Electron. Commun. Probab. **0** (2017), no. 0, 1–5.
DOI: 10.1214/ECP.vVOL-PID
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Piecewise constant local martingales with bounded numbers of jumps

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Abstract

A piecewise constant local martingale M with boundedly many jumps is a uniformly integrable martingale if and only if M_∞^- is integrable.

Keywords: Local martingale; continuous-time; discrete-time.
AMS MSC 2010: 60G42; 60G44.
Submitted to ECP on December 23, 2016, final version accepted on May 23, 2017.

1 Main theorem

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathsf{P})$ denote a filtered probability space with $\bigcup_{t\geq 0} \mathscr{F}_t \subset \mathscr{F}$. In Section 2, we shall prove the following theorem.

Theorem 1.1. *Assume for some $N \in \mathbb{N}_0$ and some stopping times $0 \leq \rho_1 \leq \cdots \leq \rho_N$ we have a local martingale M of the form*

$$M = \sum_{m=1}^{N} J_m \mathbb{1}_{[\![\rho_m,\infty[\![}, \qquad \textit{that is,} \qquad M_t = \sum_{m=1}^{N} J_m \mathbb{1}_{\{t\geq\rho_m\}}, \quad t \geq 0, \tag{1.1}$$

where J_m is $\mathscr{F}_{\rho_m}$–measurable for each $m = 1, \cdots, N$. If

$$\mathsf{E}\left[\liminf_{t\uparrow\infty} M_t^-\right] < \infty \tag{1.2}$$

then M is a uniformly integrable martingale.

In (1.2), we could replace the limit inferior by a limit since M only has finitely many jumps and hence converges to a random variable M_∞. Hence, (1.2) is equivalent to $\mathsf{E}[M_\infty^-] < \infty$.

Corollary 1.2. *Suppose the notation and assumptions of Theorem 1.1 hold, but with* (1.2) *replaced by*

$$\mathsf{E}\left[M_t^-\right] < \infty, \qquad t \geq 0.$$

Then M is a martingale.

Proof. Fix a deterministic time $T \geq 0$ and consider the local martingale $\widetilde{M} = M^T$; that is, $\widetilde{M}$ is the local martingale M stopped at time T. Then $\widetilde{M}$ satisfies the conditions of Theorem 1.1, with J_m replaced by $J_m \mathbb{1}_{\{\rho_m \leq T\}}$ for each $m = 1, \cdots, N$. Hence, $\widetilde{M}$ is a uniformly integrable martingale. Since T was chosen arbitrarily the assertion follows. □

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[1] prove the following special case of Theorem 1.1.

Proposition 1.3. *Fix $N \in \mathbb{N}_0$ and assume we have a discrete-time filtration $\mathfrak{G} = (\mathscr{G}_m)_{m=0,1,\cdots,N}$ and a $\mathfrak{G}$–local martingale $Y = (Y_m)_{m=0,1,\cdots,N}$. If $\mathsf{E}[Y_N^-] < \infty$ then Y is a $\mathfrak{G}$–uniformly integrable martingale.*

Note that Proposition 1.3 follows from Theorem 1.1. Indeed, define the continuous-time process M and the filtration $(\mathscr{F}_t)_{t\geq 0}$ by $M_t = Y_{[t]\wedge N}$ and $\mathscr{F}_t = \mathscr{G}_{[t]\wedge N}$, respectively, where $[t]$ denotes the largest integer smaller than or equal to t. Then M is a local martingale as in (1.1), with N replaced by $N+1$. To see this, set $\rho_m = m-1$ and $J_m = Y_{m-1} - Y_{m-2}$ with $Y_{-1} = 0$, for each $m = 1, \cdots, N+1$. Applying Theorem 1.1 then yields Proposition 1.3.

2 Proofs of Theorem 1.1

In the following, we will provide two proofs of Theorem 1.1. The first one assumes Proposition 1.3 is already shown and reduces the more general situation of Theorem 1.1 to the discrete-time setup of Proposition 1.3. The second proof does not assume Proposition 1.3, but instead provides a direct argument based on an induction.

Proof I, relying on Proposition 1.3. Let us set $\rho_0 = 0$ and $\rho_{N+1} = \infty$ and let $(\tau_n)_{n\in\mathbb{N}}$ denote a localization sequence of M such that M^{τ_n} is a uniformly integrable martingale for each $n \in \mathbb{N}$. For any stopping time τ we may define a sigma algebra

$$\mathscr{F}_{\tau-} = \sigma\left(\{A \cap \{t < \tau\},\, A \in \mathscr{F}_t,\, t \geq 0\} \cup \mathscr{F}_0\right).$$

Note that $\{\tau = \infty\} = \bigcap_{n\in\mathbb{N}} \{n < \tau\} \in \mathscr{F}_{\tau-}$.

Let us now define a filtration $\mathfrak{G} = (\mathscr{G}_m)_{m=0,\cdots,N}$ and a process $Y = (Y_m)_{m=0,1,\cdots,N}$ by $\mathscr{G}_m = \mathscr{F}_{\rho_m} \vee \mathscr{F}_{\rho_{m+1}-}$ and $Y_m = M_{\rho_m}$, respectively. Note that Y is adapted to $\mathfrak{G}$. Next, let us define a non-decreasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of random times, each taking values in $\{0, \cdots, N-1, \infty\}$ by

$$\sigma_n = \sum_{m=0}^{N-1} m \mathbb{1}_{\{\rho_m \leq \tau_n < \rho_{m+1} < \infty\}} + \infty \mathbb{1}_{\bigcup_{m=0}^{N} \{\rho_m \leq \tau_n\} \cap \{\rho_{m+1} = \infty\}}.$$

Then, σ_n is a $\mathfrak{G}$–stopping time for each $n \in \mathbb{N}$ since

$$\{\sigma_n = m\} = \{\rho_m \leq \tau_n < \rho_{m+1} < \infty\} \in \mathscr{F}_{\rho_m} \vee \mathscr{F}_{\rho_{m+1}-} = \mathscr{G}_m, \qquad m = 0, \cdots, N-1,$$

and, furthermore, $\lim_{n\uparrow\infty} \sigma_n = \infty$.

We now fix $n \in \mathbb{N}$ and prove that Y^{σ_n} is a $\mathfrak{G}$–martingale, which then yields that Y is a $\mathfrak{G}$–local martingale. To this end, we have, for each $m = 0, \cdots, N$,

$$\begin{aligned} Y_m^{\sigma_n} &= \sum_{k=0}^{N-1} M_{\rho_{m\wedge k}} \mathbb{1}_{\{\sigma_n = k\}} + M_{\rho_m} \mathbb{1}_{\{\sigma_n = \infty\}} \\ &= \sum_{k=0}^{N-1} M_{\rho_{m\wedge k}} \mathbb{1}_{\{\rho_k \leq \tau_n < \rho_{k+1} < \infty\}} + M_{\rho_m} \mathbb{1}_{\bigcup_{k=0}^{N} \{\rho_k \leq \tau_n\} \cap \{\rho_{k+1} = \infty\}} \\ &= M_{\rho_m}^{\tau_n}, \end{aligned}$$

yielding $\mathsf{E}[|Y_m^{\sigma_n}|] < \infty$. Now, fix $m = 1, \cdots, N$. First, for any $A \in \mathscr{F}_{\rho_m -1}$, we have

$$\mathsf{E}[Y_m^{\sigma_n} \mathbb{1}_A] = \mathsf{E}[M_{\rho_m}^{\tau_n} \mathbb{1}_A] = \mathsf{E}[M_{\rho_{m-1}}^{\tau_n} \mathbb{1}_A] = \mathsf{E}[Y_{m-1}^{\sigma_n} \mathbb{1}_A];$$

next, for any $t \geq 0$ and $A \in \mathscr{F}_t$, we have

$$\begin{aligned}\mathsf{E}[Y_m^{\sigma_n} \mathbb{1}_{A\cap\{t<\rho_m\}}] &= \mathsf{E}[M_{\rho_m}^{\tau_n} \mathbb{1}_{A\cap\{t<\rho_m\}}] = \mathsf{E}[M_t^{\tau_n} \mathbb{1}_{A\cap\{t<\rho_m\}}] = \mathsf{E}[M_{\rho_m-1}^{\tau_n} \mathbb{1}_{A\cap\{t<\rho_m\}}] \\ &= \mathsf{E}[Y_{m-1}^{\sigma_n} \mathbb{1}_{A\cap\{t<\rho_m\}}],\end{aligned}$$

yielding that $\mathsf{E}[Y_m^{\sigma_n} \mathbb{1}_A] = \mathsf{E}[Y_{m-1}^{\sigma_n} \mathbb{1}_A]$ for all $A \in \mathscr{G}_{m-1}$. Hence, Y is indeed a $\mathfrak{G}$–local martingale.

The assumptions of the theorem yield that $\mathsf{E}[Y_N^-] < \infty$; hence Y a $\mathfrak{G}$–uniformly integrable martingale by Proposition 1.3. Now, fix $t \geq 0$ and $A \in \mathscr{F}_t$. Then we get $\mathsf{E}[|M_t|] + \mathsf{E}[|M_\infty|] \leq 2\sum_{m=0}^N \mathsf{E}[|Y_m|] < \infty$ and

$$\begin{aligned}\mathsf{E}[M_\infty \mathbb{1}_A] &= \sum_{m=0}^N \mathsf{E}[Y_N \mathbb{1}_{A\cap\{\rho_m\leq t<\rho_{m+1}\}}] = \sum_{m=0}^N \mathsf{E}[Y_m \mathbb{1}_{A\cap\{\rho_m\leq t<\rho_{m+1}\}}] \\ &= \sum_{m=0}^N \mathsf{E}[M_t \mathbb{1}_{A\cap\{\rho_m\leq t<\rho_{m+1}\}}] = \mathsf{E}[M_t \mathbb{1}_A]\end{aligned}$$

since $A \cap \{\rho_m \leq t < \rho_{m+1}\} \in \mathscr{G}_m$ for each $m = 0, \cdots, N$. Hence, M is indeed a uniformly integrable martingale. □

Proof II, relying on an induction argument. We proceed by induction over N. The case $N = 0$ is clear. Hence, let us assume the assertion is proven for some $N \in \mathbb{N}_0$ and consider the assertion with N replaced by $N + 1$. Let $(\tau_n)_{n\in\mathbb{N}}$ denote a corresponding localization sequence such that M^{τ_n} is a uniformly integrable martingale for each $n \in \mathbb{N}$.

Step 1: In the first step, we want to argue that the nondecreasing sequence $(\widehat{\tau}_n)_{n\in\mathbb{N}}$, given by

$$\widehat{\tau}_n = \tau_n \mathbb{1}_{\{\tau_n<\rho_1\}} + \infty \mathbb{1}_{\{\tau_n\geq\rho_1\}} \geq \tau_n,$$

is also a localization sequence for M. To this end, fix $k \in \mathbb{N}$ and consider the process

$$\widetilde{M} = (M - M^{\tau_k})\mathbb{1}_{\{\tau_k\geq\rho_1\}}.$$

Then we have

$$\widetilde{M}^- \leq M^- + |M^{\tau_k}|;$$

hence

$$\mathsf{E}\left[\liminf_{t\uparrow\infty} \widetilde{M}_t^-\right] \leq \mathsf{E}\left[\liminf_{t\uparrow\infty} M_t^-\right] + \mathsf{E}\left[|M_\infty^{\tau_k}|\right] < \infty. \tag{2.1}$$

Next, we argue that $\widetilde{M}$ is also a local martingale, again with localization sequence $(\tau_n)_{n\in\mathbb{N}}$. Indeed, for $n \in \mathbb{N}$, $t, h \geq 0$, and $A \in \mathscr{F}_t$ note that

$$\begin{aligned}\mathsf{E}\left[\widetilde{M}_{t+h}^{\tau_n} \mathbb{1}_A\right] &= \mathsf{E}\left[\left(M_{t+h}^{\tau_n} - M_{t+h}^{\tau_n\wedge\tau_k}\right) \mathbb{1}_{A\cap\{\rho_1\leq\tau_k\leq t\}}\right] + \mathsf{E}\left[\left(M_{t+h}^{\tau_n} - M_{t+h}^{\tau_n\wedge\tau_k}\right) \mathbb{1}_{A\cap\{\rho_1\leq\tau_k\}\cap\{\tau_k>t\}}\right] \\ &= \mathsf{E}\left[\left(M_t^{\tau_n} - M_t^{\tau_n\wedge\tau_k}\right) \mathbb{1}_{A\cap\{\rho_1\leq\tau_k\leq t\}}\right] + \mathsf{E}\left[\left(M_{t+h}^{\tau_n\wedge\tau_k} - M_{t+h}^{\tau_n\wedge\tau_k}\right) \mathbb{1}_{A\cap\{\rho_1\leq\tau_k\}\cap\{\tau_k>t\}}\right] \\ &= \mathsf{E}\left[\widetilde{M}_t^{\tau_n} \mathbb{1}_A\right],\end{aligned}$$

where we used the definition of $\widetilde{M}$, $\{\rho_1 \leq \tau_k \leq t\} \in \mathscr{F}_t$, $A \cap \{\rho_1 \leq \tau_k\} \cap \{\tau_k > t\} \in \mathscr{F}_{\tau_k}$, and the martingale property of M^{τ_n}. Alternatively, we could have observed that $\widetilde{M}_\cdot =$

$\int_0^\cdot \mathbb{1}_{\{\rho_1 \le \tau_k < s\}} \mathrm{d}M_s$ (using the fact that $\mathbb{1}_{\{\rho_1 \le \tau_k\}} \mathbb{1}_{]\!]\tau_k, \infty[\![}$ is bounded and predictable since it is adapted and left-continuous). Hence, $\widetilde{M}$ is a local martingale of the form

$$\widetilde{M} = \sum_{m=2}^{N+1} \left(J_m \mathbb{1}_{\{\rho_1 \le \tau_k < \rho_m\}} \right) \mathbb{1}_{[\![\rho_m, \infty[\![},$$

satisfying (2.1), and the induction hypothesis yields that $\widetilde{M}$ is a uniformly integrable martingale. This again yields that

$$M^{\widehat{\tau}_k} = M^{\tau_k} + \widetilde{M}$$

is also a uniformly integrable martingale, proving the claim that $(\widehat{\tau}_n)_{n \in \mathbb{N}}$ is a localization sequence for M.

Step 2: We want to argue that $M_t \in \mathcal{L}^1$ for each $t \in [0, \infty]$. To this end, fix $t \in [0, \infty]$ and note

$$\begin{aligned}
\mathsf{E}[|M_t|] &\le \liminf_{n \uparrow \infty} \mathsf{E}\left[\left| M_t^{\widehat{\tau}_n} \right| \right] && (2.2)\\
&= \mathsf{E}[M_0] + 2 \liminf_{n \uparrow \infty} \mathsf{E}\left[\left(M_t^{\widehat{\tau}_n} \right)^- \right] && (2.3)\\
&\le \mathsf{E}[M_0] + 2 \liminf_{n \uparrow \infty} \mathsf{E}\left[\left(M_\infty^{\widehat{\tau}_n} \right)^- \right] && (2.4)\\
&\le \mathsf{E}[M_0] + 2\mathsf{E}[M_\infty^-] && (2.5)\\
&< \infty. && (2.6)
\end{aligned}$$

Here, the inequality in (2.2) is an application of Fatou's lemma. The equality in (2.3) relies on the fact that for any uniformly integrable martingale X we have $\mathsf{E}[|X_t|] = \mathsf{E}[X_t^+] + \mathsf{E}[X_t^-] = \mathsf{E}[X_0] + 2\mathsf{E}[X_t^-]$. The inequality in (2.4) uses that $(M^{\widehat{\tau}_n})^-$ is a uniformly integrable submartingale, thanks to Jensen's inequality, for each $n \in \mathbb{N}$. The inequality in (2.5) (which is, actually, an equality) uses the fact that $M_{\widehat{\tau}_n} \in \{0, M_\infty\}$, for each $n \in \mathbb{N}$, by construction of the localization sequence $(\widehat{\tau}_n)_{n \in \mathbb{N}}$. Finally, the inequality in (2.6) holds by assumption.

Step 3: We now argue that M is a uniformly integrable martingale. To this end, fix $t \ge 0$ and $A \in \mathscr{F}_t$. Observe that

$$\begin{aligned}
\mathsf{E}\left[M_\infty \mathbb{1}_A\right] &= \lim_{n \uparrow \infty} \left(\mathsf{E}\left[M_\infty \mathbb{1}_{A \cap \{\widehat{\tau}_n < \rho_1 < \infty\}} \right] + \mathsf{E}\left[M_\infty \mathbb{1}_{A \cap \{\widehat{\tau}_n < \rho_1\} \cap \{\rho_1 = \infty\}} \right] + \mathsf{E}\left[M_\infty \mathbb{1}_{A \cap \{\widehat{\tau}_n \ge \rho_1\}} \right] \right)\\
&= \lim_{n \uparrow \infty} \mathsf{E}\left[M_\infty^{\widehat{\tau}_n} \mathbb{1}_{A \cap \{\widehat{\tau}_n = \infty\}} \right] && (2.7)\\
&= \lim_{n \uparrow \infty} \left(\mathsf{E}\left[M_\infty^{\widehat{\tau}_n} \mathbb{1}_{A \cap \{\widehat{\tau}_n > t\}} \right] - \mathsf{E}\left[M_\infty^{\widehat{\tau}_n} \mathbb{1}_{A \cap \{t < \widehat{\tau}_n < \infty\}} \right] \right)\\
&= \lim_{n \uparrow \infty} \mathsf{E}\left[M_t^{\widehat{\tau}_n} \mathbb{1}_{A \cap \{\widehat{\tau}_n > t\}} \right] && (2.8)\\
&= \mathsf{E}\left[M_t \mathbb{1}_A \right]. && (2.9)
\end{aligned}$$

We obtained the equality in (2.7) since $\widehat{\tau}_n = \infty$ on the event $\{\widehat{\tau}_n \ge \rho_1\}$, and since the first term on the left-hand side is zero by the dominated convergence theorem and the second one thanks to the form of M. In (2.8), we used the martingale property of $M^{\widehat{\tau}_n}$ in the first term and the fact that $M_{\widehat{\tau}_n} = 0$ on the event $\{\widehat{\tau}_n < \infty\}$ in the second term, for each $n \in \mathbb{N}$. Finally, we exchanged limit and expectation in (2.9), again by an application of the dominated convergence theorem. This then concludes the proof. $\square$

3 Two examples concerning the assumptions in Theorem 1.1

Example 3.1. Assume $(\Omega, \mathscr{F}, \mathsf{P})$ allows for a sequence $(\theta_m)_{m\in\mathbb{N}}$ of independent random variables with $\mathsf{P}[\theta_1 = 2] = 1$ and $\mathsf{P}[\theta_m = -1] = 1/2 = \mathsf{P}[\theta_m = 1]$ for all $m \geq 2$. Fix families $(J_m)_{m\in\mathbb{N}}$ and $(\rho_m)_{m\in\mathbb{N}}$ of random variables with

$$J_m = 2^{m-2}\theta_m \qquad \text{and} \qquad \rho_m = (1 - 1/m)\mathbb{1}_{\bigcap_{k=2}^{m-1}\{\theta_k = 1\}} + \infty\mathbb{1}_{\bigcup_{k=2}^{m-1}\{\theta_k=-1\}}.$$

Next, define M as in (1.1) with $N = \infty$ and assume that $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by M. Then M is a local martingale, with localization sequence $(\rho_m)_{m\in\mathbb{N}}$. Indeed, M is a process that starts in one, and then, at times $1/2, 2/3, \cdots$ doubles its value or jumps to zero, each with probability $1/2$. Since it eventually jumps to zero as $\mathsf{P}[\bigcup_{m=2}^{\infty}\{\theta_m = -1\}] = 1$, we have $M_1 = 0$. In particular, M is not a true martingale, but satisfies $\mathsf{E}[M_1^-] = 0 < \infty$. Thus, the assertions of Theorem 1.1 or Corollary 1.2 are not valid if $N = \infty$, even if $\mathsf{P}[\bigcup_{m\in\mathbb{N}}\{\rho_m = \infty\}] = 1$. □

The next example illustrates that the assumptions of Corollary 1.2 are not sufficient to guarantee that M is a uniformy integrable martingale, even if there is only one jump possible, that is, even if $N = 1$. The example is adapted from [2], where it is used as a counterexample for a different conjecture.

Example 3.2. Let ρ be an $\mathbb{N} \cup \{\infty\}$–valued random variable with

$$\mathsf{P}[\rho = i] = \frac{1}{2i^2}, \qquad i \in \mathbb{N}.$$

This then yields that

$$\mathsf{P}[\rho = \infty] = 1 - \frac{\pi^2}{12}.$$

Moreover, let θ be an independent $\{-1, 1\}$ valued random variable with $\mathsf{P}[\theta = 1] = \mathsf{P}[\theta = -1] = 1/2$. Define $J = \theta\rho^2$. Then the stochastic process

$$M = J\mathbb{1}_{[\![\rho,\infty[\![},$$

along with the filtration $(\mathscr{F}_t)_{t\geq 0}$ it generates, satisfies exactly the conditions of Corollary 1.2. Indeed, ρ is an $\mathfrak{F}$–stopping time and $M_t^- \leq \rho^2\mathbb{1}_{\{\rho\leq t\}} \leq t^2$, hence $M_t^- \in \mathcal{L}^1$ for each $t \geq 0$. Thus, M is a martingale. This fact would also be very easy to check by hand.

We have $M_\infty = \lim_{t\uparrow\infty} M_t$ exists and satisfies $|M_\infty| = \rho^2\mathbb{1}_{\{\rho<\infty\}}$. Thus,

$$\mathsf{E}[|M_\infty|] = \sum_{i\in\mathbb{N}} i^2 \frac{1}{2i^2} = \infty,$$

and M cannot be a uniformly integrable martingale. □

References

[1] J. Jacod and A. N. Shiryaev, *Local martingales and the fundamental asset pricing theorems in the discrete-time case*, Finance Stoch. **2** (1998), no. 3, 259–273. MR-1809522

[2] J. Ruf, *The uniform integrability of martingales. On a question by Alexander Cherny*, Stochastic Process. Appl. **125** (2015), no. 10, 3657–3662. MR-3373298

Acknowledgments. I thank Olga Becciv, Giorgia Callegaro, Alekos Cecchin, Martino Grasselli, and Wolfgang Runggaldier for helpful discussions on the subject matter of this note. I am grateful for the generous support provided by the Oxford-Man Institute of Quantitative Finance at the University of Oxford.