B-Splines of Third Order on a Non-Uniform Grid*

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Abstract

B-splines of third order are a special representation of cubic splines and are written as weighted sum of twice differentiable basis functions consisting of four segments. They are defined via a recursive convolution formula. We present here the analytical formulas for the basis functions and list the interpolation equations for the weights in the case of natural boundary conditions. These formulas can be derived from easy computations and are well-known. However, we have not found them explicitly written out in the literature.

A thorough introduction to b-splines can be found in [1]. We work with a possibly non-equidistant grid $x_0 = x_{\min} < x_1 < \ldots < x_N = x_{\max}$ with corresponding distances $\delta(i) := x_{i+1} - x_i$. Our goal is to to interpolate the values $f(x_j)$ on these grid points with a b-spline f. In order to simplify notation, we introduce for the moment the four points $x_{-2} < x_{-1} < x_{\min}$ and $x_{N+2} > x_{N+1} > x_{\max}$ and set the corresponding distances $\delta(-2) = \delta(-1) = \delta(0)$ and accordingly, $\delta(N+1) = \delta(N) = \delta(N-1)$.

A b-spline f is defined as

$$f(x) = w_{j-1}\beta_1^{j-1}(x) + w_j\beta_0^j(x) + w_{j+1}\beta_{-1}^{j+1}(x) + w_{j+2}\beta_{-2}^{j+2}(x) \qquad (1)$$

for $x \in D_j := [x_j; x_{j+1})$ where $\{w_j\} \subset \mathbf{R}$ are weights and $\beta_{-2}^j, \beta_{-1}^j, \beta_0^j, \beta_1^j$ are the four segments of the j-th basis function β^j. More precisely,

$$b^j(x) = \beta_{-2}^j(x)\mathbf{1}_{D_{j-2}}(x) + \beta_{-1}^j(x)\mathbf{1}_{D_{j-1}}(x) + \beta_0^j(x)\mathbf{1}_{D_j}(x) + \beta_1^j(x)\mathbf{1}_{D_{j+1}}(x). \qquad (2)$$

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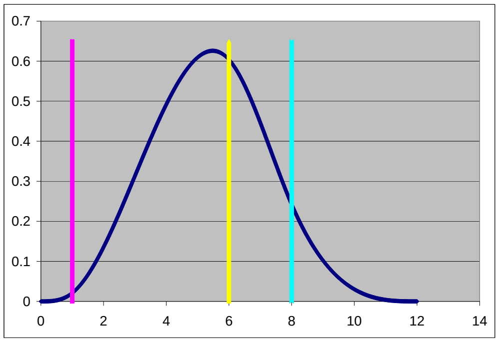

Figure 1: The b-spline basis b^2 on the non-equidistant grid $x_0 = 0$, $x_1 = 1$, $x_2 = 6$, $x_3 = 8$, $x_4 = 12$. The vertical lines indicate the positions of the inner grid points.

An advantage of b-splines is their convexity property

$$\min_{-1 \le j \le N+1} w_j \le f(x) \le \max_{-1 \le j \le N+1} w_j$$

for all $x \in [x_0; x_N]$. To wit, the weights yield lower and upper bounds for the whole spline.

Cubic b-splines are defined by iteratively convoluting lower-order b-splines (compare [1], Chapter 9, Section 13). The computations yield for the basis function b^j $(0 \le j \le N)$ of Equation (2):

$$\begin{aligned}
\beta^j_{-2}(x) &= \frac{(x - x_{j-2})^3}{\delta(j-2)(\delta(j-2) + \delta(j-1))(\delta(j-2) + \delta(j-1) + \delta(j))}, \\
\beta^j_{-1}(x) &= \frac{(x - x_{j-2})^2 (x_j - x)}{\delta(j-1)(\delta(j-2) + \delta(j-1))(\delta(j-2) + \delta(j-1) + \delta(j))} \\
&\quad + \frac{(x - x_{j-1})^2 (x_{j+2} - x)}{\delta(j-1)(\delta(j-1) + \delta(j))(\delta(j-1) + \delta(j) + \delta(j+1))} \\
&\quad + \frac{(x - x_{j-2})(x - x_{j-1})(x_{j+1} - x)}{\delta(j-1)(\delta(j-1) + \delta(j))(\delta(j-2) + \delta(j-1) + \delta(j))}, \\
\beta^j_0(x) &= \frac{(x_{j+2} - x)^2 (x - x_j)}{\delta(j)(\delta(j) + \delta(j+1))(\delta(j-1) + \delta(j) + \delta(j+1))} \\
&\quad + \frac{(x_{j+1} - x)^2 (x - x_{j-2})}{\delta(j)(\delta(j-1) + \delta(j))(\delta(j-2) + \delta(j-1) + \delta(j))} \\
&\quad + \frac{(x - x_{j-1})(x_{j+1} - x)(x_{j+2} - x)}{\delta(j)(\delta(j-1) + \delta(j))(\delta(j-1) + \delta(j) + \delta(j+1))},
\end{aligned}$$

$$\beta_1^j(x) = \frac{(x_{j+2}-x)^3}{\delta(j+1)(\delta(j)+\delta(j+1))(\delta(j-1)+\delta(j)+\delta(j+1))}.$$

The weights in Equation (1) can now be determined by solving a tridiagonal system of equations. At the interpolation points x_j $(0 \leq j \leq N)$, f takes the form

$$\begin{aligned} f(x_j) &= f_{j-1}\beta_1^{j-1}(x_j) + f_j\beta_0^j(x_j) + f_{j+1}\beta_{-1}^{j+1}(x_j) \\ &= f_{j-1}\frac{\delta(j)^2}{(\delta(j-1)+\delta(j))(\delta(j-2)+\delta(j-1)+\delta(j))} \\ &+ f_j\left(\frac{\delta(j)(\delta(j-2)+\delta(j-1))}{(\delta(j-1)+\delta(j))(\delta(j-2)+\delta(j-1)+\delta(j))}\right. \\ &+ \left.\frac{\delta(j-1)(\delta(j)+\delta(j+1))}{(\delta(j-1)+\delta(j))(\delta(j-1)+\delta(j)+\delta(j+1))}\right) \\ &+ f_{j+1}\frac{\delta(j-1)^2}{(\delta(j-1)+\delta(j))(\delta(j-1)+\delta(j)+\delta(j+1))}. \end{aligned}$$

There are $N+1$ interpolation points but $N+3$ degrees of freedom. By making the usual assumption of natural boundary conditions, that is $f''(x_0) = f''(x_N) = 0$ we get two more conditions such that we have a unique solution when solving for the unknown weights $f_0, \ldots, f_N$. More precisely, differentiating twice the b-spline f in Equation (1) with $j = 0$ at x_0 yields

$$f''(x_0) = f_{-1}\frac{1}{\delta(0)^2} + f_0\frac{-5\delta(0)-\delta(1)}{\delta(0)^2(2\delta(0)+\delta(1))} + f_1\frac{3}{\delta(0)(2\delta(0)+\delta(1))}.$$

Setting this expression equal to zero yields

$$f_{-1} = f_0\frac{5\delta(0)+\delta(1)}{2\delta(0)+\delta(1)} + f_1\frac{-3\delta(0)}{2\delta(0)+\delta(1)}.$$

Since a similar expression holds also for f_{N+1} and since $\beta_1^{-1}(x_0) = \beta_{-1}^{N+1}(x_N) = 1/6$ we can replace the first and the last equation by

$$\begin{aligned} f_0 &= f(x_0), \\ f_N &= f(x_N). \end{aligned}$$

References

[1] C. de Boor, *"A practical guide to splines"*, Revised version, Springer (2001).