{-\tilde{Y}}(du)$ *.*

Proof: The deterministic part is obvious. We start with the difference of the two Wiener processes. If $\rho = 1$, then $W \stackrel{\text{a.s.}}{=} \tilde{W}$ and $\sigma W_t - \tilde{\sigma}\tilde{W}_t \stackrel{\mathrm{d}}{=} |\sigma - \tilde{\sigma}| W_t$ which proves the statement about $\hat{\sigma}$ and $\hat{W}$. So, let be $\rho \in [-1;1)$. Then, $\sigma W_t - \tilde{\sigma}\tilde{W}_t$ has variance $(\sigma^2 + \tilde{\sigma}^2 - 2\sigma\tilde{\sigma}\rho) \cdot t = \hat{\sigma}^2 t$ and

$$\hat{W}_t := \frac{1}{\sqrt{\sigma^2 + \tilde{\sigma}^2 - 2\sigma\tilde{\sigma}\rho}} (\sigma W_t - \tilde{\sigma}\tilde{W}_t) \sim \mathcal{N}(0,t)$$

is a Wiener process with $\hat{\sigma}\hat{W} = \sigma W_t - \tilde{\sigma}\tilde{W}_t$. Let $M_t = \sum_{i=1}^{N_t} Y_i$ and $\tilde{M}_t = \sum_{i=1}^{\tilde{N}_t} (-\tilde{Y}_i)$ denote the compound Poisson processes. Since they are independent, we can simply calculate the characteristic function of their sum $\Phi_{M_t + \tilde{M}_t}$ at time t as the product of the characteristic functions Φ_{M_t} and $\Phi_{\tilde{M}_t}$, as given in Lemma 1.7:

$$\begin{aligned} \Phi_{M_t+\tilde{M}_t}(z) &= \exp\left(t \int_{\mathbb{R}} \left(e^{iuz} - 1\right) \left(\lambda \mathbb{P}_Y(du) + \tilde{\lambda}\mathbb{P}_{-\tilde{Y}}(du)\right)\right) \\ &= \exp\left(t(\lambda + \tilde{\lambda}) \int_{\mathbb{R}} (e^{iuz} - 1) \left(\frac{\lambda}{\lambda+\tilde{\lambda}} \mathbb{P}_Y(du) + \frac{\tilde{\lambda}}{\lambda+\tilde{\lambda}} \mathbb{P}_{-\tilde{Y}}(du)\right)\right). \end{aligned}$$

We notice that this corresponds to the characteristic function of a compound Poisson process with intensity $\hat{\lambda}$ and jump-size distribution $\mathbb{P}_{\hat{Y}}$ at time t. $\diamondsuit$

We shall also need the moments of a jump-diffusion process. The formula for the expectation is called Wald's equation.

Lemma 1.10 (Moments of a jump-diffusion process)
Let the notation be as in Definition 1.12.

1. *If the jump sizes are integrable, that is,* $\mathbb{E}(|Y_1|) < \infty$, *then the expectation of a jump diffusion* X *at time* t *exists and*

$$\mathbb{E}(X_t) = t\gamma_c = t\left(\gamma + \lambda \int_{\mathbb{R}} u\mathbb{P}_Y(du)\right) = t\left(\gamma + \lambda\mathbb{E}(Y_1)\right).$$

2. *If in addition the jump sizes are square integrable, that is,* $\mathbb{E}(Y_1^2) < \infty$, *then the variance of a jump diffusion* X *at time* t *exists and is given by*

$$Var(X_t) = t\left(\sigma^2 + \lambda\mathbb{E}(Y_1^2)\right).$$

Proof: Let all integrability conditions be satisfied. Since N follows a Poisson distribution, $\mathbb{E}(N_t) = \mathrm{Var}(N_t) = \lambda t$ holds. By conditioning on the number of jumps and using the independence of diffusion and jump component, we obtain

$$\begin{aligned}
\mathbb{E}(X_t) &= \mathbb{E}\left(\gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right) \\
&= \gamma t + 0 + \mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{N_t} Y_i \,\middle|\, N_t\right)\right) \\
&= \gamma t + \mathbb{E}\left(N_t \cdot \mathbb{E}(Y_1)\right) \\
&= \gamma t + \lambda t\mathbb{E}(Y_1), \\
\mathrm{Var}(X_t) &= \mathrm{Var}\left(\gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right) \\
&= \sigma^2 \mathrm{Var}(W_t) + \mathbb{E}\left(\left(\sum_{i=1}^{N_t} Y_i\right)^2\right) - \left(\mathbb{E}\left(\sum_{i=1}^{N_t} Y_i\right)\right)^2 \\
&= \sigma^2 t + \mathbb{E}\left(\mathbb{E}\left(\left(\sum_{i=1}^{N_t} Y_i\right)^2 \,\middle|\, N_t\right)\right) - t^2\lambda^2\mathbb{E}(Y_1)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^2 t + \mathbb{E}\left(\mathbb{E}\left(\sum_{i,j=1}^{N_t} Y_i Y_j \middle| N_t\right)\right) - t^2\lambda^2 \mathbb{E}(Y_1)^2 \\
&= \sigma^2 t + \mathbb{E}\left(N_t \mathbb{E}(Y_1^2) + N_t(N_t - 1)\mathbb{E}(Y_1)^2\right) - t^2\lambda^2 \mathbb{E}(Y_1)^2 \\
&= \sigma^2 t + \mathbb{E}(N_t)\mathbb{E}(Y_1^2) + \left(\mathbb{E}(N_t^2) - \mathbb{E}(N_t)\right) \mathbb{E}(Y_1)^2 - t^2\lambda^2 \mathbb{E}(Y_1)^2 \\
&= \sigma^2 t + \lambda t \mathbb{E}(Y_1^2) + \left(\lambda^2 t^2 + \lambda t - \lambda t\right) \mathbb{E}(Y_1)^2 - t^2\lambda^2 \mathbb{E}(Y_1)^2 \\
&= t\left(\sigma^2 + \lambda \mathbb{E}(Y_1^2)\right).
\end{aligned}
$$

This completes the proof. ♢

At first glance, it seems surprising that instead of the variance the second quadratic moment of Y appears in the formula for the variance of X. The reason is that the event of no jump occurring is considered, too. By looking at jumps of constant size one this becomes clear.

Chapter 2

Models

In this chapter, we discuss some background information about corporate bonds and introduce the class of structural jump-diffusion models, on which we concentrate in this thesis.[19] We also give a brief overview of an alternative approach to modelling credit risk by means of so called "reduced-form models". In the last section, we show that a firm-value process which includes large negative jumps implies credit spreads for small maturities which are strictly positive. Hence, the empirical observation that credit spreads do not vanish as maturity decreases can be captured by this class of models.

2.1 Economic background

A bond in general is a debt security which is traded on fixed-income markets. The bond holder pays a principal to the issuer, who pays back predefined coupon payments at fixed dates and a face value at the bond's maturity.[20]

Government bonds, such as American Treasury notes or German Bundesschatzbriefe, represent safe investments.[21] The holder is guaranteed to receive all claims related to the bond. These bonds are called "risk-free bonds". Since all payments are sure for such a bond, there is no difficulty at all to calculate its fair price, which

[19]For an introduction to pure diffusion models without any jumps, we refer to [BR], Chapter I.3. and [BK], Chapter 9.3.

[20]For a detailed introduction to bonds, see [BKM], Chapter 14.

[21]Of course, there are also risky sovereign bonds. For a discussion, see [DS], Chapter 6.4.1. However, we assume a risk-free investment possibility to be given and concentrate on the pricing of defaultable, corporate bonds.

is just the sum of all discounted payments. However, it is difficult to determine the fair price of a "corporate" or "defaultable bond", which is issued by another institution, such as a large corporation. The holder of such a bond cannot be sure to receive all promised payments. In case of a default by the issuing institution, a bond holder receives only a fraction of his claim or no payment at all depending on the severity of the institution's default. Pricing a corporate bond is therefore based on two stochastic influences: Whether default occurs and how severe the default is.

Investing in bonds implies more risks than just the default risk. There exist also the interest-rate risk[22], the inflation risk[23], and the liquidity risk[24]. In this thesis, however, we concentrate on the default risk and assume that there are no other risks associated to the bond. Recent cases show that the default risk is not a theoretical risk. For instance, the company WorldCom was unable to repay its debt to the investors after its bankruptcy in 2002. Therefore, a bond buyer demands for a higher interest rate than the corresponding risk-free interest rate in exchange for the risk.

Bonds are classified by different rating companies, such as S&P, according to their default risk. Table 2.1 shows the average credit spreads, that is the amount by which the interest rate of the bond exceeds the risk-free interest rate, for a given risk classification. Different maturities imply different spreads. Matching this phenomenon, called term structure of interest rates, is an essential point of modelling.

An important type of bond is the class of zero-coupon bonds.[25] The only payment associated with them takes place at the time of maturity. As the next lemma shows, every bond with predefined payment schedule can be replicated using a portfolio of zero-coupon bonds with appropriate maturities. So, we concentrate on pricing corporate zero-coupon bonds in this thesis.

[22]The interest-rate risk describes the risk of incurring opportunity costs, since there will be bonds with a higher return if the interest rate increases.

[23]The inflation risk is closely related to the interest-rate risk. If the inflation rises, an investment in tangible assets is maybe a better investment than holding a bond, since the cash flows from the bond take place in the future, when they have less worth.

[24]The liquidity risk reflects the problem of finding someone to buy the bond if the investor decides to sell it. This risk is especially high for rarely traded bonds issued by small companies.

[25]There are other types of bonds, such as callable bonds, which include the right to reduce the bond's maturity. We do not pay attention to this class of bonds. For a description of different types of bonds and more sophisticated credit derivatives, such as credit default swaps, we refer to [Bo].

Maturity	Risk-free	Defaultable	Credit Spread
3 months	4.63	5.18	0.55
6 months	4.73	5.26	0.53
1 year	4.73	5.30	0.57
2 years	4.67	5.36	0.69
3 years	4.67	5.42	0.75
4 years	4.65	5.48	0.83
5 years	4.63	5.55	0.92
7 years	4.60	5.64	1.04
8 years	4.60	5.72	1.12
9 years	4.60	5.75	1.14*
10 years	4.60	5.80	1.20
15 years	4.72	6.02	1.31*
20 years	4.71	6.04	1.33
25 years	4.63	6.12	1.49
30 years	4.50	6.11	1.61

* was caused by a rounding error.

Table 2.1: Credit spreads for different maturities for February 28th 2006. The data set is provided by Bloomberg L.P. It consists of the risk-free (US Treasure Composite) interest rates, the average defaultable (US Industrial BBB) interest rates, and the corresponding credit spreads.

Lemma 2.1 (Representation of coupon bonds)
For any coupon bond there exists a portfolio of zero-coupon bonds with the same payoff structure. More precisely, the price of a coupon bond $\phi(0, t_1, t_2, \ldots, t_n, q)$ *with maturity* T*, face value* F*, and promised coupon payments* $q = (q_1, q_2, \ldots, q_n)$ *at* $0 < t_1 < t_2 < \ldots < t_n = T$ *satisfies*

$$\phi(0, t_1, t_2, \ldots, t_n, q) = \sum_{j=1}^{n} q_j \phi(0, t_j) + F\phi(0, t_n),$$

where $\phi(0, t_j)$ *denotes the price of a zero-coupon bond with face value one and maturity* t_j*.*

Proof: The statement follows directly from the no-arbitrage condition. $\diamondsuit$

2.2 Structural jump-diffusion models

In a structural credit-risk model, the value of a company, the so called firm value, and a default threshold are explicitly modelled with stochastic processes. For this thesis, both processes are defined by the exponentials $V = \{V_t\}_{t\geq 0}$ and $D = \{D_t\}_{t\geq 0}$ of the jump diffusions $X = \{X_t\}_{t\geq 0}$ and $d = \{d_t\}_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. More precisely,

$$\begin{aligned} V_t &= v_0 \exp(X_t), \quad v_0 \in \mathbb{R}, && (2.1)\\ X_t &= \gamma^X t + \sigma^X W_t^X + \sum_{i=1}^{N_t^X} Y_i^X, && (2.2)\\ D_t &= d_0 \exp(d_t), \quad d_0 \in \mathbb{R},\, d_0 < v_0, \quad \text{and} && (2.3)\\ d_t &= \gamma^d t + \sigma^d W_t^d + \sum_{i=1}^{N_t^d} Y_i^d. && (2.4)\end{aligned}$$

$\gamma^X, \gamma^d, \sigma^X,$ and σ^d are constant over time and defined as in Definition 1.12, along with W^X, W^d, N^X, N^d, Y^X, and Y^d. Additionally, we assume that the covariance of W and $\tilde{W}$ satisfies $\mathrm{Cov}(W_t, \tilde{W}_t) = \rho\sigma\tilde{\sigma}t$, where $\rho \in [-1;1]$ is constant over time, and that all other appearing random variables are mutually independent.

We assume that $\mathbb{P}$ is a risk-neutral measure.[26] We denote by $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ the natural filtration generated by the processes V and D[27], that is, $\mathcal{F}_t = \sigma(V_s, D_s : 0 \leq s \leq t) = \sigma(X_s, d_s : 0 \leq s \leq t)$, augmented to satisfy the usual conditions of right-continuity and $\mathbb{P}$-completeness[28].

The firm value can represent the company's accumulated assets, which equal the sum of the company's equity and debt, or the sum of the company's discounted future cash flows. Also, there are different interpretations for the default threshold: It can stand for the sum of the company's short-term liabilities plus a fraction (for instance 50%) of the long-term liabilities, as in the models of the financial consulting company KMV.[29] Black and Cox interpret the threshold in [BC] as the minimal firm value required to operate the company.

[26] But the discounted firm-value and default-threshold processes do not have to be martingales, since they do not always represent tradeable assets.

[27] This implies that we can observe both the actual firm value and the actual default threshold. [Schm] discusses a model where the default threshold cannot be observed.

[28] See Definition 1.1.

[29] See [CB], Section 2.

The stochastic processes of the model are described by the following parameters[30]:

$\gamma^X,\ \gamma^d$	:	The linear trends of the diffusion components.
$\sigma^X,\ \sigma^d$	:	The volatilities of the diffusion components.
$\lambda^X,\ \lambda^d$	:	The jump intensities.
$\mathbb{P}_{Y^X},\ \mathbb{P}_{Y^d}$	:	The jump distributions.
$v_0,\ d_0$	:	The initial firm value and the initial value of the default threshold.

We prove the results and introduce the algorithms for general jump-size distributions as far as possible. If necessary, as for the implementation, we concentrate on two-sided exponentially distributed jump sizes introduced at the end of this section.

While in the financial literature generalizations to stochastic interest rates were made[31], we concentrate on a constant interest rate, which we denote by r.

The company issues a zero-coupon bond with maturity T at time 0. We have got two similar ways to model how default is triggered: In the classical approach by Black, Scholes, [BlS] (1973) and Merton, [M1] (1974, without jumps) and [M2] (1976, with jumps), a credit event is triggered if at the bond's maturity the firm is not able to meet its obligations, that is, the firm value is below the default threshold. The second approach is to assume that default is triggered at the first time τ, when the firm value crosses[32] the default threshold.[33] These models are called first-passage time models.[34] This idea was first presented by Black

[30]In Theorem 2.1 we see that the two stochastic processes can be combined to one. The new stochastic process will be the quotient of the firm value and the default threshold. Hence, we shall not need the parameters $\gamma^d, \sigma^d, \lambda^d, d_0$.

[31]For instance, Longstaff and Schwartz, [LS] extended the Black-Cox model to include stochastic interest rates. The short-rate process follows the mean-reverting process suggested by Vasicek, [V] in 1977 and can be correlated with the firm-value process V.

[32]We use the expression "to cross" although we define default as the event that the firm value crosses OR touches the default barrier. It can be shown that having a real Gaussian component, that is, $\sigma > 0$ for σ from Theorem 2.1, the two definitions are equivalent. Compare to [Bi], Chapter 37, page 507.

[33]If the default barrier is constant, that is, $d_t \equiv 0$, then it was shown in Lemma 1.1 that τ is a stopping time. If $\{d_t\}_{t\geq 0}$ is a jump-diffusion process, then τ is also a stopping time, since the relevant filtration $\mathbb{F}$ is the natural filtration of the two stochastic processes V and D, which represent the firm value and default threshold. This property can be proved by applying Theorem 2.1.

[34]By setting the default barrier to zero for the time until the bond's maturity we realize that the first structural models are special cases of the more general first-passage time models. Since it is not possible to manipulate the default barrier in such a way in the restricted jump-diffusion setup, this does not hold true for our structural jump-diffusion models.

and Cox, [BC] in 1976. In general, the time of default is endogenously included in structural models. While a corporate bond in [BlS] and [M1] resembles a European put, in the first-passage time models it is more like a barrier option. Altogether, it is possible to adapt pricing techniques of equity securities to price bonds and other debt securities. In this thesis, we concentrate on the first-passage time setup.

Only since the late nineties of the twentieth century, firm-value processes with a diffusion and jump component have been combined with first-passage time models. They were first introduced by Schönbucher, [Schö], Section 3 in 1996 and Zhou, [Z1] in 1997.[35] Allowing the firm-value process to jump provides a natural model of the default severity, which is now endogenously given by the model: If a company defaults by a jump, its value process falls below its default threshold. This random undershot is used to specify the default severity, and hence the recovery rate. Therefore, we model the recovery rate as a function of the firm-value to default-threshold ratio at the time of default. The bond holder receives the fraction $w(V_\tau/D_\tau)$ of the face value[36] in case of a default, where w is a positive, non-decreasing and measurable function, defined on the unit interval $[0;1]$.[37] We consider two different recovery-rate functions:

$$\begin{aligned} w_1^R(x) &= R, \quad R \in [0;1], \\ w_2^R(x) &= R \cdot x, \quad R \in [0;1]. \end{aligned}$$

Using w_1^R implies not to pay any attention to the undershot $1 - V_\tau/D_\tau$ of the firm-value process over the default barrier, that is, the same amount of money independent of the firm value at time of default is paid back to the investor at time of default. When using w_2^R the investor's payout at time of default depends linearly on the firm value.[38]

The recovery payoff can be timed in two different ways: at time of default or at the bond's maturity.

Remark 2.1 (Summary of the model parameters)
The model is determined by the specification of the jump-diffusion processes including the jump-size distribution, by fixing the interest rate and the bond's maturity, by setting the recovery rates, and by deciding about the recovery-payoff timing.

[35] Already in 1981, there was a model presented in [MB] which combined jump processes with first-passage time models. However, there was no diffusion component included.

[36] This is one unit of money in our setup.

[37] This definition slightly differs from Zhou's suggestion in [Z2].

[38] For empirical studies about the recovery rate, see for instance [AK].

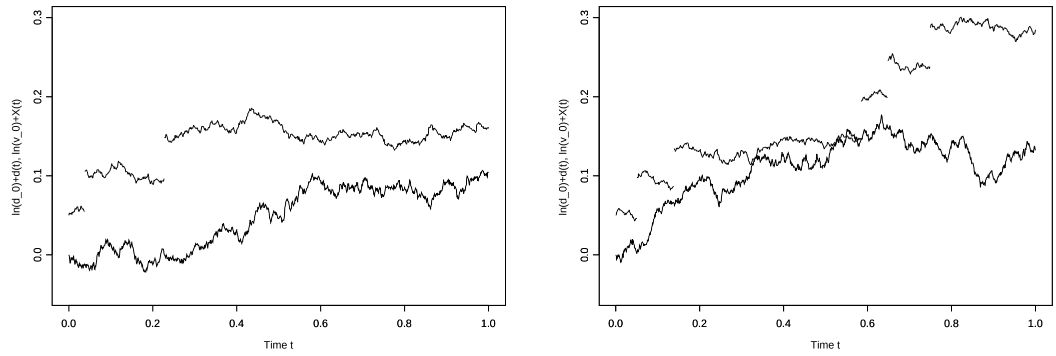

Figure 2.1: Two sample paths of $X + \ln(v_0)$ and $d + \ln(d_0)$ with parameters $\gamma^X = 0.025$, $\sigma^X = 0.05$, $\lambda^X = 3$, $\gamma^d = 0.025$, $\sigma^d = 0.08$, $v_0 = e^{0.05}$, and $d_0 = 1$. While X has constant jumps of size 0.05, d is a pure diffusion. On the left picture no default occurs. On the right picture, however, default occurs at time $\tau = 0.32$. If the maturity of the corresponding bond is $T = 1$, in the Black-Scholes-Merton setup no default would occur in either sample, since $V_1 > D_1$ holds.

We illustrate the influence of the recovery-payoff timing on the zero-coupon bond valuation in the next lemma.

Lemma 2.2 (Price of a zero-coupon bond)
With $\phi(t,T)$ (resp. $\tilde{\phi}(t,T)$) denoting the price of a corporate bond at time t with recovery payout at time of default (resp. at maturity), we obtain for $\tau > t$:

$$\phi(t,T) = e^{-r(T-t)} \mathbb{P}(\tau > T | \mathcal{F}_t) + \mathbb{E}\left(e^{-r(\tau-t)} w\left(\frac{V_\tau}{D_\tau}\right) \mathbf{1}_{\{t<\tau\le T\}} \middle| \mathcal{F}_t \right), \quad (2.5)$$

$$\tilde{\phi}(t,T) = e^{-r(T-t)} \left(\mathbb{P}(\tau > T | \mathcal{F}_t) + \mathbb{E}\left(w\left(\frac{V_\tau}{D_\tau}\right) \mathbf{1}_{\{t<\tau\le T\}} \middle| \mathcal{F}_t \right) \right). \quad (2.6)$$

Proof: The fair price of a defaultable bond is given as the expectation of its discounted payout with respect to the risk-neutral probability measure $\mathbb{P}$. The discount factor[39] for the time interval $[t_1; t_2]$ is exactly $e^{-r(t_2-t_1)}$. If no default occurs the bond holder receives one unit of money. In case of a default, $w(V_\tau/D_\tau)$ is paid. This completes the proof. $\diamondsuit$

The last lemma yields the representation of a defaultable zero-coupon bond as the sum of two securities. One security pays one unit of money at maturity if

[39] This is also the price of a risk-free zero-coupon bond.

the company does not default, the other security pays the (possibly stochastic) recovery rate w if the company defaults before or at maturity.

If we have to calibrate the model[40] the number of parameters is too high.[41] However, this problem can be reduced, since we can decrease the number of parameters by merging the two stochastic processes V and D to only one jump diffusion $\tilde{V}$. This is illustrated in the next theorem.

Theorem 2.1 (Reducing the model's complexity)
Let V and D be defined as in Equations (2.1) to (2.4). Define

$$\begin{aligned}\tilde{V}_t &:= \tilde{v}_0 \exp(\tilde{X}_t) \quad \textit{with} \\ \tilde{X}_t &:= \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \quad \textit{and} \\ \tilde{D}_t &:= 1,\end{aligned}$$

where $\tilde{v}_0 := \frac{v_0}{d_0}$ and $\tilde{X}$ is a jump-diffusion process with parameters $\gamma = \gamma^X - \gamma^d$, $\sigma = \sqrt{(\sigma^X)^2 + (\sigma^d)^2 - 2\sigma^X \sigma^d \rho}$, intensity $\lambda = \lambda^X + \lambda^d$, and jump-size distribution $\mathbb{P}_Y(du) = \lambda^X/(\lambda^X + \lambda^d)\mathbb{P}_{Y^X}(du) + \lambda^d/(\lambda^X + \lambda^d)\mathbb{P}_{-Y^d}(du)$. Then, the model specified by the firm-value process $\tilde{V}$ and default threshold $\tilde{D}$ has the same dynamics as the model specified by V and D, in particular both times of default (τ and $\tilde{\tau}$) and both undershots ($1 - V_\tau/D_\tau$ and $1 - \tilde{V}_{\tilde{\tau}}$) are identically distributed. Thus, the same bond prices and credit spreads are generated from the two model specifications.

Proof: In Lemma 1.9 we have shown that $\tilde{X}_t \stackrel{\mathrm{d}}{=} X_t - d_t$. So,

$$\begin{aligned}\tilde{V}_t &= \tilde{v}_0 \exp(\tilde{X}_t) \stackrel{\mathrm{d}}{=} \frac{v_0 \exp(X_t)}{d_0 \exp(d_t)} = \frac{V_t}{D_t} \quad \text{and} \\ \tilde{\tau} &= \inf\{s : \tilde{V}_s \leq \tilde{D}_s\} = \inf\{s : \tilde{V}_s \leq 1\} \stackrel{\mathrm{d}}{=} \inf\{s : V_s/D_s \leq 1\} = \tau,\end{aligned}$$

which completes the proof. ◇

Theorem 2.1 is useful not only for calibrating but also for simplifying simulation algorithms. Obviously, it is faster to simulate only one stochastic process, even if the intensity of the jump component increases and the jump-size distribution

[40] That is, to adapt the model parameters to given market prices.

[41] Another problem linked to this one is the fact that a change of different parameters can have the same effect, for instance increasing the default drift is almost equivalent to decreasing the firm-value drift.

occasionally becomes more complex. Thus, from now on we assume a constant default threshold $D_t \equiv 1$. This yields $V = \tilde{V}$.

We need the following definition:

Definition 2.1 (Distance to default)
At any time $t \geq 0$, the distance to default for X, that is, how far X_t is away from the critical barrier is given by

$$x_t := \ln(V_t) = \ln(v_0) + X_t.$$

x_0 from the last definition has a special meaning: $-x_0$ is the barrier which the stochastic process X crosses when default occurs, as shown in the next remark.

Remark 2.2 (Further possible simplification of the model)
Default occurs if $1 \geq V_t = v_0 \exp(X_t)$. That is equivalent to

$$X_t \leq -\ln(v_0) \Longleftrightarrow \frac{X_t}{\ln(v_0)} \leq -1.$$

If we standardize $\gamma^{new} = \frac{\gamma}{\ln(v_0)}$, $\sigma^{new} = \frac{\sigma}{\ln(v_0)}$, and $Y^{new} = \left\{\frac{Y_i}{\ln(v_0)}\right\}_{i\geq 1}$, we would not need the parameter v_0 any more. However, in this thesis, we abstain from this simplification, since firstly, the parameter v_0 is often exogenously given and does not have to be calibrated and secondly, when comparing bond values and credit spreads of companies in different financial situations it is easier to modify v_0 than to modify γ, σ, and the jump-size distribution $\mathbb{P}_Y$.

There are infinitely many choices for selecting the jump-size distribution. Merton, [M2] and Zhou, [Z2] chose normally distributed jump sizes, that is $Y_i \sim \mathcal{N}(\mu, \delta^2)$ for $\mu \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$. This choice has the advantage that the probability-density function of the jump-diffusion process X can be analytically represented.[42] In contrast, Scherer, [Sche] used two-sided exponentially distributed jump sizes, which were studied by Kou and Wang, [KW]. Due to the memoryless property of exponentially distributed random variables, some expressions depending on the first-passage times can sometimes be analytically obtained. As long as possible, we state the theorems in terms of general jump-size distributions. However, we implemented the algorithms with two-sided exponentially distributed jump sizes.

[42] For details, we refer to [CT], page 111.

Definition 2.2 (Two-sided exponential distribution)
The distribution

$$\mathbb{P}_Y(dx) = p\lambda_\oplus e^{-\lambda_\oplus x}\mathbf{1}_{\{x\geq 0\}}dx + (1-p)\lambda_\ominus e^{\lambda_\ominus x}\mathbf{1}_{\{x<0\}}dx$$

with parameters $p \in [0;1]$ *and* $\lambda_\oplus, \lambda_\ominus \in \mathbb{R}^+$ *is called a two-sided exponential distribution.*

Jumps following this distribution are positive with probability p and negative with probability $1-p$. Their size is exponentially distributed with means $1/\lambda_\oplus$ and $1/\lambda_\ominus$, respectively.

2.3 Alternative approaches

There are alternative structural models. The simplest one, a pure diffusion model, is included as a special case in the jump-diffusion model by only allowing jumps of size zero.[43] Also, by setting $\sigma = 0$ we obtain a pure jump model, as presented in [MB]. Cariboni and Schoutens, [CS] discuss a so called variance-Gamma approach. Here, the firm-value process is the exponential of a stochastic process X which is completely driven by jumps, that is, X is built by infinitely many jumps in any finite interval and its paths are of finite variation. This implies that it does not have a Brownian-motion component. To obtain results, time-consuming Monte-Carlo simulations have to be done or partial-differential integral equations have to be solved.

Many different interpretations of the default boundary are discussed in the financial literature. One approach differing from ours is to model the ratio of the firm value to the default threshold as a mean-reverting process in order to include a company's policy to receive a fixed leverage ratio as suggested by Collin-Dufresne and Goldstein, [CG].[44] In Leland's model, [L], the owner of the company chooses the default threshold.

In reduced-form models[45], the firm value does not serve as a reference. Instead, there is an exogenously specified Poisson process P with random intensity H.

[43]Thus, there are still jumps but they are of size zero. We chose this interpretation of pure diffusion models in order to remain in the class of jump-diffusion models. For $\lambda = 0$, N is not a Poisson process any more.

[44]Since the mean-reverting component does not allow us to reformulate the quotient of firm value and threshold as a jump-diffusion process, we do not deal with this interpretation in this thesis. For further details in a pure diffusion setup see the original paper [CG].

[45]Also intensity-based models called.

Default is triggered by the first jump of P. This jump is not predictable with respect to the underling filtration $\mathbb{F}$. The reduced-form models mainly differ in the definition of the default-intensity process H, which is also called a hazard-rate process. In reduced-form models, the recovery rate is also given exogenously. For an introduction to these models, we refer to [DS], Chapter 3.5. Another alternative is to combine structural and intensity-based models to so called hybrid models. The default-intensity process H is determined by the firm value. Duffie and Lando, [DL] discuss such a model which additionally assumes periodic and imperfect accounting information.

The orientation mainly at the fundamentals firm value and liabilities is both advantageous and disadvantageous at the same time for the structural approach. On the one hand, a link between the firm's assets and liabilities and default risk is explicitly given and has not to be constructed as in the reduced-form approach. Also, the value of the company at time of default is endogenously included and can be used to determine the recovery rate. On the other hand, it is not possible to always observe the firm value and liabilities. Their structure can be very complex (outstanding wages, tax payments, etc.) and it can be difficult to calculate them exactly, since balance sheets often include noisy information, as the case of the company Enron shows. Furthermore, reduced-form models can be better adjusted to fit observed credit spreads, since they are not restricted by the connection of fundamentals and default risk.

The default times in pure diffusion models are predictable, that is, there always exist increasing sequences of stopping times which converge to them. This implies that short-term credit spreads are extremely low, contradictory to reality.[46] In contrast, default times in reduced-form models occur by surprise and are unpredictable. A jump-diffusion model can have both kinds of stopping times: While the diffusion creates predictable stopping times, the jump component generates unexpected ones.

For a detailed comparison of the structural and reduced-form approach, we refer to the paper of Jarrow and Protter, [JP].

2.4 Credit spreads for small maturities

As stated in the last section, credit spreads for short-maturity bonds in a traditional pure diffusion model are smaller than observed credit spreads on the

[46] See for instance [JMR] or [Fo].

markets. This is a huge drawback of pure diffusion models, since they are not able to mirror this property of short-term bonds and so, cannot be used to price them. We now show that including large negative jumps results in credit spreads which do not vanish as maturity decreases to zero.

We start with the formal definition of credit spreads:

Definition 2.3 (Credit spread)
Let $\phi(0,T)$ denote the price of a zero-coupon bond with maturity T. Then its corresponding credit spread is the real number η_T that solves the relation

$$\phi(0,T) = e^{-(r+\eta_T)T}.$$

The local default rate is an important component of the exact limit of credit spreads as maturity tends to zero:

Definition 2.4 (Local default rate of τ)
The local default rate of τ, abbreviated as LDR_τ, is defined by

$$LDR_\tau = \lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(\tau \leq h).$$

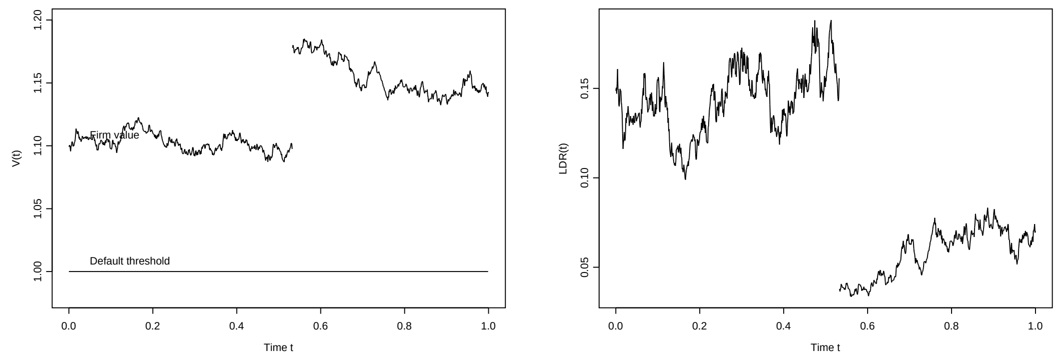

Figure 2.2: A sample path of V and the corresponding local default rate LDR_τ. The jump sizes are assumed to be two-sided exponentially distributed. We chose as parameters $\gamma = 0.025$, $\sigma = 0.05$, $p = 0.5$, $\lambda = 2$, $\lambda_\oplus = \lambda_\ominus = 20$, and $v_0 = 1.1$.

We first concentrate on the pure diffusion setup. The next result shows that the local default rate in pure diffusion models is zero. Furthermore, it helps to obtain results for the jump-diffusion setup.

Lemma 2.3 (The local default rate of τ in a pure diffusion model)
If there are no jumps, that is, $Y \equiv 0$*, then* $LDR_\tau = 0$*.*

Proof: The case $\sigma = 0$ is obvious. So without loss of generality, $\sigma > 0$ holds. Lemma 1.3 yields the cumulative distribution function of the minimum of a Brownian motion. We obtain the cumulative distribution function of the first-passage time τ with $x = 0$ and $b = -x_0 = -\ln(v_0)$, where x_0 denotes the distance to default:

$$\mathbb{P}(\tau \leq h) = \Phi\left(\frac{-x_0 - \gamma h}{\sigma\sqrt{h}}\right) + \exp\left(-\frac{2\gamma x_0}{\sigma^2}\right)\Phi\left(\frac{-x_0 + \gamma h}{\sigma\sqrt{h}}\right),$$

where Φ represents the cumulative normal distribution. Hence,

$$\begin{aligned}
\mathrm{LDR}_\tau &= \lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(\tau \leq h) \\
&= \lim_{h\downarrow 0}\frac{1}{h}\left(\Phi\left(\frac{-x_0 - \gamma h}{\sigma\sqrt{h}}\right) + \exp\left(-\frac{2\gamma x_0}{\sigma^2}\right)\Phi\left(\frac{-x_0 + \gamma h}{\sigma\sqrt{h}}\right)\right) \\
&= \lim_{h\downarrow 0}\frac{x_0}{\sqrt{2\pi}\sigma h^{\frac{3}{2}}}\exp\left(-\frac{(x_0 + \gamma h)^2}{2\sigma^2 h}\right) \\
&= 0.
\end{aligned}$$

In the second to last equation, we applied l'Hospital's rule. The derivative was calculated in Lemma 1.3. ♢

Now, we can handle the jump-diffusion setup. We were able to generalize a result from Scherer.[47] For absolutely continuous[48] jump-size distributions, the local default rate is determined by the Lévy measure of the logarithm of the firm-value process and the distance to default.

Theorem 2.2 (The local default rate of τ in a jump-diffusion model)
Let F_Y *denote the cumulative jump-size distribution function. The distance to default for* X *is given by* x_0*. We obtain*

$$LDR_\tau = \begin{cases} \lambda F_Y((-x_0)-) + \lambda \cdot \frac{1}{2}\mathbb{P}(Y = -x_0) & \text{if } \sigma > 0, \\ \lambda F_Y((-x_0)-) + \lambda \cdot \mathbf{1}_{\gamma \leq 0} \cdot \mathbb{P}(Y = -x_0) & \text{if } \sigma = 0. \end{cases}$$

If the jump-size distribution is absolutely continuous, this simplifies to

$$LDR_\tau = \lambda F_Y(-x_0) = \nu([-\infty; -x_0]),$$

[47] See [Sche], Theorem 4.1.

[48] That is, the distribution $\mathbb{P}_Y$ has a representation $\mathbb{P}_Y(dx) = g(x)dx$ for a measurable function $g : \mathbb{R} \to \mathbb{R}$. This also implies that Y's cumulative distribution function F_Y satisfies $F_Y(x) = F_Y(x-)$ for all $x \in \mathbb{R}$.

where ν *denotes the Lévy measure of* X *.*

Proof: We condition on the number N_h of jumps which occurred up to time h and denote the first jump time by $\tau(h)$. We obtain

$$
\begin{aligned}
&\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(\tau\leq h)\\
&\quad= \lim_{h\downarrow 0}\frac{1}{h}\sum_{n=0}^{\infty}\mathbb{P}(N_h=n)\mathbb{P}\left(\inf_{0\leq s\leq h}X_s\leq -x_0\middle|N_h=n\right)\\
&\quad= \lim_{h\downarrow 0}\frac{e^{-\lambda h}}{h}\mathbb{P}\left(\inf_{0\leq s\leq h}(\gamma s+\sigma W_s)\leq -x_0\right)+\\
&\qquad \lim_{h\downarrow 0}\lambda e^{-\lambda h}\mathbb{P}\left(\inf_{0\leq s\leq h}\left(\gamma s+\sigma W_s+\mathbf{1}_{\{s\geq\tau(h)\}}Y_1\right)\leq -x_0\right)+\\
&\qquad \lim_{h\downarrow 0}\frac{1}{h}\sum_{n=2}^{\infty}\frac{e^{-\lambda h}(\lambda h)^n}{n!}\mathbb{P}\left(\inf_{0\leq s\leq h}\left(\gamma s+\sigma W_s+\sum_{j=1}^{N_s}Y_j\right)\leq -x_0\middle|N_h=n\right).
\end{aligned}
$$

Lemma 2.3 yields that the first limit is zero. Considering the last limit, the probability can be limited by one and a dominated convergence argument allows us to interchange limit and summation, establishing that this limit also equals zero. We now examine the second limit, the case of exactly one jump. Writing $B_s:=\gamma s+\sigma W_s$, $A_t(x):=\{\omega\in\Omega:\inf_{0\leq s<t}B_s(\omega)\leq x\}$ and $A_t^C(x):=\Omega\setminus A_t(x)$ for brevity, we obtain by conditioning

$$
\begin{aligned}
&\mathbb{P}\left(\inf_{0\leq s\leq h}\left(B_s+\mathbf{1}_{\{s\geq\tau(h)\}}Y_1\right)\leq -x_0\right)\\
&\quad= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{\left\{\inf_{0\leq s\leq h}\left(B_s+\mathbf{1}_{\{s\geq\tau(h)\}}Y_1\right)\leq -x_0\right\}}\middle|Y_1\right)\right)\\
&\quad= \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{A_{\tau(h)}(-x_0)}+\mathbf{1}_{A^C_{\tau(h)}(-x_0)\cap\left\{\inf_{\tau(h)\leq s\leq h}B_s+Y_1\leq -x_0\right\}}\middle|Y_1\right)\right)\\
&\quad= \mathbb{P}\left(A_{\tau(h)}(-x_0)\right)+\\
&\qquad \mathbb{E}\left(\mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{A^C_{\tau(h)}(-x_0)\cap\left\{\inf_{\tau(h)\leq s\leq h}B_s+Y_1\leq -x_0\right\}}\middle|B_{\tau(h)},Y_1\right)\middle|Y_1\right)\right)\\
&\quad= \mathbb{P}\left(A_{\tau(h)}(-x_0)\right)+\\
&\qquad \mathbb{E}\left(\mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{A^C_{\tau(h)}(-x_0)\cap\tilde{A}_{h-\tau(h)}(-x_0-Y_1-B_{\tau(h)})}\middle|B_{\tau(h)},Y_1\right)\middle|Y_1\right)\right),
\end{aligned}
$$

where $\tilde{A}_t(x)$ is defined as $A_t(x)$ with B being replaced by the independent Brownian motion $\tilde{B}_s:=B_{\tau(h)+s}-B_{\tau(h)}$. Since $\tau(h)\leq h$ holds, the limit of the first term tends to zero with h, as in the pure diffusion setup. Lemma 2.3 also yields for the second term: If $Y_1>-x_0$, then the conditional expectation tends to zero, since so does $\mathbb{P}(B_{\tau(h)} \leq -x_0 - y)$ for all $y > -x_0$ for h going to zero, due to the continuity of the diffusion part. If $Y_1 < -x_0$, then the conditional expectation tends to one, since so does $\mathbb{P}(B_{\tau(h)} \leq -x_0 - y)$ for all $y < -x_0$ for h going to zero and since $\mathbb{P}(\tilde{A}_t(x)) = 1$ for all $t, x \in \mathbb{R}^+$. If $Y_1 = -x_0$, then the conditional expectation tends to zero if $B_{\tau(h)} > 0$ and to one if $B_{\tau(h)} \leq 0$ with h. Thus, we obtain for $\sigma > 0$,

$$
\begin{aligned}
\lim_{h \downarrow 0} \mathbb{P} & \left(\inf_{0 \leq s \leq h} \left(B_s + \mathbf{1}_{\{s \geq \tau(h)\}} Y_1 \right) \leq -x_0 \right) \\
&= 0 + F_Y((-x_0)-) + \mathbb{P}(Y_1 = -x_0) \cdot \lim_{h \downarrow 0} \mathbb{E} \left(\mathbb{E} \left(\mathbf{1}_{\{B_{\tau(h)} \leq 0\}} \middle| \tau(h) \right) \right) \\
&= F_Y((-x_0)-) + \mathbb{P}(Y_1 = -x_0) \cdot \lim_{h \downarrow 0} \mathbb{E} \left(\Phi \left(-\frac{\gamma \sqrt{\tau(h)}}{\sigma} \right) \right) \\
&= F_Y((-x_0)-) + \frac{1}{2} \mathbb{P}(Y_1 = -x_0).
\end{aligned}
$$

For $\sigma = 0$, the corresponding statements can be shown equivalently. $\diamondsuit$

The result can be interpreted as follows: If a negative jump exceeds the distance to default with positive probability, that is, $F_Y((-x_0)-) > 0$, then the local default rate is positive. Based on this local default rate, we are now able to calculate the exact limit of credit spreads as maturity decreases to zero.

Theorem 2.3 (Credit spreads at time zero)
We assume a constant recovery rate, that is, we choose $w_1^R(x) \equiv R$. Then, the limit of credit spreads at time zero is given by

$$\lim_{h \downarrow 0} \eta_h = (1 - R) LDR_\tau.$$

Proof: The theorem does not specify the recovery-payoff timing. First, we calculate the credit spread of a zero-coupon bond $\tilde{\phi}$ with maturity h whose payoff in case of a default takes place at maturity h. We obtain from Equation (2.6) in Lemma 2.2:

$$\tilde{\phi}(0, h) \quad = \quad e^{-rh} \left(\mathbb{P}(\tau > h) + R \cdot \mathbb{P}(\tau \leq h) \right).$$

By the definition of credit spreads,

$$
\begin{aligned}
\lim_{h \downarrow 0} \tilde{\eta}_h &= \lim_{h \downarrow 0} -\frac{1}{h} \ln \left(\tilde{\phi}(0, h) \right) - r \\
&= \lim_{h \downarrow 0} -\frac{1}{h} \ln \left(\mathbb{P}(\tau > h) + R \cdot \mathbb{P}(\tau \leq h) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\lim_{h\downarrow 0} \frac{\ln\left(\mathbb{P}(\tau > h) + R\cdot\mathbb{P}(\tau \le h)\right) - \ln\left(\mathbb{P}(\tau > 0) + R\cdot\mathbb{P}(\tau \le 0)\right)}{h} \\
&= -\frac{\delta}{\delta h} \ln\left(\mathbb{P}(\tau > h) + R\cdot\mathbb{P}(\tau \le h)\right)\Big|_{h=0} \\
&= \frac{\frac{\delta}{\delta h}\mathbb{P}(\tau \le h) - R\cdot\frac{\delta}{\delta h}\mathbb{P}(\tau \le h)}{1 - \mathbb{P}(\tau \le h) + R\cdot\mathbb{P}(\tau \le h)}\Bigg|_{h=0} \\
&= \mathrm{LDR}_\tau - R\cdot\mathrm{LDR}_\tau \\
&= \mathrm{LDR}_\tau(1-R).
\end{aligned}
$$

Hence, a bond $\tilde{\phi}$ with recovery payoff at maturity satisfies the statement. Let us now look at a bond ϕ whose payout in case of a default takes place at time of default. We can give a lower bound for its payout $R\cdot e^{-r\tau}$ by $R\cdot e^{-rh}$ and an upper bound by R. The lower bound is represented by $\tilde{\phi}$, the upper bound is represented by a bond $\hat{\phi}$ with value

$$\hat{\phi}(0,h) \quad = \quad e^{-rh}\cdot\mathbb{P}(\tau > h) + 1\cdot R\cdot\mathbb{P}(\tau \le h).$$

Calculating the limit of credit spreads at time zero as for $\tilde{\phi}$, we again obtain

$$\lim_{h\downarrow 0} \hat{\eta}_h = \mathrm{LDR}_\tau(1-R).$$

Credit spreads are monotone decreasing functions of the bond value, that is,

$$\lim_{h\downarrow 0} \hat{\eta}_h \le \lim_{h\downarrow 0} \eta_h \le \lim_{h\downarrow 0} \tilde{\eta}_h.$$

Hence, the limit of credit spreads at time zero for ϕ is given by

$$\lim_{h\downarrow 0} \eta_h = \mathrm{LDR}_\tau(1-R).$$

This completes the proof. $\diamondsuit$

Remark 2.3 (Interpretation of Theorem 2.3)
We found that the limit of credit spreads is the product of the local default rate and the fractional loss at time of default. This is economically reasonable, as a potential loss is decreasing in the recovery rate, resulting in a smaller credit spread. Moreover, the local default rate approximates the probability of a default within a small time interval. Therefore, credit spreads of bonds with a small maturity merely depend on the probability of a sudden default, in other words, credit spreads are increasing in the local default rate.

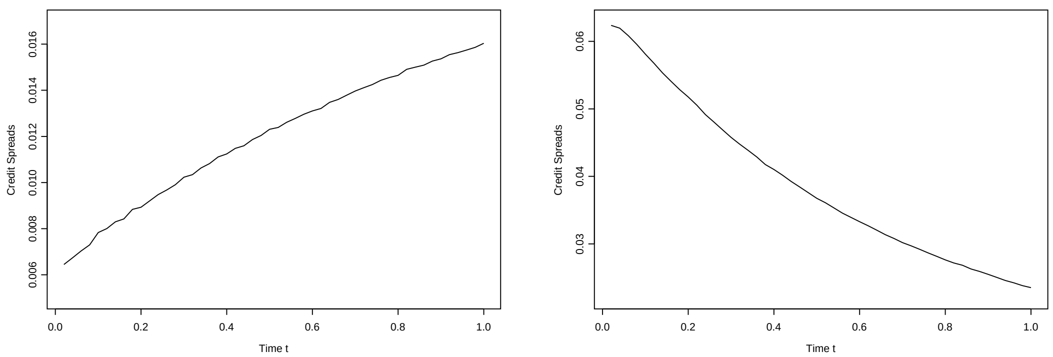

Figure 2.3: The credit spreads of 2×50 corporate bonds $\phi^i(0, t_j)$, $i = 1, 2$, $j = 1, \ldots, 50$ with maturities $t_j = j/50$ issued by two companies identified with the firm-value processes V^1 and V^2 with two-sided exponentially distributed jump sizes and parameters $\gamma^1 = 0.025$, $\gamma^2 = 0.2$, $\sigma^1 = \sigma^2 = 0.05$, $\lambda^1 = \lambda^2 = 2$, $p^1 = p^2 = 1/2$, $\lambda^1_\oplus = \lambda^2_\oplus = \lambda^1_\ominus = \lambda^2_\ominus = 20$, and $v^1_0 = 1/0.8$ and $v^2_0 = 1/0.9$, respectively. The interest rates are chosen to be constant $r = 0.02$, and so is the recovery-rate function $w^{0.5}_1(x) \equiv 0.5$. The credit spreads were calculated by the later introduced Algorithm 3.2 with ten million simulations per bond. As we can see, as maturity goes to zero both credit spreads tend to $(1 - R)LDR^i_\tau(0) = 1/2 \cdot \lambda^i p^i \cdot \exp(-\lambda^i_\ominus \cdot \ln(v^i_0))$, which is approximately 0.00576 for company V^1 and 0.06079 for company V^2, respectively. The reason for the higher credit spread of the second company close to zero is the fact that it is closer to default at time zero, since $v^2_0 < v^1_0$. The different shapes are caused by the different life cycles of V^1 and V^2. So, V^2 represents a company which is close to default at time zero but has high growth perspectives represented by a high γ^2. Its credit spread therefore narrows down. In contrast, V^1 is a company which has a lower leverage at time zero but also a lower expected growth. This fits exactly to empirical observations, see for instance [Fo].

Chapter 3

Bond pricing

As briefly explained in Lemma 2.1, we can represent every coupon bond with predefined payment schedule as a weighted sum of zero-coupon bonds with appropriate maturities. Thus, we restrict our focus to the valuation of zero-coupon bonds. In the first three sections, we introduce methods to calculate their prices, namely Zhou's algorithm, the Brownian-bridge pricing technique, and the Laplace-transform approach. All three methods are designed to evaluate the pricing formula within the jump-diffusion setting. Further, we discuss differences and biases of the three algorithms in the fourth section.

3.1 Zhou's algorithm

Zhou suggests in [Z2] a simple Monte-Carlo simulation to obtain bond prices. His idea is to discretize the time interval $[0;T]$ into a fine grid and then to sample trajectories of the firm-value process on this grid. Default can only occur at the grid points. Since there is always a small probability of a default occurring strictly between the grid points, this algorithm underestimates the risk of default. Hence, the calculated bond prices are biased, as we will see. In this section, we assume that the jump sizes can be simulated numerically and that the recovery-rate function w is continuous.

The following theorem, which appears in [Z2] as Theorem 1, serves as the theoretical justification for Zhou's algorithm, which we present afterwards. Zhou only presents a short outline of the proof. We slightly modify the theorem by setting the recovery payoff to the time of default, which we assume to be at the middle of the time interval before default is observed. Then, we prove the statement for all jump-size distributions. In Zhou's original paper, the recovery payoff takes place at maturity. The proof for this setup can be easily derived from our proof.

Theorem 3.1 (Price of a zero-coupon bond)
The price of a zero-coupon bond with maturity T, $\phi(0,T)$, can be expressed as

$$\phi(0,T) = \exp(-rT)\mathbb{P}(\tau > T) + \lim_{n\to\infty}\sum_{i=1}^{n}\exp\left(-rt_{i-\frac{1}{2}}\right)\mathbb{E}\left(w\left(V_{t_i}^*\right)|\Omega_i^*\right)\mathbb{P}(\Omega_i^*),$$

where for $i = 1,\dots,n$

$$\begin{aligned} t_i &:= \frac{i}{n}T, \quad t_{i-\frac{1}{2}} = \frac{i-\frac{1}{2}}{n}T, \quad \textit{and} \\ \Omega_i^* &:= \left\{V_{t_i}^* \le 1 \ \textit{and}\ V_{t_j}^* > 1,\, \forall j < i\right\}. \end{aligned}$$

Moreover, $V_{t_i}^*$ *is defined recursively as*

$$\begin{aligned} V_{t_0}^* &:= v_0, \\ \ln\left(V_{t_i}^*\right) &:= \ln\left(V_{t_{i-1}}^*\right) + x_i + \pi_i\cdot y_i, \quad i = 1,\dots,n. \end{aligned}$$

Here, x_i, y_i, *and* π_i *are mutually and serially independent random variables drawn from*

$$\begin{aligned} x_i &\sim \mathcal{N}\left(\gamma\cdot\frac{T}{n}, \sigma^2\cdot\frac{T}{n}\right), \\ y_i &\sim \mathbb{P}_Y, \quad \textit{and} \\ \pi_i &= \begin{cases} 0 & \textit{with prob. } 1-\lambda\cdot T/n, \\ 1 & \textit{with prob. } \lambda\cdot T/n. \end{cases} \end{aligned}$$

Proof: In the proof, we add the letter n to the index of the variables to emphasize the dependence of the sets on n. For the case that a set A has probability zero we define for a random variable Z: $\mathbb{E}(Z|A) := 0$. Since the recovery-rate function w is continuous, it has a maximum on the relevant interval $[0;1]$ and we denote $w_{\max} = \max\{w(x) : x \in [0;1]\}$.

Furthermore, we use the notation $N_{i,n} := N_{t_i} - N_{t_{i-1}}$. Since N is a Poisson process, its increments $\{N_{i,n}\}_i$ are mutually independent and exponentially distributed, as shown in Lemma 1.4. We denote by $\Omega_{i,n} := \{\omega \in \Omega : t_{i-1} < \tau(\omega) \le t_i\}$ the set of all states $\omega \in \Omega$ such that the company defaults in $(t_{i-1}; t_i]$. We define $A_n := \{\omega \in \Omega : \exists i \in \{1, 2, \dots, n\}$ with $N_{i,n}(\omega) \ge 2\}$ as the set of the states $\omega \in \Omega$ such that the process $X_t = \ln(V_t/v_0)$ has at least two jumps in one interval $(t_{i-1}; t_i]$, $i \leq n$. We denote by $A_n^C = \Omega \setminus A_n$ the complement of A_n. For A_n,

$$\mathbb{P}(A_n) \leq n \cdot c\frac{1}{n^2} = c\frac{1}{n} \tag{3.1}$$

holds for a constant $c \in \mathbb{R}_0^+$.

From Equation (2.5) in Lemma 2.2 we conclude

$$\begin{aligned} \phi(0,T) &= e^{-rT}\mathbb{P}(\tau > T) + \\ & \lim_{n\to\infty}\sum_{i=1}^{n} \mathbb{E}(e^{-r\tau}w(V_\tau)|\Omega_{i,n}\cap A_n^C)\mathbb{P}(\Omega_{i,n}\cap A_n^C) + \\ & \lim_{n\to\infty}\sum_{i=1}^{n} \mathbb{E}(e^{-r\tau}w(V_\tau)|\Omega_{i,n}\cap A_n)\mathbb{P}(\Omega_{i,n}\cap A_n). \end{aligned}$$

The third limit is zero, since

$$\begin{aligned} 0 &\leq \lim_{n\to\infty}\sum_{i=1}^{n} \mathbb{E}(e^{-r\tau}w(V_\tau)|\Omega_{i,n}\cap A_n)\mathbb{P}(\Omega_{i,n}\cap A_n) \\ &\leq w_{\max}\cdot \lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{P}(\Omega_{i,n}|A_n)\cdot\mathbb{P}(A_n) \\ &\leq w_{\max}\cdot c\cdot \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(\Omega_{i,n}|A_n) \\ &\leq w_{\max}\cdot c\cdot \lim_{n\to\infty}\frac{1}{n}\cdot 1 \\ &= 0. \end{aligned}$$

In the next step, we replace the stochastic discount factor by a deterministic one. We observe that on $\Omega_{i,n}$

$$\left|\exp(-r\tau(\omega)) - \exp\left(-rt_{i-\frac{1}{2}}\right)\right| \leq \exp\left(r\frac{T}{2n}\right) - 1$$

holds. Hence, with $\tilde{\Omega}_{i,n} := \Omega_{i,n}\cap A_n^C$,

$$\begin{aligned} &\left|\sum_{i=1}^{n}\mathbb{E}(\exp(-r\tau)w(V_\tau)|\tilde{\Omega}_{i,n})\mathbb{P}(\tilde{\Omega}_{i,n}) - \sum_{i=1}^{n}\mathbb{E}\left(\exp\left(-rt_{i-\frac{1}{2}}\right)w(V_\tau)\middle|\tilde{\Omega}_{i,n}\right)\mathbb{P}(\tilde{\Omega}_{i,n})\right| \\ &\quad\leq \sum_{i=1}^{n}\left(\exp\left(r\frac{T}{2n}\right)-1\right)w_{\max}\mathbb{P}(\Omega_{i,n}) \end{aligned}$$

$$\leq \left(\exp\left(r\frac{T}{n}\right) - 1\right) w_{\max} \longrightarrow 0 \quad (n \to \infty).$$

This shows that the limits of both sums are equal.

In the next step, we approximate $w(V_\tau)$ by $w(V_{t_i})$. We know with τ_i representing the time of the potential jump in $(t_{i-1};t_i]$ and w being uniformly continuous on the compact set $[0;1]$

$$\begin{aligned}
&\left|\mathbb{E}\left((w(V_\tau) - w(V_{t_i}))\,\mathbf{1}_{\tilde{\Omega}_{i,n}}\right)\right| \\
&\quad\leq \mathbb{E}\left(|w(V_\tau) - w(V_{t_i})| \cdot \mathbf{1}_{\tilde{\Omega}_{i,n}} \cdot \mathbf{1}_{\{N_{i,n}=1 \text{ and } \tau_i > \tau\}}\right) + \\
&\qquad \mathbb{E}\left(|w(V_\tau) - w(V_{t_i})| \cdot \mathbf{1}_{\tilde{\Omega}_{i,n}} \cdot \mathbf{1}_{\{N_{i,n}=0 \text{ or } \tau_i \leq \tau\}}\right) \\
&\quad\leq 2w_{\max} \cdot \mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{N_{i,n} = 1\} \cap \{\tau_i > \tau\}\right) + o(1) \cdot \mathbb{P}\left(\tilde{\Omega}_{i,n}\right) \\
&\quad\leq 2w_{\max} \cdot \mathbb{P}(N_{i,n} = 1) \cdot \mathbb{P}\left(\tilde{\Omega}_{i,n}\right) + o(1) \cdot \mathbb{P}\left(\tilde{\Omega}_{i,n}\right) \\
&\quad= o(1) \cdot \mathbb{P}\left(\tilde{\Omega}_{i,n}\right), \qquad (3.2)
\end{aligned}$$

since the increments of a Lévy process are independent and $\mathbb{P}(N_{i,n} = 1)$ is of complexity $o(1/n)$ [49].

In the next step, we prove that

$$\mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{\omega \in \Omega : V_{t_i}(\omega) > 1\}\right) = o\left(\frac{1}{n^2}\right) \qquad (3.3)$$

by conditioning on the number of jumps in the interval $(t_{i-1};t_i]$:

$$\begin{aligned}
\mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{V_{t_i} > 1\}\right) &= \sum_{j=0}^{\infty} \mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{V_{t_i} > 1\} \middle| \{N_{i,n} = j\}\right) \cdot \mathbb{P}(N_i = j) \\
&\leq \mathbb{P}\left(\Omega_{i,n} | \{N_{i,n} = 0\}\right) + \\
&\quad \mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} > 1\} | \{N_{i,n} = 1\}\right) \cdot \frac{\lambda T}{n} + 0.
\end{aligned}$$

The third term is zero, since $\tilde{\Omega}_{i,n} \subset A_n^C$. As in Lemma 2.3 by applying l'Hospital rule a second time, the first term is of complexity $o(1/n^2)$. It remains to show

[49] We remind the reader that a function $f : \mathbb{N} \to \mathbb{R}$ is of complexity $o(g(n))$ if $f(n)/g(n)$ tends to zero as n goes to ∞. Throughout the proof we omit to show that every time $\sum_{i=1}^{n} o(1/n) = o(1)$ and $\sum_{i=1}^{k_n} o(1)\mathbb{P}(C_{i,n}) = o(1)$ hold, where $C_{1,n}, \ldots, C_{k_n,n}$ are, for every n, mutually disjoint subsets of Ω. This can be easily verified for all used o-notations, but it does not hold in general.

that $\mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_{i,n}} > 1\} \middle| \{N_{i,n} = 1\}\right)$ is of complexity $o(1/n)$. We denote the jump time by τ_i. Then we are able to bound the probability by examining whether the default occurs before or after the jump:

$$\begin{aligned}
&\mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} > 1\} | \{N_{i,n} = 1\}\right) \\
&\quad = \mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} > 1\} \cap \{\tau < \tau_i\} | \{N_{i,n} = 1\}\right) + \\
&\qquad \mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} > 1\} \cap \{\tau \geq \tau_i\} | \{N_{i,n} = 1\}\right) \\
&\quad \leq \mathbb{P}\left(\Omega_{i,n} \cap \{\tau < \tau_i\} | \{N_{i,n} = 1\}\right) + \\
&\qquad \mathbb{P}\left(\{V_{t_i} > 1\} \cap \{t_i \geq \tau \geq \tau_i\} | \{N_{i,n} = 1\}\right) \\
&\quad \leq \mathbb{P}\left(\left\{V_{t_{i-1}} > 1\right\} \cap \left\{\inf_{t_{i-1} \leq s < \tau_i} V_s \leq 1\right\} \middle| \{N_{\tau_i -} - N_{t_{i-1}} = 0\}\right) + \\
&\qquad \mathbb{P}\left(\{V_{t_i} > 1\} \cap \left\{\inf_{\tau_i \leq s < t_i} V_s \leq 1\right\} \middle| \{N_{t_i} - N_{\tau_i} = 0\}\right).
\end{aligned}$$

As in the case with no jumps, by conditioning on V_{τ_i} in the second term and using a similar formula for the maximum of a Brownian motion as for the minimum in Lemma 1.3, we again see that both terms are of complexity $o(1/n)$. Hence, so is $\mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} > 1\} | \{N_{i,n} = 1\}\right)$ and Equation (3.3) is proven.

We also need that

$$\mathbb{P}(\Omega_{i,n} | A_n^C) \cdot \mathbb{P}(A_n) = o\left(\frac{1}{n}\right). \tag{3.4}$$

Due to Equation (3.1) it suffices to show that $\mathbb{P}\left(\Omega_{i,n} | A_n^C\right)$ tends to zero as n goes to ∞:

$$\begin{aligned}
\mathbb{P}\left(\Omega_{i,n} | A_n^C\right) &= \mathbb{P}\left(\Omega_{i,n} \cap \{N_{i,n} = 0\} | A_n^C\right) + \mathbb{P}\left(\Omega_{i,n} \cap \{N_{i,n} = 1\} | A_n^C\right) \\
&= \mathbb{P}\left(\Omega_{i,n} | A_n^C \cap \{N_{i,n} = 0\}\right) \cdot \frac{\mathbb{P}\left(A_n^C \cap \{N_{i,n} = 0\}\right)}{\mathbb{P}\left(A_n^C\right)} + \\
&\quad \mathbb{P}\left(\Omega_{i,n} | A_n^C \cap \{N_{i,n} = 1\}\right) \cdot \frac{\mathbb{P}\left(A_n^C \cap \{N_{i,n} = 1\}\right)}{\mathbb{P}\left(A_n^C\right)} \\
&\leq \mathbb{P}\left(\Omega_{i,n} | A_n^C \cap \{N_{i,n} = 0\}\right) + \frac{\frac{\lambda T}{n}}{1 + \frac{\lambda T}{n}}.
\end{aligned}$$

The limit of the second term is obviously zero and so is the limit of the first term, which can be proven as in Lemma 2.3. Thus, Equation (3.4) holds.

Now,

$$\mathbb{P}(\tilde{\Omega}_{i,n}) = \mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{V_{t_i} \leq 1\}\right) + \mathbb{P}\left(\tilde{\Omega}_{i,n} \cap \{V_{t_i} > 1\}\right)$$

$$
\begin{aligned}
&\overset{(3.3)}{=} \quad \mathbb{P}\left(A_n^C \cap \Omega_{i,n} \cap \{V_{t_i} \leq 1\}\right) + o\left(\frac{1}{n}\right) \\
&= \quad \mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} \leq 1\} \middle| A_n^C\right) \cdot (1 - \mathbb{P}(A_n)) + o\left(\frac{1}{n}\right) \\
&\overset{(3.4)}{=} \quad \mathbb{P}\left(\Omega_{i,n} \cap \{V_{t_i} \leq 1\} \middle| A_n^C\right) + o\left(\frac{1}{n}\right) \\
&= \quad \mathbb{P}\left(\{V_{t_i} \leq 1\} \cap \{V_{t_j} > 1, \forall j < i\} \middle| A_n^C\right) + o\left(\frac{1}{n}\right), \qquad (3.5)
\end{aligned}
$$

where the last equation follows from the fact that

$$
\mathbb{P}\left(\{\tau \leq t_{i-1}\} \cap \{V_{t_j} > 1, \forall j < i\} | A_n^C\right) = n \cdot o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n}\right),
$$

which follows directly from Equation (3.3). Thus, by writing briefly $B_{i,n} := \{V_{t_i} \leq 1\} \cap \{V_{t_j} > 1, \forall j < i\}$,

$$
\begin{aligned}
\left| \mathbb{E}\left(w(V_{t_i}) \mathbf{1}_{\tilde{\Omega}_{i,n}}\right) - \mathbb{E}\left(w(V_{t_i}) \mathbf{1}_{B_{i,n}} \middle| A_n^C\right) \right| \quad &\leq \quad w_{\max} \left| \mathbb{P}\left(\tilde{\Omega}_{i,n}\right) - \mathbb{P}\left(B_{i,n} \middle| A_n^C\right) \right| \\
&\overset{(3.5)}{=} \quad o\left(\frac{1}{n}\right). \qquad (3.6)
\end{aligned}
$$

We now replace V by V^*. We need

$$
\begin{aligned}
f_i \quad &: \quad \{0,1\}^i \to \mathbb{R}^+, \\
f_i(k_1, \ldots, k_i) \quad &= \quad \frac{\mathbb{P}\left(\{N_1 = k_1\} \cap \ldots \cap \{N_i = k_i\} | A_n^C\right)}{\mathbb{P}\left(\{\pi_1 = k_1\} \cap \ldots \cap \{\pi_i = k_i\}\right)} \\
&= \quad \frac{\left(\frac{\lambda \frac{T}{n}}{1+\lambda \frac{T}{n}}\right)^{\sum_{j=1}^i k_j} \left(\frac{1}{1+\lambda \frac{T}{n}}\right)^{i - \sum_{j=1}^i k_j}}{\left(\lambda \frac{T}{n}\right)^{\sum_{j=1}^i k_j} \left(1 - \lambda \frac{T}{n}\right)^{i - \sum_{j=1}^i k_j}} \\
&= \quad \left(1 - \lambda^2 \frac{T^2}{n^2}\right)^{-i} \left(1 - \lambda \frac{T}{n}\right)^{\sum_{j=1}^i k_j}.
\end{aligned}
$$

Let $K \in \mathbb{N}$ denote an arbitrary constant (independent of i). Then, $f_i(k_1, \ldots, k_i) = 1 + o(1)$ for $\sum_{j=1}^i k_j \leq K$ and we obtain

$$
\begin{aligned}
&\mathbb{E}\left(w(V_{t_i}) \mathbf{1}_{B_{i,n}} \mathbf{1}_{\{N_T \leq K\}} \middle| A_n^C\right) \\
&\quad = \sum_{\substack{k_1 \in \{0,1\}, \ldots, k_i \in \{0,1\} \\ \sum_{j=1}^i k_j \leq K}} \mathbb{E}\left(w(V_{t_i}) \mathbf{1}_{B_{i,n}} \middle| A_n^C \cap \{N_1 = k_1\} \cap \ldots \cap \{N_i = k_i\}\right) \cdot \\
&\qquad \mathbb{P}\left(\{N_1 = k_1\} \cap \ldots \cap \{N_i = k_i\} | A_n^C\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{k_1\in\{0,1\},\ldots,k_i\in\{0,1\}\\ \sum_{j=1}^i k_j\le K}} \mathbb{E}\left(w(V^*_{t_i})\mathbf{1}_{\Omega^*_{i,n}}\middle|\{\pi_1=k_1\}\cap\ldots\cap\{\pi_i=k_i\}\right)\cdot \\
&\qquad \mathbb{P}\left(\{\pi_1=k_1\}\cap\ldots\cap\{\pi_i=k_i\}\right)\cdot f_i(k_1,\ldots,k_i) \\
&= \sum_{\substack{k_1\in\{0,1\},\ldots,k_i\in\{0,1\}\\ \sum_{j=1}^i k_j\le K}} \mathbb{E}\left(w(V^*_{t_i})\mathbf{1}_{\Omega^*_{i,n}\cap\{\pi_1=k_1\}\cap\ldots\cap\{\pi_i=k_i\}}\right)(1+o(1)) \\
&= \mathbb{E}\left(w(V^*_{t_i})\mathbf{1}_{\Omega^*_{i,n}}\mathbf{1}_{\{\sum_{i+1}^n\pi_i\le K\}}\right)+o(1)\cdot\mathbb{P}(\Omega^*_{i,n}). \qquad (3.7)
\end{aligned}
$$

For the event that more than K jumps happen until time T, we obtain

$$
\begin{aligned}
g_1(K) &:= \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\mathbb{E}\left(w(V_{t_i})\mathbf{1}_{B_{i,n}}\mathbf{1}_{\{N_T>K\}}\middle| A_n^C\right) \\
&\le w_{\max}\mathbb{P}(N_T>K), \\
g_2(K) &:= \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\mathbb{E}\left(w(V^*_{t_i})\mathbf{1}_{\Omega^*_{i,n}}\mathbf{1}_{\{\sum_{i=1}^n\pi_i>K\}}\right) \\
&\le w_{\max}\lim_{n\to\infty}\mathbb{P}\left(\sum_{i=1}^n\pi_i>K\right) \\
&\le w_{\max}\lim_{n\to\infty}\begin{pmatrix} n \\ K\end{pmatrix}\left(\frac{\lambda T}{n}\right)^K \\
&\le \frac{w_{\max}(\lambda T)^K}{K!},
\end{aligned}
$$

and thus

$$
g_1(K)-g_2(K)\to 0 \quad (K\to\infty). \qquad (3.8)
$$

Summarized, we obtain

$$
\begin{aligned}
&\Phi(0,T) \\
&= \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\mathbb{E}\left(w(V_\tau)\middle|\tilde{\Omega}_{i,n}\right)\mathbb{P}\left(\tilde{\Omega}_{i,n}\right) \\
&= \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\mathbb{E}\left(w(V_\tau)\mathbf{1}_{\tilde{\Omega}_{i,n}}\right) \\
&\overset{(3.2)}{=} \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\left(\mathbb{E}\left(w(V_{t_i})\mathbf{1}_{\tilde{\Omega}_{i,n}}\right)+o(1)\cdot\mathbb{P}\left(\tilde{\Omega}_{i,n}\right)\right) \\
&\overset{(3.6)}{=} \lim_{n\to\infty}\sum_{i=1}^n \exp\left(-rt_{i-\frac12}\right)\left(\mathbb{E}\left(w(V_{t_i})\mathbf{1}_{B_{i,n}}\middle| A_n^C\right)+o\left(\frac1n\right)\right)+\lim_{n\to\infty}o(1)\cdot 1
\end{aligned}
$$

$$\overset{(3.7)}{=} \lim_{n\to\infty} \sum_{i=1}^{n} \exp\left(-rt_{i-\frac{1}{2}}\right) \left(\mathbb{E}\left(w(V^*_{t_i}) \mathbf{1}_{\Omega^*_{i,n}} \right) + o(1) \cdot \mathbb{P}(\Omega^*_{i,n}) \right) + g_1(K) - g_2(K)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \exp\left(-rt_{i-\frac{1}{2}}\right) \mathbb{E}\left(w\left(V^*_{t_i}\right) |\Omega^*_{i,n}\right) \mathbb{P}(\Omega^*_{i,n}) + g_1(K) - g_2(K).$$

Letting K go to ∞ and applying Equation (3.8) completes the proof of Theorem 3.1. $\diamondsuit$

Based on the last theorem, we now introduce Zhou's algorithm. As model specification, we choose an arbitrary jump-diffusion process with jump sizes Y which can be simulated numerically, a constant interest rate r, a strictly positive maturity $T > 0$, recovery payoff at time of default[50], and an arbitrary but continuous recovery-rate function $w: [0;1] \to \mathbb{R}_0^+$.

Algorithm 3.1 (Zhou's algorithm)
Choose the number of simulation runs K and the fineness[51] of the grid N and approximate $\phi(0,T)$ by

$$\phi(0,T) \approx \frac{1}{K} \sum_{j=1}^{K} \phi_j^N(0,T),$$

where each $\phi_j^N(0,T)$ is calculated by these steps:

1. *Denote $t_i \equiv T \cdot i/N$ for all $i \in \{0,1,\dots,N\}$.*
2. *Generate a series of mutually and serially independent random vectors (x_i, π_i, y_i) for $i \in \{1,2,\dots,N\}$ with*

$$\begin{aligned} x_i &\sim \mathcal{N}\left(\gamma \cdot \frac{T}{N}, \sigma^2 \cdot \frac{T}{N}\right), \\ y_i &\sim \mathbb{P}_Y, \quad \text{and} \\ \pi_i &= \begin{cases} 0 & \text{with prob. } 1 - \lambda \cdot T/N, \\ 1 & \text{with prob. } \lambda \cdot T/N. \end{cases} \end{aligned}$$

3. *Let $V^*_{t_0} = v_0$ and calculate $V^*_{t_i}$ according to the formula*

$$V^*_{t_i} = V^*_{t_{i-1}} \cdot \exp(x_i + \pi_i \cdot y_i), \quad i \in \{1,2,\dots,N\}.$$

[50] An algorithm for recovery payoff at maturity can be obtained by a slight change.
[51] This is here the number of intervals.

4. Find the smallest integer $i \leq N$ *such that* $V^*_{t_i} \leq 1$ *. If such an* i *exists, let*

$$\phi_j^N(0,T) = \exp\left(-r\left(t_i - \frac{1}{2N}\right)\right) \cdot w(V^*_{t_i}).$$

If such an i *does not exist, set*

$$\phi_j^N(0,T) = \exp(-rT).$$

This algorithm[52] is simple but has serious drawbacks. It returns biased bond prices, since it underestimates the risk of default when assuming that default can only occur at the times t_i, $i = 1, \dots, n$. Also, increasing the number of intervals in order to obtain a smaller bias results in a long running time. In Section 3.4, we present some numerical results to substantiate these statements.

Remark 3.1 (Sums of zero bonds)
To price more complex coupon bonds with payouts at discrete time points we replicate them with a weighted sum of zero-coupon bonds with appropriate maturities and price them separately, as shown in Lemma 2.1. However, we now work with pathwise simulations and it is not self-evident that this approach does not produce a systematic bias. To check this concern, we simulated the price of a coupon bond directly and compared it to the sum of the prices of the corresponding zero-coupon bonds. The results, which are listed in the appendix, Result B.1, show that the error caused by the transition to zero-coupon bonds is negligible and that no systematic bias is generated.

3.2 Brownian-bridge pricing technique

Metwally and Atiya, [MA] presented in 2002 another algorithm, which is again based on a Monte-Carlo simulation. This algorithm is designed for pricing barrier options in a jump-diffusion model. In contrast to Zhou's algorithm, it not only produces unbiased results but also is significantly faster than Zhou's algorithm. The principal idea is to condition on the number of jumps, the jump times, and the values of the jump-diffusion process at these times. Scherer, [Sche] adapted the algorithm to calculate bond prices with constant recovery rates. By a slight modification, we generalize this algorithm to include stochastic recovery rates, depending on the value of the company at the time of default. Besides, we

[52] See the appendix, Remark B.1 for additional information about its implementation.

significantly improve an approximation of the corresponding integral suggested in [MA] in terms of precision.

If $\sigma = 0$ holds, then there exists no random influence after having conditioned on the number of jumps, the jump times, and the jump sizes. Having all this information about the jump structure, calculating the non-random default time is trivial. For simplicity, we assume throughout this section that there exists a real diffusion component, that is, $\sigma > 0$.

An essential stochastic ingredient of this algorithm is the Brownian bridge:

Definition 3.1 (Brownian bridge)
A Brownian bridge over $[t_0; t_1]$ with volatility σ, pinned at $x \in \mathbb{R}$ and $y \in \mathbb{R}$, is an almost surely continuous stochastic process $W^{BB(x,y,t_0,t_1,\sigma)}$ whose finite-dimensional distributions are given by

$$\begin{aligned}&\mathbb{P}_{W_{s_1}^{BB(x,y,t_0,t_1,\sigma)},\dots,W_{s_n}^{BB(x,y,t_0,t_1,\sigma)}}(d(x_1,\dots,x_n))\\ &\quad = \prod_{i=1}^{n} p((s_i - s_{i-1})\sigma^2, x_{i-1}, x_i) \cdot \frac{p((t_1 - s_n)\sigma^2, x_n, y)}{p((t_1 - t_0)\sigma^2, x, y)} d(x_1, \dots x_n),\end{aligned}$$

where $t_0 = s_0 < \cdots < s_n < t_1$, $x_0 = x$, $(x_1, \dots, x_n) \in \mathbb{R}^n$, and

$$p(t, a, b) := \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-(a-b)^2}{2t}\right).$$

Lemma 3.1 (Spanned diffusion process is Brownian bridge)
Let X denote a jump-diffusion process without jumps in $[t_0; t_1]$. Then Z, defined by

$$Z_t \;:=\; I\!E(X_t | X_{t_0} = x, X_{t_1} = y, N_{t_1} - N_{t_0} = 0), \quad t \in [t_0; t_1], \tag{3.9}$$

is a Brownian bridge over $[t_0; t_1]$ with volatility σ, pinned at x and y, where σ is the volatility of the diffusion component of X. Vice versa, for every Brownian bridge Z there is a Brownian motion X, such that Equation (3.9) is satisfied.

Proof: We refer to [KS], Section 5 B for the statement if X does not have a drift. If X has a drift γ then $Z_t - \gamma t$ is the Brownian bridge corresponding to the drift-free process $X_t - \gamma t$. It is easily verified that

$$\mathbb{P}_{Z_{s_1} - \gamma s_1, \dots, Z_{s_n} - \gamma s_n}(d(x_1, \dots, x_n)) \;=\; \mathbb{P}_{Z_{s_1}, \dots, Z_{s_n}}(d(x_1 + \gamma s_1, \dots, x_n + \gamma s_n))$$

$$= \mathbb{P}_{Z_{s_1},\ldots,Z_{s_n}}(d(x_1,\ldots,x_n)).$$

Thus, $Z = \{(Z_t - \gamma t) + \gamma t\}_{t\in[t_0;t_1]}$ is a Brownian bridge, since adding a drift does not change the continuity of the paths. The second statement is obvious. $\diamondsuit$

Remark 3.2 (Diffusion component)
As we have just seen, it is irrelevant for a Brownian bridge of a spanned diffusion process X whether X has a drift component or not. Thus, in all calculations with a Brownian motion pinned at the endpoints, we can assume without loss of generality that its drift is zero.

For the Brownian-bridge pricing technique, we need the probability-density function of the minimum of a Brownian bridge.[53] It is given in the next lemma.

Lemma 3.2 (Probability-density function of τ in Brownian bridge)
Let X denote a Brownian bridge over $[t_0;t_1]$ with volatility σ, pinned at X_{t_0} and X_{t_1}. Let $b \in \mathbb{R}$ denote an arbitrary barrier. Then, we obtain[54] for

$C_t := \{\omega \in \Omega : \{X_s(\omega)\}_{t_0\le s\le t_1}$ *passes* b *for the first time in* $[t;t{+}dt]\}$, $t \in (t_0;t_1]$,

$$\begin{aligned} g(t)dt &:= \mathbb{P}(C_t|X_{t_0},X_{t_1}) \\ &= \mathbf{1}_{\{X_{t_0}>b\}} \frac{X_{t_0}-b}{2y\pi\sigma^2(t-t_0)^{3/2}(t_1-t)^{1/2}} \cdot \\ &\quad \exp\left(-\frac{(X_{t_1}-b)^2}{2(t_1-t)\sigma^2} - \frac{(X_{t_0}-b)^2}{2(t-t_0)\sigma^2}\right) dt, \end{aligned} \tag{3.10}$$

where

$$y = \frac{1}{\sqrt{2\pi\sigma^2(t_1-t_0)}} \exp\left(-\frac{(X_{t_1}-X_{t_0})^2}{2\sigma^2(t_1-t_0)}\right).$$

Proof: See [MA], Section II. $\diamondsuit$

We can calculate the cumulative distribution function corresponding to g at t_1:

Lemma 3.3 (Minimum of a Brownian bridge)
The probability of a Brownian bridge X over $[t_0;t_1]$ with volatility σ, pinned at X_{t_0} and X_{t_1}, falling below a barrier $b \in \mathbb{R}$ is given by

$$\tilde{\Phi}_b^{BB}(X_{t_0},X_{t_1},t_1-t_0) := \mathbb{P}\left(\min_{t_0\le s\le t_1} X_s \le b\right)$$

[53] Compare to Lemma 1.3, where the probability-density function of the minimum of a Brownian motion was calculated.

[54] In [MA] and [Sche], the same probability-density function is given but not simplified.

$$
\begin{aligned}
&= \mathbf{1}_{\{X_{t_0}\le b \text{ or } X_{t_1}\le b\}} + \\
&\quad \mathbf{1}_{\{X_{t_0}> b \text{ and } X_{t_1}> b\}} \exp\left(-\frac{2(X_{t_0}-b)(X_{t_1}-b)}{(t_1-t_0)\sigma^2}\right).
\end{aligned}
$$

Proof: The statement directly follows by integrating g in Equation (3.10). One possible way to integrate is to calculate the Laplace transform of the integral and invert this transform, as done in Lemma 3.4.[55] ◇

The Brownian-bridge pricing technique works as follows: First, we generate the number of jumps and the jump times. Then, we generate the value of X immediately before and after each jump. More precisely, if $(\tau_1, \tau_2, \ldots)$ denotes the sequence of jump times, the value of X immediately before the first jump is a sample drawn from a Gaussian distribution with mean $\gamma\tau_1$ and variance $\sigma^2\tau_1$. The value at the first jump time is obtained by adding a realization of the jump-size distribution to this number. For the value immediately before the second jump time, we add another sample drawn from a Gaussian distribution with mean $\gamma(\tau_2-\tau_1)$ and variance $\sigma^2(\tau_2-\tau_1)$, and so on. We can now check whether the company defaults at one of the jump times. The probability of the company not defaulting between two jumps is the probability of a Brownian bridge not crossing a certain barrier. This probability is given in Lemma 3.3.

We now present a theorem which justifies the algorithm which we introduce later.

Theorem 3.2 (Price of a zero-coupon bond)
The price of a zero-coupon bond with maturity and recovery payoff at time of default, $\phi(0,T)$, can be expressed as

$$
\begin{aligned}
\phi(0,T) &= I\!E\left(I\!E\left(\mathbf{1}_{\{\tau>T\}}e^{-rT} + w(V_\tau)\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}\,\middle|\,\mathcal{F}^*\right)\right) \qquad (3.11)\\
&= \sum_{k=0}^{\infty} \int\limits_{\substack{(\tau_1,\ldots,\tau_k)\\ \in[0;T]^k}} \int\limits_{\substack{(x_1,\ldots,x_{k+1})\\ \in(-\infty;\infty)^{k+1}}} \int\limits_{\substack{(y_1,\ldots,y_k)\\ \in(-\infty;\infty)^k}} I\!E\left(\mathbf{1}_{\{\tau>T\}}e^{-rT} + w(V_\tau)\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}\,\middle|\,\mathcal{F}^*\right)\cdot\\
&\quad \prod_{j=1}^{k} I\!P_Y(dy_j)\cdot\prod_{j=1}^{k+1}\varphi_{\gamma\Delta\tau_j,\sigma^2\Delta\tau_j}(x_j)dx_j\,\cdot\\
&\quad \mathbf{1}_{\{0<\tau_1<\cdots<\tau_k<T\}}\frac{k!}{T^k}d(\tau_1,\ldots,\tau_k)\cdot\frac{(\lambda T)^k}{k!}e^{-\lambda T},
\end{aligned}
$$

where

$$\mathcal{F}^* := \sigma\left\{N_T;\ 0<\tau_1<\cdots<\tau_{N_T}<T;\ X_{\tau_1-}, X_{\tau_1},\ldots,X_{\tau_i-},X_{\tau_i},\ldots,X_T\right\}$$

[55] By setting $r=0$ in Lemma 3.4, we realize that Lemma 3.3 is a special case of the approximation in Lemma 3.4.

is the σ -algebra representing the information from the number of jumps, their location and the values of X immediately before the jump times, at the jump times, and at maturity. The function $\varphi_{\gamma\Delta\tau_j,\sigma^2\Delta\tau_j}$ represents the probability-density function of the normal distribution with mean $\gamma(\tau_j - \tau_{j-1})$ and variance $\sigma^2(\tau_j - \tau_{j-1})$, where $\tau_0 = 0$ and $\tau_{N_T+1} = T$.

For $b = -\ln(v_0)$, the conditional expectation satisfies

$$
\begin{aligned}
& I\!E\left(\mathbf{1}_{\{\tau>T\}}e^{-rT} + w(V_\tau)\mathbf{1}_{\{\tau\le T\}}e^{-r\tau}\,\middle|\,\mathcal{F}^*\right) \\
&= w(1)\sum_{i=1}^{U}\prod_{j=1}^{i-1}\Phi_b^{BB}(j)\int_{\tau_{i-1}}^{\tau_i}e^{-rs}g_i(s)ds + \\
&\quad w(V_{\tau_I})\mathbf{1}_{\{I\neq 0\}}e^{-r\tau_I}\prod_{j=1}^{I}\Phi_b^{BB}(j) + \mathbf{1}_{\{I=0\}}e^{-rT}\prod_{j=1}^{N_T+1}\Phi_b^{BB}(j),
\end{aligned}
\tag{3.12}
$$

where

$$I := \min\{i \in \{1,\dots,N_T\} : X_{\tau_i} \le b\}, \quad \min\emptyset := 0,$$

denotes the index of the first jump time such that X_{τ_I} crosses the barrier,

$$U := \begin{cases} I & \text{if } I \neq 0, \\ N_T + 1 & \text{if } I = 0, \end{cases}$$

$\Phi_b^{BB}(j) := 1 - \tilde{\Phi}_b^{BB}(X_{\tau_{j-1}}, X_{\tau_j-}, \tau_j - \tau_{j-1})$ represents the probability of the company not defaulting within the interval $(\tau_{j-1};\tau_j)$, and $g_i(t)dt = I\!P(C_t | X_{\tau_{i-1}}, X_{\tau_i-})$ is defined as in Equation (3.10) as the probability of the company defaulting in $[t;t+dt]$ for $t \in (\tau_{i-1};\tau_i)$.

The price of a zero-coupon bond with recovery payoff at maturity, $\tilde{\phi}(0,T)$, is calculated on the same way after having replaced $e^{-r\tau}$ by e^{-rT} in all equations. The term $w(1)\sum_{i=1}^{U}\prod_{j=1}^{i-1}\Phi_b^{BB}(j)\int_{\tau_{i-1}}^{\tau_i}e^{-rs}g_i(s)ds$ in Equation (3.12) simplifies to $w(1)e^{-rT}\sum_{i=1}^{U}\prod_{j=1}^{i}\Phi_b^{BB}(j)$.

Proof: The price formula for a zero-coupon bond $\tilde{\phi}$ with recovery payoff at maturity is a direct consequence of the price formula for the more sophisticated bond ϕ with recovery payoff at time of default. Hence, it is sufficient to prove the statements for ϕ.

The first equality in Equation (3.11) follows directly from Equation (2.5) in Lemma 2.2 and the basic properties of the conditional expectation. The formulas for the probability distribution of the number of jumps and the probability-density function of the jump locations in the second equation are given in the

Lemmas 1.4 and 1.5. Furthermore, we use the fact that the jump sizes are independent from each other and all other appearing random variables and the increments of the Lévy process X are independent and Gaussian between the jump times.

It remains to prove Equation (3.12). If the company does not die from a jump of the underlying stochastic process, then $I = 0$ and $U = N_T + 1$ hold. However, it can die from the diffusion. In this case, the value of the company at time of default equals one and the recovery paid to the bond holder is $w(1)$. The probability of the company dying after the i^{th} and before the $(i+1)^{th}$ jump at time $t \in (\tau_i; \tau_{i+1})$ is the probability of the company not dying in the intervals before, that is $\prod_{j=1}^{i} \Phi_b^{BB}(j)$, times the probability of the firm-value process V first crossing the default bound b within $(\tau_i; \tau_{i+1})$ at time t which is described by $g_{i+1}(t)$. The probability of the firm-value process V dying from diffusion is given by $\prod_{j=1}^{N_T+1} \Phi_b^{BB}(j)$. Even if the case $I > 0$ occurs, default by diffusion can happen before time τ_I. The probability of this event is again given by the first term in Equation (3.12). Default by diffusion does not occur with probability $\prod_{j=1}^{I} \Phi_b^{BB}(j)$. In this case, the recovery is $w(V_{\tau_I})$. Combining the two cases $I = 0$ and $I > 0$ completes the proof. $\diamondsuit$

Based on this theorem, we now formally introduce the Brownian-bridge pricing algorithm. Similar to Zhou's algorithm, we choose as model specification an arbitrary jump-diffusion process with jump sizes Y which can be simulated numerically, a constant interest rate r, a strictly positive maturity $T > 0$, recovery payoff at time of default[56], and an arbitrary (not necessarily continuous) recovery-rate function $w: [0;1] \to \mathbb{R}_0^+$.

Algorithm 3.2 (Brownian-bridge pricing algorithm)
Choose the number of simulation runs K and approximate $\phi(0,T)$ by

$$\phi(0,T) \approx \frac{1}{K} \sum_{j=1}^{K} \phi_j(0,T),$$

where each $\phi_j(0,T)$ is calculated by these steps:

1. *Simulate the number of jumps N_T according to Equation (1.1).*
2. *Simulate the jump times $\tau_1 < \tau_2 < \cdots < \tau_{N_T}$ according to Equation (1.2) independent from N_T. Set $\tau_0 = 0$ and $\tau_{N_T+1} = T$.*

[56] Again, an algorithm for a recovery payoff at maturity can be obtained by a slight change.

3. *Generate two series of mutually independent random variables* $x_1, \ldots, x_{N_T+1}$ *and* $y_1, \ldots, y_{N_T}$ *, independent from* N_T *and* $\tau_1, \ldots, \tau_{N_T}$ *, with*

$$\begin{aligned} x_i &\sim \mathcal{N}\left(\gamma(\tau_i - \tau_{i-1}), \sigma^2(\tau_i - \tau_{i-1})\right) \quad \textit{and} \\ y_i &\sim \mathbb{P}_Y . \end{aligned}$$

4. *Calculate inductively* $X_0, X_{\tau_1 -}, X_{\tau_1}, X_{\tau_2 -}, \ldots, X_{\tau_{N_T}}, X_{\tau_{N_T+1}-} = X_{\tau_{N_T+1}}$ *by*

$$\begin{aligned} X_{\tau_0} &= 0, \\ X_{\tau_i -} &= X_{\tau_{i-1}} + x_i, \quad \forall\, i \in \{1, \ldots, N_T + 1\}, \\ X_{\tau_i} &= X_{\tau_i -} + y_i, \quad \forall\, i \in \{1, \ldots, N_T\}. \end{aligned}$$

5. *Determine* I *,* U *, and* b *as in Theorem 3.2.*

6. *Calculate* $\phi_j(0,T) = \mathbb{E}\left(\mathbf{1}_{\{\tau > T\}} e^{-rT} + w(V_\tau)\mathbf{1}_{\{\tau \le T\}} e^{-r\tau} \middle| \mathcal{F}^*\right)$ *as in Equation (3.12) of Theorem 3.2.*

The speed of this algorithm[57] depends strongly on the expected number of jumps, that is λT. The higher the jump intensity λ, the more samples have to be drawn and the more integrals have to be calculated. We illustrate this in Section 3.4, where we compare the speed of this algorithm for different parameter sets.

Remark 3.3 (Sums of zero bonds)
As for Zhou's algorithm, we also check for the Brownian-bridge pricing technique whether a systematic bias is caused due to pricing a coupon bond with payouts at discrete time points according to Lemma 2.1. The results are listed in the appendix, Result B.2. Again, they show that the error caused by the transition to zero-coupon bonds is negligible and that no bias is generated.

If we assume recovery payoff at time of default we have to calculate the integrals which appear in the conditional expectation in step 6 of Algorithm 3.2. This is the most time-consuming step. Metwally and Atiya, [MA] suggest an approximation of these integrals, which we introduce below. Important tools which are used are the Laplace transform of a function and its inverse. Their definitions are given in the appendix. The idea is to calculate the Laplace transform of such an integral (which can be represented as the convolution of two functions), to interpret this transform as a function of r, to expand it into a Taylor series, and

[57] See the appendix, Remark B.2 for additional information about its implementation.

to calculate the Laplace inverse of the second-order approximation. Our calculations yield a different result from that in the original paper [MA], as discussed in Remark 3.4. However, numerical experiments, which we illustrate in Figure 3.1, show that our approximation is closer to the correct value.

Lemma 3.4 (Approximation of the integral[58])
We assume that $X_{\tau_{i-1}} > b$. The integral in Equation (3.12) can be approximated by[59]

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s)ds = e^{-r\tau_{i-1}}\left(\exp\left(-\frac{2(X_{\tau_{i-1}}-b)(X_{\tau_i-}-b)}{\Delta\tau_i\sigma^2}\right) + \frac{r(X_{\tau_{i-1}}-b)}{4\sigma}(A_1+C_1B)\right) + O(r^3) \quad (3.13)$$

if $X_{\tau_i-} > b$ and by

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rs} g_i(s)ds = e^{-r\tau_{i-1}}\left(1 + \frac{r(X_{\tau_{i-1}}-b)}{4\sigma}(A_2+C_2B)\right) + O(r^3) \quad (3.14)$$

if $X_{\tau_i-} \leq b$, where

$$\begin{aligned}
\Delta\tau_i &= \tau_i - \tau_{i-1},\\
\Delta X_i &= X_{\tau_i-} - X_{\tau_{i-1}},\\
A_1 &= -\frac{r}{\sigma}\Delta\tau_i\Delta X_i \exp\left(-\frac{2(X_{\tau_{i-1}}-b)(X_{\tau_i-}-b)}{\Delta\tau_i\sigma^2}\right),\\
C_1 &= -\sqrt{2\pi\Delta\tau_i}\exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right)\Phi\left(\frac{2b-X_{\tau_i-}-X_{\tau_{i-1}}}{\sqrt{\Delta\tau_i\sigma^2}}\right),\\
B &= 4 - r\Delta\tau_i - \frac{r}{\sigma^2}\Delta X_i\left(X_{\tau_i-}+X_{\tau_{i-1}}-2b\right),\\
A_2 &= \frac{r}{\sigma}\Delta\tau_i\left(X_{\tau_i-}+X_{\tau_{i-1}}-2b\right), \quad \textit{and}\\
C_2 &= -\sqrt{2\pi\Delta\tau_i}\exp\left(\frac{(\Delta X_i)^2}{2\Delta\tau_i\sigma^2}\right)\Phi\left(\frac{\Delta X_i}{\sqrt{\Delta\tau_i\sigma^2}}\right)
\end{aligned}$$

with Φ denoting the cumulative normal distribution function.

Proof: A substitution and some calculations show that the integral can be written as the convolution of two functions. We find

$$\int_{\tau_{i-1}}^{\tau_i} e^{-rx} g_i(x)dx = e^{-r\tau_{i-1}}\int_0^{\Delta\tau_i} f(x)h(\Delta\tau_i - x)dx,$$

[58] See the appendix, Remark B.3 for additional information about its implementation.

[59] We remind the reader that a function $f:\mathbb{R}\to\mathbb{R}$ is of complexity $O(g(x))$, if $f(x)/g(x)$ is bounded.

where

$$\begin{aligned} f(x) &= \frac{e^{-rx}\Delta_0}{\sqrt{2\pi\sigma^2}} x^{-\frac{3}{2}} \exp\left(-\frac{\Delta_0^2}{2x\sigma^2}\right) \quad \text{and} \\ h(x) &= \frac{1}{\sqrt{2\pi\sigma^2 y}} x^{-\frac{1}{2}} \exp\left(-\frac{\Delta_1^2}{2x\sigma^2}\right) \end{aligned}$$

with y as in Lemma 3.2, $\Delta_0 := X_{\tau_{i-1}} - b$, and $\Delta_1 := X_{\tau_i -} - b$.

We now calculate the Laplace transform of the integral. Being a convolution, its Laplace transform is the product of the Laplace transforms of f and h. The Formulas 5.28 and 5.30 of [OB], Chapter 1.5, page 41 yield

$$\begin{aligned} \left(\mathcal{L}\left(ax^{-\frac{3}{2}}\exp\left(-\frac{a^2}{4x}\right)\right)\right)(s) &= 2\sqrt{\pi}\exp(-a\sqrt{s}), \quad a \in \mathbb{R}^+ \quad \text{and} \\ \left(\mathcal{L}\left(x^{-\frac{1}{2}}\exp\left(-\frac{a^2}{4x}\right)\right)\right)(s) &= \sqrt{\pi}\frac{\exp(-|a|\sqrt{s})}{\sqrt{s}}, \quad a \in \mathbb{R}. \end{aligned}$$

By these formulas, the shift theorem, and some simple calculations,

$$\begin{aligned} \left(\mathcal{L}\left(f\right)\right)(s) &= \exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right) \quad \text{and} \\ \left(\mathcal{L}\left(h\right)\right)(s) &= \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{2}\sigma\sqrt{s}y}. \end{aligned}$$

Thus, the explicit form of the Laplace transform is given by

$$\begin{aligned} l_r(s) &:= \left(\mathcal{L}\left(\int_0^t f(x)h(t-x)dx\right)\right)(s) \\ &= \left(\mathcal{L}(f)\right)(s) \cdot \left(\mathcal{L}(h)\right)(s) \\ &= \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sigma y\sqrt{2s}} \cdot \exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right). \end{aligned}$$

Using the representation of y in Lemma 3.2, that is,

$$y = \frac{1}{\sqrt{2\pi\sigma^2\Delta\tau_i}} \exp\left(-\frac{\Delta X_i^2}{2\sigma^2\Delta\tau_i}\right),$$

we obtain

$$\alpha := \frac{1}{\sqrt{2}\sigma y} = \sqrt{\pi\Delta\tau_i}\exp\left(\frac{(\Delta X_i)^2}{2\sigma^2\Delta\tau_i}\right).$$

We now interpret $l_r(s)$ as a function in r and expand it into a Taylor series. Therefore, we calculate its derivatives and obtain

$$\begin{aligned}
l_r^1(s) := \frac{\delta}{\delta r} l_r(s) &= -\frac{\Delta_0 \alpha}{\sqrt{2}\sigma} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right)}{\sqrt{s+r}}, \\
l_r^2(s) := \frac{\delta^2}{\delta^2 r} l_r(s) &= \frac{\Delta_0^2 \alpha}{2\sigma^2} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right)}{s+r} + \\
&\quad \frac{\Delta_0 \alpha}{2^{\frac{3}{2}}\sigma} \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right)}{(s+r)^{\frac{3}{2}}}, \\
l_r^3(s) := \frac{\delta^3}{\delta^3 r} l_r(s) &= \sum_{j=3}^{5} c_j \cdot \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\left(\sqrt{s+r}\right)\right)}{(s+r)^{\frac{j}{2}}}
\end{aligned} \tag{3.15}$$

for some constants c_3, c_4, and c_5. Hence, the second-order Taylor expansion of $l_r(s)$ around zero is given by

$$l_r(s) = l_0(s) + r \cdot l_0^1(s) + \frac{r^2}{2} l_0^2(s) + \frac{r^3}{6} l_{r_s^*}^3(s), \tag{3.16}$$

where $r_s^* \in (0; r)$ depends on s.

The inverse Laplace transform is linear, allowing us to examine each component of Equation (3.16) separately. We start with the last term. Simple calculations show that

$$\begin{aligned}
&\left| \left(\mathcal{L}^{-1} \left(\frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r_s^*}\right)}{(s+r_s^*)^{\frac{j}{2}}} \right) \right)(t) \right| \\
&\leq \left| \int_{y-i\infty}^{y+i\infty} \exp(st) \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r_s^*}\right)}{(s+r_s^*)^{\frac{j}{2}}} ds \right| \\
&\leq \exp(yt) \int_{y-i\infty}^{y+i\infty} \left| \frac{\exp\left(-\frac{\sqrt{2}|\Delta_1|}{\sigma}\sqrt{s}\right)}{\sqrt{s}} \cdot \frac{\exp\left(-\frac{\sqrt{2}\Delta_0}{\sigma}\sqrt{s+r_s^*}\right)}{(s+r_s^*)^{\frac{j}{2}}} \right| ds \\
&\leq \exp(yt) \int_{y-i\infty}^{y+i\infty} \frac{\exp\left(-\frac{|\Delta_1|}{\sigma}\sqrt{|s|}\right)}{\sqrt{|s|}} \cdot \frac{\exp\left(-\frac{\Delta_0}{\sigma}\sqrt{|s+r_s^*|}\right)}{(|s+r_s^*|)^{\frac{j}{2}}} ds \\
&\leq \exp(yt) \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{\Delta_0+|\Delta_1|}{\sigma}\sqrt{|x|}\right)}{|x|^{\frac{j+1}{2}}} dx
\end{aligned}$$

is uniformly (with respect to r_s^*) bounded for all $j \in \{3,4,5\}$. This proves that the error caused by truncating the Taylor-series expansion of the Laplace transform $l_r(s)$ after the second term is of size $O(r^3)$.

We use the Formulas 5.87, 5.89, and 5.92 of [OB], Chapter 2.5, pages 258 and 259 and Lemma A.6, which yield for $a \in \mathbb{R}^+$

$$
\begin{aligned}
\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{\sqrt{s}}\right)\right)(t) &= \frac{\exp(-\frac{a^2}{t})}{\sqrt{\pi t}}, \\
\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s}\right)\right)(t) &= 2\left(1-\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\right), \\
\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s^{\frac{3}{2}}}\right)\right)(t) &= \frac{2\sqrt{t}\exp(-\frac{a^2}{t})}{\sqrt{\pi}} - 4a\left(1-\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\right), \quad \text{and} \\
\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s^2}\right)\right)(t) &= -\frac{2a\sqrt{t}\exp(-\frac{a^2}{t})}{\sqrt{\pi}} + (2t+4a^2)\left(1-\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\right).
\end{aligned}
$$

In the next steps, we apply these equations with $a := (\Delta_0 + |\Delta_1|)/(\sqrt{2}\sigma)$ on the first three terms of Equation (3.16), denoting $\beta := 1 - \Phi(\sqrt{2}a/\sqrt{\Delta\tau_i})$.

$$
\begin{aligned}
l_0(s) &= \alpha \cdot \frac{\exp\left(-\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\sqrt{s}\right)}{\sqrt{s}}. \\
\left(\mathcal{L}^{-1}(l_0)\right)(\Delta\tau_i) &= \exp\left(\frac{(\Delta X_i)^2}{2\sigma^2\Delta\tau_i}\right) \cdot \exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right) \\
&= \begin{cases} 1, & X_{\tau_i-} \le b, \\ \exp\left(\frac{-2(X_{\tau_{i-1}}-b)(X_{\tau_i-}-b)}{\sigma^2\Delta\tau_i}\right), & X_{\tau_i-} > b. \end{cases} \\
l_0^1(s) &= -\frac{\Delta_0\alpha}{\sqrt{2}\sigma} \cdot \frac{\exp\left(-\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\sqrt{s}\right)}{s}. \\
\left(\mathcal{L}^{-1}(l_0^1)\right)(\Delta\tau_i) &= -\sqrt{2}\frac{\Delta_0}{\sigma} \cdot \alpha \cdot \beta. \\
l_0^2(s) &= \frac{\Delta_0^2\alpha}{2\sigma^2}\frac{\exp\left(-\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\sqrt{s}\right)}{s^{\frac{3}{2}}} + \frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma}\frac{\exp\left(-\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\sqrt{s}\right)}{s^2}. \\
\left(\mathcal{L}^{-1}(l_0^2)\right)(\Delta\tau_i) &= \frac{2\sqrt{\Delta\tau_i}\exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right)}{\sqrt{\pi}}\left(\frac{\Delta_0^2\alpha}{2\sigma^2} - \frac{\Delta_0+|\Delta_1|}{\sqrt{2}\sigma}\cdot\frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma}\right) + \\
&\quad 2\beta\left(\frac{\Delta_0\alpha}{2^{\frac{3}{2}}\sigma}\left(\Delta\tau_i + \frac{(\Delta_0+|\Delta_1|)^2}{\sigma^2}\right) - \frac{\Delta_0^2\alpha}{2\sigma^2}\frac{\sqrt{2}(\Delta_0+|\Delta_1|)}{\sigma}\right)
\end{aligned}
$$

$$= \frac{\sqrt{\Delta\tau_i}\exp\left(-\frac{(\Delta_0+|\Delta_1|)^2}{2\sigma^2\Delta\tau_i}\right)\alpha\Delta_0(\Delta_0-|\Delta_1|)}{2\sqrt{\pi}\sigma^2}+$$
$$\beta\frac{\Delta_0\alpha}{\sqrt{2}\sigma}\Delta\tau_i+\beta\frac{\Delta_0\alpha}{\sqrt{2}\sigma^3}(\Delta_1^2-\Delta_0^2).$$

By combining these equations with the representation of $l_r(s)$ in Equation (3.16) and using

$$\Phi(-x) = 1-\Phi(x),$$
$$\int_{\tau_{i-1}}^{\tau_i} e^{-rx}g_i(x)dx = e^{-r\tau_{i-1}}\left(\mathcal{L}^{-1}(l_r)\right)(\Delta\tau_i),$$

and the linearity of the inverse Laplace transform we obtain the approximation of the integral. ♢

Remark 3.4 (Comparison with result from [MA])
Our approximation of the integrals slightly differs from the approximation in [MA]. They obtain one factor $e^{-\Delta\tau_i}$ which we do not need and they evaluate Φ at a different position.[60] *Since the error term in Equation (3.15) is negative, we expect that the bond prices generated by both approximations are slightly larger than the real ones and that the corresponding credit spreads are slightly lower.*

We implemented both approximations and compared the quality of the approximation for different parameter sets and different interest rates. Our simulations show that our approximation implies a lower relative pricing error almost always[61]*. While the pricing error of the original algorithm increases significantly in r the pricing error of our modification remains small when the interest rate increases. Comparing the different scenarios, we find that a lower intensity implies a higher pricing error in the original approximation. Our formulas always create a bias as expected, namely bond prices which are slightly to high. In contrast, the original formulas create bond prices which are too low. These observations led us to believe that there is an error in the calculations of the original paper, [MA].*[62] *We strongly recommend to use our formulas instead of the original ones.*

[60] To be more precise: In [MA], the second term in the sum of Equations (3.13) and (3.14) is multiplied by $e^{-\Delta\tau_i}$ and Φ is evaluated at $(2b - X_{\tau_i-} - X_{\tau_{i-1}}/(\sqrt{2\Delta\tau_i\sigma^2})$ (resp. $(\Delta X_i)/(\sqrt{2\Delta\tau_i\sigma^2})$) in C_1 (resp. C_2).

[61] Except for the third scenario when r equals 2.5%.

[62] However, the credits for this excellent idea go to Metwally and Atiya who firstly introduced this algorithm and discussed it. We just want to stress that our formulas correct an error and help to reduce a bias significantly, especially when higher interest rates are used.

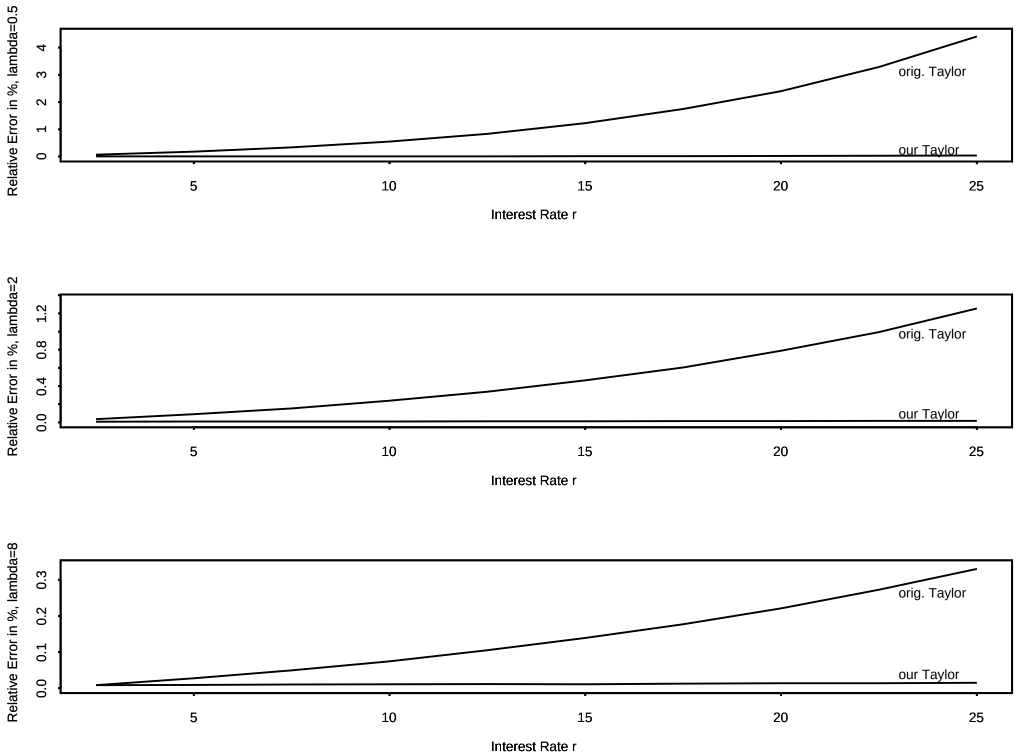

Figure 3.1: The relative pricing error of the original and our approximation in different scenarios.

Figure 3.1 shows the absolute values of the relative pricing error of both approximations in different scenarios. More precisely, it shows $|(p_u - p_a)/p_u|$ *, where* p_u *(resp.* p_a *) represents the unbiased (resp. approximated) bond price. We calculated the pricing error in three scenarios, assuming a drift of* $\gamma = 0.025$ *, volatility of diffusion* $\sigma = 0.05$ *, and two-sided exponentially distributed jumps. The first graph shows the scenario with a low jump intensity* $\lambda = 0.5$ *and parameters* $p = 0.5$ *,* $\lambda_\oplus = \lambda_\ominus = 10$ *, the second one shows the pricing error for* $\lambda = 2$ *,* $p = 0.5$ *, and* $\lambda_\oplus = \lambda_\ominus = 20$ *, and the third graph shows the relative pricing error when many small jumps are expected, that is,* $\lambda = 8$ *,* $p = 0.5$ *, and* $\lambda_\oplus = \lambda_\ominus = 40$ *.*[63] *The recovery rate was set constant to* 0.4 *at time of default. We chose* $T = 5$ *as maturity. For each scenario and each interest rate, we performed ten million simulations.*

For the scenario with a low number of large jumps expected, the relative pricing

[63]The scenarios were chosen so that the jump structure changes but the overall volatility stays constant, compare to Lemma 1.10.

error of the original Taylor approximation increases up to 4.41%, when the interest rate tends to 25%[64]*. In the scenario with an average number of two jumps per year, the pricing error increases to 1.25% and even in the scenario with a high number of jumps, the pricing error still exceeds 0.33%. In contrast, applying our formulas, the pricing error remains relatively small, namely at about 0.036%, 0.018%, and 0.015%, respectively. The exact numbers underlying this graph are given in the appendix, Result B.3.*

3.3 Laplace-transform approach

The Algorithms 3.1 and 3.2 take too long when used for calibrating the model parameters if the grid size and the number of expected jumps, respectively, is high. We briefly discuss an alternative method which is not based on a Monte-Carlo simulation and therefore much faster. A drawback of this method is that it only works for two-sided exponentially distributed jump sizes and a constant recovery rate. We introduce the basic ideas and theorems. For a detailed discussion see [KW] and [Sche]. In Section 3.4 we compare results and running time of this method and the already introduced Monte-Carlo simulations. The idea of the method discussed in this section is to find an analytical representation F of the Laplace transform of $\mathbb{P}(\tau \leq t)$, to invert F numerically, and to calculate the bond price based on this inversion.[65]

The next lemma is helpful for calculating the Laplace transform.

Lemma 3.5 (Representation of the moment-generating function)
Let the jumps of the Lévy process X_t be two-sided exponentially distributed as in Definition 2.2. The moment-generating function $\tilde{\Phi}_{X_t}$ of X at time t can be represented as

$$\tilde{\Phi}_{X_t}(x) := \mathbb{E}(e^{xX_t}) = e^{tG(x)}, \quad x \in \mathbb{R},$$

where $G(x) = \psi(-ix)$ for ψ defined in Equation (1.5) of Theorem 1.2. $G(x)-\alpha$ has for all $\alpha > 0$ exactly four roots. We denote them by $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, and $-\beta_{4,\alpha}$. All of them are real and they satisfy

$$0 < \beta_{1,\alpha} < \lambda_{\oplus} < \beta_{2,\alpha} < \infty \quad \textit{and} \quad 0 < \beta_{3,\alpha} < \lambda_{\ominus} < \beta_{4,\alpha} < \infty,$$

[64] Of course, $r = 25\%$ is not realistic. Nevertheless, simulations with high interest rates illustrate how fast the original approximation becomes inaccurate. However, even for $r = 5\%$, the relative pricing error of the original approach is 0.18%, which exceeds the relative pricing error of 0.007% in our approximation by far.

[65] We have to detour, since there is no analytical solution of $\mathbb{P}(\tau \leq t)$ known.

where $\lambda_\ominus$ and $\lambda_\oplus$ are defined as in Definition 2.2. $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $\beta_{3,\alpha}$, and $\beta_{4,\alpha}$ can be found as the roots of a polynomial.

Proof: See Theorem 1.2, Remark 1.2, [KW], Lemma 2.1, and its adapted version, [Sche], Lemma 3.3. ♢

We are able to express the Laplace transform F of $\mathbb{P}(\tau \leq t)$ in terms of the zeros of $G(x) - \alpha$.

Theorem 3.3 (The Laplace transform of $\mathbb{P}(\tau \leq t)$)
Fix $\alpha > 0$. For $b = -ln(v_0)$, we obtain

$$\begin{aligned} \mathbb{E}(e^{-\alpha\tau}) &= A_2 e^{b\beta_{3,\alpha}} + B_2 e^{b\beta_{4,\alpha}} \quad \textit{and} \\ F(\alpha) := \mathcal{L}\left(\mathbb{P}(\tau \leq t)\right)(\alpha) &= \frac{1}{\alpha}\left(A_2 e^{b\beta_{3,\alpha}} + B_2 e^{b\beta_{4,\alpha}}\right), \end{aligned}$$

where[66]

$$A_2 = \frac{\lambda_\ominus - \beta_{3,\alpha}}{\lambda_\ominus} \cdot \frac{\beta_{4,\alpha}}{\beta_{4,\alpha} - \beta_{3,\alpha}} \quad \textit{and} \quad B_2 = \frac{\beta_{4,\alpha} - \lambda_\ominus}{\lambda_\ominus} \cdot \frac{\beta_{3,\alpha}}{\beta_{4,\alpha} - \beta_{3,\alpha}}.$$

Proof: See the proofs of [KW], Theorem 3.1 and [Sche], Theorem 3.1 for the expression of $\mathbb{E}(e^{-\alpha\tau})$. The representation of the Laplace transform F is a direct consequence obtained by a partial integration:

$$F(\alpha) = \int_0^\infty e^{-\alpha t}\mathbb{P}(\tau \leq t)dt = \frac{1}{\alpha}\int_0^\infty e^{-\alpha t} d\mathbb{P}(\tau \leq t) = \frac{1}{\alpha}\mathbb{E}(e^{-\alpha\tau}).$$

This shows the second part of the statement. ♢

Since the Laplace transform F depends on $\beta_{3,\alpha}$ and $\beta_{4,\alpha}$, which are not explicitly given, no explicit inversion of the Laplace transform is known and we need a numerical inversion algorithm. Kou and Wang, [KW], Section 5 suggest the Gaver-Stehfest algorithm, which is a direct consequence of the next lemma.

Lemma 3.6 (Gaver-Stehfest)
Let F denote the Laplace transform of $\mathbb{P}(\tau \leq t)$. Then,

$$\mathbb{P}(\tau \leq t) = \lim_{n\to\infty} \sum_{k=1}^{n} w(k,n)\tilde{F}_k(t),$$

[66] We chose these indices to stay consistent with [KW] and [Sche].

where

$$\begin{aligned} w(k,n) &= (-1)^{n-k}\frac{k^n}{k!(n-k)!} \quad \textit{and} \\ \tilde{F}_n(t) &= \frac{\ln(2)}{t}\cdot\frac{(2n)!}{n!(n-1)!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}F\left(\frac{(n+k)\ln(2)}{t}\right). \end{aligned}$$

Proof: See [AW], Proposition 8.2. $\diamondsuit$

As discussed in [KW], Section 5, we can increase the numeric stability by skipping $\tilde{F}_1$ and $\tilde{F}_2$. For constant recovery rates $w_1^R(x) \equiv R$, Lemma 2.2 gives us the formula[67]

$$\phi(0,T) = e^{-rT}\mathbb{P}(\tau > T) + R\int_0^T e^{-rt}d\mathbb{P}(\tau \le t).$$

Together with the presented lemmas and theorems in this section this leads to the Laplace-transform algorithm[68] for bonds with recovery payoff at time of default.

Algorithm 3.3 (Laplace-transform algorithm)

1. *Choose a number K to partition the interval $[0;T]$ in K equidistant subintervals.*
2. *Choose $N > 0$ and calculate for all $t_j := j/K \cdot T$, $j \in \{1,2,\dots,K\}$*

$$\mathbb{P}^N(\tau \le t_j) \approx \sum_{k=1}^{N} w(k,N)\tilde{F}_{k+2}(t_j),$$

   where w and $\tilde{F}_k$ are defined as in Lemma 3.6.

3. *Calculate the approximated bond price $\phi^{K,N}(0,T)$ by*

$$\begin{aligned} \phi^{K,N}(0,T) &= e^{-rT}(1-\mathbb{P}^N(\tau \le T)) + \\ & R\sum_{j=1}^{K}\exp\left(-r\frac{j-0.5}{n}T\right)\mathbb{P}^N\left(\tau \in \left(\frac{(j-1)T}{n};\frac{jT}{n}\right]\right). \end{aligned}$$

The drawbacks for this algorithm are its restrictions to a constant recovery rate and to two-sided exponentially distributed jump sizes. The big advantage of this algorithm is its speed. In the next section, we compare calculated bond prices and running time of this method and the other introduced algorithms.

[67] The second term in this equation is not directly related to the Laplace transform, since the integration interval is $[0;T]$ and not $[0;\infty)$. That is the reason why we first have to apply the Laplace inversion.

[68] See the appendix, Remark B.4 for additional information about its implementation.

3.4 Comparison

In this section, we provide a comparison of all presented algorithms.

While the Laplace-transform approach implies the highest restrictions to the model assumptions, namely a constant recovery rate and two-sided exponentially distributed jumps, Zhou's Monte-Carlo simulation requires only continuous recovery-rate functions and the Brownian-bridge pricing technique even works for an arbitrary recovery-rate function.

We implemented all algorithms in C.[69] Concerning Zhou's algorithm, we used three different discretizations: N in Algorithm 3.1 was set to $10T$, $100T$, and $1000T$ (where T denotes the maturity in years), which corresponds to checking whether the bond defaulted about once per month, every three days, and three times per day. As parameters, we chose $r = 0.02$, $\gamma = 0.025$, $\sigma = 0.05$, and $T = 5$. The jump sizes are assumed to be two-sided exponentially distributed. Negative and positive jumps occur with the same probability, that is, $p = 0.5$. v_0 is set to $1/0.8$, that is, default occurs when the value of a company falls under 80% of its value at time zero. We performed all simulations in four different scenarios: In the first three scenarios, the recovery rate is kept constant, $w_1^{0.5}(x) \equiv 0.5$. In the scenario titled "Low", we expect only $\lambda = 0.5$ jumps per year but they are expected to be large, that is, $\lambda_\ominus = \lambda_\oplus = 10$. The scenario "Middle" corresponds to $\lambda = 2$ and $\lambda_\ominus = \lambda_\oplus = 20$. In the scenario "High", $\lambda = 8$ jumps per year are expected with $\lambda_\ominus = \lambda_\oplus = 40$. The scenario "Stochastic" has the same same jump structure as the scenario "Middle", but the recovery rate is stochastic, $w_2^{0.7}(x) = 0.7x$. Lemma 1.10 yields that the volatility and expectation of the underlying Lévy process remain the same in all scenarios. We performed one million simulations per algorithm, except for the Laplace-transformation approach, which needs only one run. Since the Brownian-bridge pricing technique generates an unbiased price, we additionally performed ten million runs of this algorithm and interpreted the result as the correct price.

Table 3.1 includes, besides the running time and the generated credit spread of each algorithm and scenario, the relative error of the credit spread which we define as (spread − generated spread)/spread[70]. The data set shows that Zhou's algorithm produces a significant bias. When simulating with only 10 grid points

[69] See Appendix B for details about the implementation and the machines we worked on.

[70] While "spread" denotes the credit spread obtained from the Brownian-bridge simulation with ten million runs, "generated spread" represents the credit spread from the corresponding algorithm.

		Low	Middle	High	Stochastic
	Spread	.013171	.016213	.018184	.009742
Zhou (10)	Rel. error	7.2465	8.9464	9.0436	7.5799
	Time	0:5:28.23	0:5:22.09	0:7:36.21	0:5:48.35
	Spread	.013751	.017162	.019251	.010081
Zhou (100)	Rel. error	3.1620	3.6168	3.7065	4.3639
	Time	0:52:11.84	0:48:26.94	0:54:00.39	0:51:31.67
	Spread	.013942	.017428	.019567	.010193
Zhou (1000)	Rel. error	1.8169	2.1229	2.1259	3.3014
	Time	8:48:31.82	8:00:06.00	8:44:07.81	9:19:27.21
	Spread	.014209	.017813	.019969	.010548
Brown. bridge	Rel. error	-0.0634	-0.0393	0.1150	-0.0664
	Time	0:6:45.78	0:19:24.65	0:58:23.12	0:19:33.96
	Spread	.014323	.017853	.020037	.010624
Taylor (orig.)	Rel. error	-0.8662	-0.2640	-0.2251	-0.7874
	Time	0:0:42.18	0:2:17.55	0:8:03.13	0:2:19.27
	Spread	.014162	.017815	.019989	.010523
Taylor (our)	Rel. error	0.2676	-0.0505	0.0150	0.1708
	Time	0:0:39.74	0:2:06.30	0:8:03.13	0:2:19.25
	Spread	.014168	.017776	.019972	-
Laplace	Rel. error	0.2254	0.1685	0.1000	-
	Time	0:0:0.14	0:0:0.14	0:0:0.14	-
Brown. bridge	10 Mil.	.014200	.017806	.019992	.010541

Table 3.1: Comparison of Zhou's algorithm (with discretization $N = 10T$, $N = 100T$, $N = 1000T$), the Brownian-bridge pricing technique, the original and our Taylor approximation, and the Laplace-transform approach in four different jump-structure and recovery settings. The relative error is given as percentage.

per year, the relative error exceeds 7%. Even with 1000 grid points per year, the relative error is at least 1.8%, which is still too high for an application in real life. The error results from the fact that the algorithm not only omits to check for default between the grid points but also allows at most one jump between the grid points. Thus, the default risk is significantly underestimated and the resulting credit spreads are too low. The restriction to at most one jump in between the grid and the fact that a growing jump intensity increases the probability of more than one jump in between the grid also explain the observation that the relative error increases in the jump intensity. Another drawback of Zhou's algorithm is its inefficiency in terms of running time, especially for fine grids (and thus, small biases).

Examining the running time of the Brownian-bridge pricing technique shows that it depends strongly on the expected number of jumps. The reason therefore is that the number of random variables that have to be drawn and the number of integrals which have to be calculated depends on the number of jumps. The same statement holds for the Taylor approximations, where the calculations of the approximations are significantly faster but still have a strong impact on the running time. While the Taylor approximation already significantly reduces the running time, the Laplace-transformation approach is unbeatable. Besides, both the Laplace-transformation approach and especially our Taylor approximation yield very good results which can be used to obtain credit spreads. Hence, for calibrations with constant (resp. stochastic) recovery rates, the Laplace-transform approach (resp. our Taylor approximation) is an adequate choice.

Appendix A

Laplace transform

Essential tools in the Sections 3.2 and 3.3 are the Laplace transform and its inverse. In this chapter, we give a short definition and introduce important properties which we apply in some proofs. We abstain from introducing the Laplace transform in full generality; instead we define it briefly without specifying its domain, as this simplification is totally sufficient for our purposes. In the second part of this chapter, we prove a representation of a Laplace inversion which is used in Lemma 3.4. For an introduction to Laplace-transform theory, we refer to [Schi] and [D]. As a table for well-known Laplace transforms and inverses we use [OB].

Definition A.1 (Laplace transform, Laplace inverse)
Let f denote a continuous function defined on $\mathbb{R}_0^+$ such that $\int_0^\infty e^{-st} f(t)dt$ exists for some $s \in \mathbb{R}$. Then, the Laplace transform of f, $\mathcal{L}(f) : U_f \to \mathbb{C}$, where $U_f := \{s \in \mathbb{C} : Re(s) > c_f\}$ for $c_f := \inf\{\alpha \in \mathbb{R} : \int_0^\infty e^{-\alpha t} f(t)dt \text{ exists}\}$, is defined by

$$(\mathcal{L}(f))(s) := \int_0^\infty e^{-st} f(t)dt.$$

The Laplace inverse $\mathcal{L}^{-1}$ is the inverse function of the Laplace transform.

Lemma A.1 (Representation of the Laplace inverse)
Let $F = \mathcal{L}(f)$ denote the Laplace transform of a function f. It holds

$$f(t) = \left(\mathcal{L}^{-1}(F)\right)(t) = \frac{1}{2\pi i} \int_{y-i\infty}^{y+i\infty} e^{ts} F(s)ds, \quad \forall\, t > 0$$

with $y \in \mathbb{R}$ chosen so that all singularities of F are to the left of y.

Proof: See [Schi] or [D]. $\diamondsuit$

We use some properties of the Laplace transform and its inverse which we now present.

Lemma A.2 (Linearity of the Laplace transform)
The Laplace transform and its inverse are linear.

Proof: The statement follows directly from Definition A.1 and Lemma A.1. $\diamondsuit$

Lemma A.3 (Shift theorem)
If f has a Laplace transform defined on $U_f = \{s \in \mathbb{C} : Re(s) > c_f\}$, then for all $a \in \mathbb{R}$, $e^{-at} f(t)$ has a Laplace transform defined on $\tilde{U} = \{s \in \mathbb{C} : Re(s) > c_f - a\}$. It holds

$$\left(\mathcal{L}(e^{-at} f(t))\right)(s) = \left(\mathcal{L}(f)\right)(s+a).$$

Proof: For s with $Re(s) > c_f - a$,

$$\left(\mathcal{L}(f)\right)(s+a) = \int_0^\infty e^{-(s+a)t} f(t) dt = \int_0^\infty e^{-st} e^{-at} f(t) dt = \left(\mathcal{L}(e^{-at} f(t))\right)(s)$$

holds by the definition. $\diamondsuit$

Lemma A.4 (Convolution)
If f and h have Laplace transforms defined on $U_f = \{s \in \mathbb{C} : Re(s) > c_f\}$ and $U_h = \{s \in \mathbb{C} : Re(s) > c_h\}$, then their convolution, that is $(f * h)(t) := \int_0^t f(x) h(t-x) dx$, has a Laplace transform, defined on $U_{f*h} = \{s \in \mathbb{C} : Re(s) > \max\{c_f, c_h\}\}$. It holds

$$\left(\mathcal{L}(f * h)\right)(s) = \left(\mathcal{L}(f)\right)(s) \cdot \left(\mathcal{L}(h)\right)(s).$$

Proof:

$$\begin{aligned}
\left(\mathcal{L}(f * h)\right)(s) &= \int_0^\infty e^{-st} \int_0^t f(x) h(t-x) dx dt \\
&= \int_0^\infty \int_x^\infty e^{-st} f(x) h(t-x) dt dx \\
&= \int_0^\infty \int_0^\infty e^{-st} e^{-sx} f(x) h(t) dt dx \\
&= \left(\mathcal{L}(f)\right)(s) \cdot \left(\mathcal{L}(h)\right)(s),
\end{aligned}$$

where the second to last equation follows by a substitution. ♢

In order to calculate a Laplace inversion in Lemma 3.4, the following functions are useful.[71] We do not specify their area of domain.

Definition A.2 (Some important functions)

- *The Gamma function* Γ *is defined by*

$$\Gamma(z) := \lim_{n\to\infty} \frac{n! n^z}{z(z+1)_n},$$

where $(z)_n := z \cdot (z+1) \cdot \ldots \cdot (z+n-1)$ *.*

- *The Kummer's function* ${}_1F_1$ *is given by*

$${}_1F_1(a,b,z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}.$$

- *The Whittaker's functions* $M_{k,m}$ *and* $W_{k,m}$ *are defined by*

$$\begin{aligned} M_{k,m}(z) &:= z^{m+\frac{1}{2}} e^{-\frac{1}{2}z} {}_1F_1(\frac{1}{2}+m-k, 2m+1, z) \\ &= z^{m+\frac{1}{2}} e^{\frac{1}{2}z} {}_1F_1(\frac{1}{2}+m+k, 2m+1, -z) \quad \textit{and} \\ W_{k,m}(z) &:= \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)} M_{k,-m}(z), \end{aligned}$$

where the equality follows from ${}_1F_1(a,b,z) = e^z {}_1F_1(b-a,b,-z)$ *.*

- *The Parabolic cylinder function* D_k *is given by*

$$D_k(z) := 2^{\frac{1}{4}+\frac{1}{2}k} z^{-\frac{1}{2}} W_{\frac{1}{4}+\frac{1}{2}k,\frac{1}{4}}\left(\frac{1}{2}z^2\right).$$

For the Laplace inversion we are doing in Lemma 3.4, we have to evaluate the Gamma function for different values. The following lemma is helpful.

Lemma A.5 (Some properties of the Gamma function)
The Gamma function Γ *satisfies*

$$\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R}^+$$

[71] We want to emphasize that the functions introduced here are only a very small subset of functions which appear in the context of Laplace transforms and inversions.

$$\begin{aligned}\Gamma(1) &= 1,\\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}, \quad \textit{and}\\ \Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi}.\end{aligned}$$

Proof: The statement follows from an integration by parts over an alternative representation of Γ and from an induction. See [O], Chapter 2.1. $\diamondsuit$

Lemma A.6 (A useful Laplace inversion)
It holds for $a \in \mathbb{R}^+$

$$\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s^2}\right)\right)(t) = (2t+4a^2)\left(1-\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\right) - \frac{2a\sqrt{t}}{\sqrt{\pi}}e^{-\frac{a^2}{t}}.$$

Proof: By Formula 5.94 in [OB], Section 2.5, page 259,

$$\begin{aligned}\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s^2}\right)\right)(t) &= \frac{2^{\frac{3}{2}}t}{\sqrt{\pi}}e^{-\frac{a^2}{2t}}D_{-3}\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\\ &= \frac{2^{\frac{3}{2}}t}{\sqrt{\pi}}e^{-\frac{a^2}{2t}}\frac{2^{-\frac{5}{4}}t^{\frac{1}{4}}}{2^{\frac{1}{4}}a^{\frac{1}{2}}}W_{-\frac{5}{4},\frac{1}{4}}\left(\frac{a^2}{t}\right)\\ &= \frac{t^{\frac{5}{4}}}{\sqrt{a\pi}}e^{-\frac{a^2}{2t}}W_{-\frac{5}{4},\frac{1}{4}}\left(\frac{a^2}{t}\right)\\ &= \frac{t^{\frac{5}{4}}}{\sqrt{a\pi}}e^{-\frac{a^2}{2t}}\left(-4M_{-\frac{5}{4},\frac{1}{4}}\left(\frac{a^2}{t}\right)+\sqrt{\pi}M_{-\frac{5}{4},-\frac{1}{4}}\left(\frac{a^2}{t}\right)\right)\\ &= \frac{t^{\frac{5}{4}}}{\sqrt{a\pi}}e^{-\frac{a^2}{2t}}\left(-4\left(\frac{a^2}{t}\right)^{\frac{3}{4}}e^{\frac{a^2}{2t}}{}_1F_1\left(-\frac{1}{2},\frac{3}{2},-\frac{a^2}{t}\right)+\right.\\ &\qquad\left.\sqrt{\pi}\left(\frac{a^2}{t}\right)^{\frac{1}{4}}e^{-\frac{a^2}{2t}}{}_1F_1\left(\frac{3}{2},\frac{1}{2},\frac{a^2}{t}\right)\right)\\ &= -4\frac{a\sqrt{t}}{\sqrt{\pi}}{}_1F_1\left(-\frac{1}{2},\frac{3}{2},-\frac{a^2}{t}\right)+te^{-\frac{a^2}{t}}{}_1F_1\left(\frac{3}{2},\frac{1}{2},\frac{a^2}{t}\right).\end{aligned}$$

We calculate

$${}_1F_1\left(\frac{3}{2},\frac{1}{2},z\right) = \sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n}\frac{z^n}{n!}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(3+2n-2)\frac{z^n}{n!} \\
&= e^z + 2z\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!} \\
&= e^z(1+2z)
\end{aligned}
$$

and for $z \in \mathbb{R}^-$

$$
\begin{aligned}
{}_1F_1\left(-\frac{1}{2},\frac{3}{2},z\right) &= \sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{-1}{(3+2n-4)(3+2n-2)}\frac{z^n}{n!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{z^n}{n!} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{2n-1}\frac{z^n}{n!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(\sqrt{-z})^{2n}}{2n+1}\frac{(-1)^n}{n!} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(\sqrt{-z})^{2n}}{2n-1}\frac{(-1)^n}{n!} + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\sum_{n=0}^{\infty}\frac{(\sqrt{-z})^{2n+1}}{2n+1}\frac{(-1)^n}{n!} - \frac{\sqrt{-z}}{2}\sum_{n=1}^{\infty}\frac{(\sqrt{-z})^{2n-1}}{2n-1}\frac{(-1)^n}{n!} + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\sum_{n=0}^{\infty}\int_0^{\sqrt{-z}} x^{2n}dx \cdot \frac{(-1)^n}{n!} - \\
&\quad \frac{\sqrt{-z}}{2}\sum_{n=1}^{\infty}\int_0^{\sqrt{-z}} x^{2n-2}dx \cdot \frac{(-1)^n}{n!} + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\int_0^{\sqrt{-z}}\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}dx - \frac{\sqrt{-z}}{2}\int_0^{\sqrt{-z}}\frac{1}{x^2}\sum_{n=1}^{\infty}\frac{(-x^2)^n}{n!}dx + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\int_0^{\sqrt{-z}} e^{-x^2}dx - \frac{\sqrt{-z}}{2}\int_0^{\sqrt{-z}}\frac{e^{-x^2}-1}{x^2}dx + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\int_0^{\sqrt{-z}} e^{-x^2}dx + \\
&\quad \frac{\sqrt{-z}}{2}\left(\left.\frac{e^{-x^2}-1}{x}\right|_0^{\sqrt{-z}} + \int_0^{\sqrt{-z}}\frac{2xe^{-x^2}}{x}dx\right) + \frac{1}{2} \\
&= \frac{1}{2\sqrt{-z}}\int_0^{\sqrt{-z}} e^{-x^2}dx + \frac{e^z-1}{2} + \sqrt{-z}\int_0^{\sqrt{-z}} e^{-x^2}dx + \frac{1}{2} \\
&= \frac{1-2z}{2\sqrt{-z}}\int_0^{\sqrt{-z}} e^{-x^2}dx + \frac{e^z}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1-2z)\sqrt{\pi}}{2\sqrt{-z}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{-2z}} e^{-\frac{x^2}{2}}\,dx + \frac{e^z}{2} \\
&= \frac{(1-2z)\sqrt{\pi}}{4\sqrt{-z}} \left(2\Phi\left(\sqrt{-2z}\right) - 1\right) + \frac{e^z}{2}.
\end{aligned}
$$

Combining these results, we obtain

$$
\begin{aligned}
&\left(\mathcal{L}^{-1}\left(\frac{\exp(-2a\sqrt{s})}{s^2}\right)\right)(t) \\
&\quad = -4\frac{a\sqrt{t}}{\sqrt{\pi}}\left(\frac{(1+\frac{2a^2}{t})\sqrt{\pi}}{\frac{4a}{\sqrt{t}}}\left(2\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)-1\right) + \frac{e^{\frac{-a^2}{t}}}{2}\right) + t\left(1+\frac{2a^2}{t}\right) \\
&\quad = (2t+4a^2)\left(1-\Phi\left(\frac{\sqrt{2}a}{\sqrt{t}}\right)\right) - \frac{2a\sqrt{t}}{\sqrt{\pi}}e^{-\frac{a^2}{t}}.
\end{aligned}
$$

This completes the proof. $\diamondsuit$

Appendix B

Details about the implementation

In this appendix, we provide some further information about the programs which do the simulations and obtain the data we present in this thesis. All programs can be found on a CD attached to this thesis.

We implemented all algorithms with two-sided exponentially distributed jump sizes, a constant interest rate, and recovery functions w_1^R and w_2^R, as specified in Section 2.2. In case of a default, the recovery was paid at time of default. We used the C-programming language and routines from the NAG-software library[72]. We worked on a Sun computer equipped with an UltraSPARC-III+ processor (900MHz). To provide a benchmark of the running time the output user time of the Unix command `timex` was chosen.

Remark B.1 (Implementation of Zhou's algorithm (Algorithm 3.1))
As opposed to the procedure presented in Algorithm 3.1, we decided not to begin with the simulation of a whole sample path of V^* but instead to simulate it piece by piece and to decide at every time step whether a default occurred. After the time of default, the path of V^* is irrelevant and this small modification allowed us to save the time which would be otherwise spent on simulating unnecessary samples. The NAG routines we used were `g05ccc()` *for the random generator,* `g05ddc()` *for simulating the Gaussian part,* `g05cac()` *for deciding whether a jump occurred and whether it was negative or positive, and* `g05dbc()` *for obtaining the jump size.*

Result B.1 (Sum of zero bonds in Zhou's algorithm[73])

[72]See http://www.nag.co.uk.

[73]See Lemma 2.1 and Remark 3.1.

We calculated the price of a coupon bond $\phi(0, t_1, t_2, T, q)$ *with maturity* $T = 10$ *, face value* 1 *, and coupon payments* $q = (1, 1, 0)$ *at the times* $t_1 = 2$ *and* $t_2 = 3$ *with an adapted Zhou's algorithm and compared it to the sum of the prices of three zero-coupon bonds* $\phi(0, t_1)$, $\phi(0, t_2)$ *, and* $\phi(0, T)$ *with maturities* t_1, t_2 *, and* T *. The number of grid points a year was set to* 100 *. The other parameters were* $r = 0.02$ *,* $\gamma = 0.025$ *,* $\sigma = 0.05$ *, and* $v_0 = 1/0.8$ *. The jumps were two-sided exponentially distributed with parameters* $p = 0.5$ *,* $\lambda = 2$ *, and* $\lambda_\oplus = \lambda_\ominus = 20$ *. Recovery at time of default was* $w_1^{0.5} \equiv 0.5$ *(times one for the zero-coupon bonds and times the sum of the outstanding coupon payments and the face value for the coupon bond). We performed ten runs, each run with* $200,000$ *path simulations. The obtained differences* $(\phi(0, t_1) + \phi(0, t_2) + \phi(0, T)) - \phi(0, t_1, t_2, T)$ *are listed in Table B.1.*

Run	Difference	Run	Difference
1	0.000058	6	-0.000593
2	0.001739	7	-0.000101
3	0.001972	8	0.000716
4	-0.000879	9	0.002380
5	-0.001974	10	-0.000897

Table B.1: The difference of the sum of $\phi(0, t_1)$, $\phi(0, t_2)$, and $\phi(0, T)$ and $\phi(0, t_1, t_2, T)$.

As we can see, the differences are small and both negative and positive.

Remark B.2 (Implementation of Brownian-bridge pricing Alg. 3.2)
We used an implementation from [Sche] and modified it to include stochastic recovery rates. As for Zhou's algorithm, we did not simulate X_{τ_i} *for all jump times* τ_i *at the beginning but compared simultaneously whether a default by jump or a sure default by diffusion occurred, that is,* $X_{\tau_i -}$ *has crossed the default barrier* $-\ln(v_0)$ *, and stopped if this was confirmed. We used the NAG routines* `g05ccc()` *for the random generator,* `g05ecc()` *and* `g05eyc()` *for simulating the number of jumps,* `g05cac()` *and* `m01cac()` *for the jump times,* `g05ddc()` *for obtaining the Gaussian increments,* `g05cac()` *and* `g05dbc()` *for simulating the jump size, and* `d01ajc()` *for calculating the integral.*

Result B.2 (Sum of zero bonds in Brownian-bridge pricing alg.[74])
In order to calculate the price of a coupon bond $\phi(0, t_1, t_2, T, q)$ *with maturity*

[74] See Lemma 2.1 and Remark 3.3.

$T = 10$, face value 1, and coupon payments $q = (1,1,0)$ at the times $t_1 = 2$ and $t_2 = 3$ we modified the Brownian-bridge pricing algorithm slightly: We conditioned on the values of X not only at the jump times but also at the times of coupon payments t_1 and t_2, since the recovery changes at these times. Then, we compared $\phi(0,t_1,t_2,T,q)$ to the sum of the prices of the zero-coupon bonds $\phi(0,t_1), \phi(0,t_2)$, and $\phi(0,T)$ with maturities t_1, t_2, and T. The parameters were the same as in the corresponding Zhou simulation: $r = 0.02$, $\gamma = 0.025$, $\sigma = 0.05$, and $v_0 = 1/0.8$. The jumps were two-sided exponentially distributed with parameters $p = 0.5$, $\lambda = 2$, and $\lambda_\oplus = \lambda_\ominus = 20$. Recovery at time of default was $w_1^{0.5} \equiv 0.5$ (times one for the zero-coupon bonds and times the sum of the outstanding coupon payments and of the face value for the coupon bond). We performed ten runs, each run with $200,000$ path simulations. The obtained differences $(\phi(0,t_1) + \phi(0,t_2) + \phi(0,T)) - \phi(0,t_1,t_2,T)$ are listed in Table B.2.

Run	Difference	Run	Difference
1	0.000111	6	-0.000820
2	0.000276	7	-0.002072
3	0.000209	8	-0.000955
4	-0.000798	9	-0.000891
5	0.000854	10	-0.000331

Table B.2: The difference of the sum of $\phi(0,t_1), \phi(0,t_2)$, and $\phi(0,T)$ and $\phi(0,t_1,t_2,T)$.

Similar to the data in Table B.1, the differences are small and both negative and positive.

Remark B.3 (Implementation of the approximation in Lemma 3.4)
We only had to modify the implementation for the calculation of the unbiased version slightly. We exchanged the NAG routine `d01ajc()`*, used for calculating the integral over g, by* `g01eac()` *to evaluate the cumulative normal distribution Φ.*

Result B.3 (Relative error of the approximation[75])
For each scenario, we calculated the relative pricing error for different interest rates. We list the exact numbers in Table B.3.

Remark B.4 (Implementation of the Laplace-transform Algorithm 3.3)

[75] See Lemma 3.4 and Remark 3.4.

	$\lambda = 0.5$		$\lambda = 2$		$\lambda = 8$	
r	Original	Our	Original	Our	Original	Our
2.5	0.072560	-0.006450	0.035636	-0.008159	0.008500	-0.008627
5.0	0.183640	-0.007161	0.089252	-0.008701	0.027612	-0.009191
7.5	0.339830	-0.008382	0.154848	-0.009499	0.049408	-0.010083
10.0	0.550475	-0.009074	0.239234	-0.010398	0.074507	-0.010582
12.5	0.836835	-0.010999	0.338649	-0.011148	0.105069	-0.011337
15.0	1.228662	-0.012840	0.462425	-0.012113	0.139050	-0.010996
17.5	1.747642	-0.016792	0.603955	-0.013367	0.177395	-0.012765
20.0	2.400023	-0.021394	0.788351	-0.014252	0.221226	-0.013789
22.5	3.288684	-0.028168	0.993531	-0.015914	0.272949	-0.013913
25.0	4.414416	-0.036117	1.253171	-0.017627	0.330520	-0.014954

Table B.3: The relative pricing errors for approximations with the original and our formula. All numbers are given as percentages.

We used an implementation from [Sche]. Scherer calculates the roots of $G(x) - \alpha$ *by reformulating the problem as an equivalent polynomial equation*[76] *and using the Pegasus algorithm*[77] *to find its roots. Choosing* $N = 9$ *and* $K = 100T$ *yields a sufficiently good result.*

[76]See [Sche], Lemma 3.3.
[77]See [ER] for a description of this algorithm.

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Summary

In this thesis, we present a structural default model and methods to price corporate bonds within this framework. The model is based on two jump-diffusion processes: While the first process represents the company's value, the second one models a default threshold. A bond holder receives all promised payments as long as the company's value process does not cross the default threshold. If the company defaults, the investor receives a fraction of the outstanding payments. Since only the difference between the two stochastic processes, the so called distance to default, is relevant for pricing a bond, we can simplify the model to a setup with only one stochastic process.

With the possibility of jumps, the model captures empirical observations, such as the fact that credit spreads for short maturities do not vanish. However, including jumps makes it difficult to find analytical solutions for fair bond prices in this setup. To overcome this problem, we present and compare different Monte-Carlo simulations. In Zhou's algorithm, the time to maturity is discretized and at every grid point default is checked for. Not being able to check for default between the grid points, however, causes biased bond prices, which are too high as the default probabilities are underestimated. The Brownian-bridge pricing technique avoids this bias by conditioning on the jumps. An analytical formula can be found for the probabilities of default between the jumps, when the jump-diffusion process behaves like a Brownian bridge. In the pricing formula, these probabilities form an integral which is numerically expensive to calculate but possible to approximate. If the jump sizes are two-sided exponentially distributed and the bond holder receives a non-stochastic payment in case of a default, we can renounce a Monte-Carlo simulation but use the Laplace-transform approach, where the bond is priced by means of the analytically known Laplace transform of the default probabilities. This approach turns out to be the fastest pricing algorithm available.

Contribution

We introduce a jump-diffusion model with an arbitrary jump-size distribution and a stochastic default threshold, which is modelled as another jump-diffusion process. The model is a generalization of Zhou's and Scherer's model, [Z2] and [Sche], where the default threshold is deterministic and constant, respectively, and the jumps are normally distributed and two-sided exponentially distributed, respectively. As in [Z2], stochastic recoveries are allowed. We show that the model can be simplified to a model with a constant default threshold.

Assuming constant recovery rates, we give an analytical formula for the local default rate and prove that in the jump-diffusion model the credit spreads do not vanish as maturity decreases. Both results are non-trivial generalizations of [Sche], where equivalent statements are proven but only for two-sided exponentially distributed jump sizes. This generalization was suggested by and worked out together with Matthias Scherer.

We thoroughly prove a theorem which justifies Zhou's algorithm for all jump-size distributions. In [Z1] and [Z2], where the jump sizes are restricted to normally distributed random variables, only a short outline of the proof is given.

We generalize the Brownian-bridge pricing technique from [Sche] and [MA] to include stochastic recovery rates. Furthermore, we significantly improve an approximation for an integral which is used in the algorithm in terms of precision. An extensive simulation of the original approximation as introduced in [MA] and [Sche] and of our approximation shows by how much the error of the approximation is reduced if our approximation is used.

By implementing all algorithms and performing several simulations we systematically compare resulting bond prices and running times of the different algorithms. We also check for Zhou's algorithm and for the Brownian-bridge pricing technique whether coupon bonds can be priced by means of a portfolio of zero-coupon bonds without a bias.

Ehrenwörtliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig angefertigt und nur die angegebenen Quellen benutzt habe.

Wörtlich oder inhaltlich übernommenes Gedankengut wurde nach bestem Wissen und Gewissen als solches kenntlich gemacht.

Diese Arbeit wurde bisher keinem anderen Prüfungsgremium vorgelegt und auch noch nicht veröffentlicht.

Ulm, 01. Juni 2006