

# Local and Stochastic Volatility

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## Some first remarks

- This mini course only touches on a few themes in the world of local and stochastic volatility models.
- The course puts more emphasis on models used for pricing and hedging than on models used for estimation.
- This presentation is partially based on notes by Michael Monoyios and Sergey Nadtochiy.



## Estimation: spot volatility

- Usually, the volatility is easier to estimate than the drift.
- Large amount of current research on high-frequency / *tick* data.
- In a simple setup: assume discrete price observations  $s_i = S_{i\Delta t}(\omega)$  for  $i = 0, \dots, n$  with  $n\Delta t = T$ .
- Define logarithmic returns  $r_i := \log(s_i/s_{i-1})$  for  $i = 1, \dots, n$ .
- Itô's formula yields

$$d \log(S_t) = \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dB_t.$$

- Thus,

$$R_i = \log \left( \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} \right) = \int_{(i-1)\Delta t}^{i\Delta t} \left( \mu_t - \frac{1}{2} \sigma_t^2 \right) dt + \int_{(i-1)\Delta t}^{i\Delta t} \sigma_t dB_t.$$

## Estimation: spot volatility II

- Assume, for a moment, that  $\mu_t \equiv \mu$  and  $\sigma_t \equiv \sigma$ .
- Then,

$$R_i \sim \mathcal{N} \left( \left( \mu - \frac{1}{2} \sigma^2 \right) \Delta t, \sigma^2 \Delta t \right).$$

- Thus, maximum likelihood estimator  $\hat{\sigma}^2$  of  $\sigma^2$  is, with  $\bar{r} = \sum_{i=1}^n r_i / n$ ,

$$\hat{\sigma}^2 = \frac{1}{\Delta t} \cdot \frac{1}{n} \sum_{i=1}^n (r_i - \bar{r})^2 = \frac{1}{T} \sum_{i=1}^n (r_i - \bar{r})^2.$$

- How can we find an estimator for  $\sigma_t^2$  if  $\sigma_t$  is not a constant?

## Estimation: spot volatility III

- Remember:

$$\int_0^T \sigma_t^2 dt = [\log(S)]_T.$$

- Thus,

$$\int_0^T \sigma_t^2 dt \approx \sum_{i=1}^n r_i^2,$$

where the right-hand side is the approximate quadratic variation.

- If  $\sigma_t \equiv \sigma$ , this then yields an estimator  $\tilde{\sigma}^2$  of  $\sigma^2$ :

$$\tilde{\sigma}^2 = \frac{1}{T} \sum_{i=1}^n r_i^2 \approx \frac{1}{T} \sum_{i=1}^n (r_i - \bar{r})^2 = \hat{\sigma}^2.$$

## Estimation: spot volatility IV

- General theory yields (remember, we did not allow for jumps) that

$$\sum_{i=1}^n r_i^2 \xrightarrow{\mathbb{P}} \int_0^T \sigma_t^2 dt$$

as  $\Delta t \downarrow 0$ .

- Thus, we can estimate  $\int_0^T \sigma_t^2 dt$  consistently with high-frequency data.
- However, keep in mind that model assumptions (diffusion) do not describe well super-high frequency data (ticker size, ...).
- If we know  $\int_0^T \sigma_t^2 dt$  for all  $T > 0$ , then we can determine  $\sigma_t^2$  Lebesgue-almost everywhere. (More cannot be expected.) However the integrated version seems to be the more natural quantity in any case.

## Estimation: spot volatility $V$

- Consider again the case  $\sigma_t \equiv \sigma$ .
- One then needs to decide on a choice of  $T$ , often 30 or 180 days when using daily observations.
- Development of more sophisticated methods, e.g. (exponentially weighted) moving averages (RiskMetrics)
- See notes for an example with Dow Jones data.

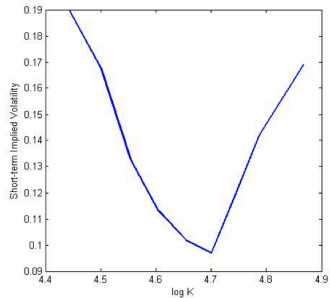
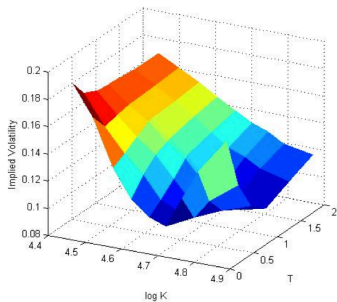
## Realized versus implied volatility

- Realized volatility estimate: based on historical data (past observations).
- Implied volatility: based on current market prices.
- Observe that BS-pricing formula (for calls and puts) implies, as a function of  $\sigma$ , an inverse function. For each price (in its range), there exists a unique  $\sigma$ , which, when put into the BS formula, yields that price.
- Given a market price  $C_t^{\text{MKT}}(T, K)$ , the implied volatility  $\Sigma_t(T, K)$  is the unique volatility, that solves

$$C_t^{\text{BS}}(\Sigma_t(T, K)^2, S_t, T, K) = C_t^{\text{MKT}}(T, K).$$

- The function  $\Sigma_t$  is called *volatility surface*.
- If BS model was correct,  $\Sigma_t(\cdot, \cdot)$  would be constant.

# Implied volatility of SP500 index options



## Black-Scholes and implied volatility

- BS assumption of constant implied volatility clearly does not hold in markets where calls and puts are liquidly traded (otherwise, implied volatilities cannot be observed).
- Graphs illustrate that implied volatilities today change as maturity and strike changes.
- Moreover, implied volatilities for fixed maturities and strikes also change over time; that is,  $\Sigma_t(T, K)$  as a function of  $t$  is not constant.

## VIX



## VIX II



## Modeling of implied volatilities

- A word of warning: In this course, we shall not model implied volatilities;
- instead, we shall model the process  $\sigma_t$ .
- Implied volatilities correspond, in some sense, to averages of future realizations of paths of  $\sigma_t$ .
- The direct modeling of implied volatilities is highly complex; in particular to check whether these models satisfy standard no-arbitrage conditions, e.g.
  - call prices are convex functions in strike  $K$ ,
  - call prices are increasing functions in maturity  $T$ .

## Stylized facts

- Volatility clustering and persistence: small price moves follow small moves, large moves follow large moves (high autocorrelation of volatility measures).
- Thick tails: distribution of asset returns have heavier tails than normal distribution (*leptokurtic distribution*).
- Negative correlation between prices and volatility: when prices go down, volatility tends to rise (*leverage effect*).
- Mean reversion: volatility tends to revert to some long-run level.

BS model does not capture these stylized facts.



## Various volatility models

- Pricing and hedging models. In increasing generality:
  - deterministic models:  $\sigma_t = \sigma(t)$  only function of  $t$ ,
  - *local volatility models*:  $\sigma_t = \sigma(t, S_t)$  function of  $t$  and  $S_t$ ,
  - stochastic volatility models: additional stochastic factors, e.g. SABR, Heston.
- Econometric models (mainly for estimation / forecasting): ARCH, GARCH, EGARCH, IGARCH, ARMA-EGARCH, ...

## Econometric models

- Usually formulated in discrete time, suitable for statistical estimation (“time series” analysis).
- Motivated by an attempt to model volatility clustering.
- Heteroskedastic = variance / volatility can change.
- *Autoregressive Conditional Heteroskedastic (ARCH)* models:  
With  $P_i$  denoting the log-price at times  $i\Delta t$ ,

$$P_i = P_{i-1} + \alpha + \eta_i \varepsilon_i,$$

where  $\alpha$  represents the trend,  $\{\varepsilon_i\}_{i \in \mathbb{N}}$  is a family of i.i.d. standard normally distributed random variables and

$$\eta_i^2 = \beta_0 + \sum_{j=1}^k \beta_j (P_{i-j} - P_{i-j-1} - \alpha)^2.$$

Thus, large observed recent price changes increase volatility of next price change.

## Econometric models II

- *Generalized ARCH (GARCH)* models depend also on past values of  $\eta_i$ .
- Usually a GARCH model needs less parameters. (ARCH models often need large  $k$  to get good fit.)
- Analytic expressions for maximum-likelihood estimators or forecasted volatilities are available.
- In EGARCH models, one distinguishes between positive and negative returns.
- For details, see for example books by Lai and Xing, *Statistical Models and Methods for Financial Markets* (Springer) or Tsay, *Analysis of Financial Time Series* (Wiley).

# Outline from now on

Pricing and hedging models:

1. deterministic models:  $\sigma_t = \sigma(t)$  only function of  $t$
2. *local volatility models*:  $\sigma_t = \sigma(t, S_t)$  function of  $t$  and  $S_t$
3. stochastic volatility models: additional stochastic factors

## Deterministic volatility models

- Simplest generalization of BS:

$$dS_t = \mu S_t dt + \sigma(t) S_t dB_t,$$

where  $\sigma(\cdot)$  is a deterministic function of  $t$ .

- Pricing and hedging of contingent claims is basically the same as in BS.
- Market is *complete* (all contingent claims can be replicated by trading in the underlying).
- Contingent claim price  $v(t, S_t)$  corresponding to terminal payoff  $h(S_T)$  usually satisfies BS PDE (assume no dividends)

$$v_t(t, s) + rsv_s(t, s) + \frac{1}{2}\sigma(t)^2 s^2 v_{s,s}(t, s) - rv(t, s) = 0,$$

$$v(T, s) = h(s).$$

## Deterministic volatility models II

- By the Feynman-Kac theorem,

$$v(t, s) = E^{\mathbb{Q}} \left[ e^{-r(T-t)} h(S_T) | S_t = s \right],$$

where  $S$  has  $\mathbb{Q}$ -dynamics

$$dS_t = rS_t dt + \sigma(t)S_t dB_t^{\mathbb{Q}}.$$

- Now, solve SDE to obtain

$$\log(S_T) = \log(S_t) + r(T-t) - \frac{1}{2} \int_t^T \sigma(u)^2 du + \int_t^T \sigma(u) dB_u^{\mathbb{Q}}$$

to observe that distribution of  $S_T$ , given  $S_t = s$ , is normal:

$$\log(S_T) \sim \mathcal{N} \left( \log(s) + \left( r - \frac{1}{2} \bar{\sigma}_t^2 \right) (T-t), \bar{\sigma}_t^2 (T-t) \right)$$

where

$$\bar{\sigma}_t^2 = \frac{1}{T-t} \int_t^T \sigma(u)^2 du.$$

## Deterministic volatility models III

- Thus, in all BS pricing formulas for European, path-independent contingent claims, just replace  $\sigma$  by  $\bar{\sigma}_t$ .
- E.g., price of a call option at time  $t$  if  $S_t = s$  is given by

$$C_t^{\text{BS}}(\bar{\sigma}_t^2, s, T, K) = s\mathcal{N}(d_1) - e^{-r(T-t)}K\mathcal{N}(d_2),$$

where

$$d_1 = \frac{\log(s/K) + (r + \bar{\sigma}_t^2)(T - t)}{\bar{\sigma}_t\sqrt{T - t}},$$

$$d_2 = d_1 - \bar{\sigma}_t\sqrt{T - t},$$

$$\bar{\sigma}_t^2 = \frac{1}{T - t} \int_t^T \sigma(u)^2 du.$$

## Definition: local volatility model

- Further generalization of BS:

$$dS_t = \mu S_t dt + \sigma(t, S_t) S_t dB_t.$$

- The deterministic function

$$(t, s) \rightarrow \sigma(t, s)$$

is called *local volatility*.

- Model is complete.
- Option price  $v(t, S_t)$  for terminal payoff  $h(S_T)$  usually satisfies BS PDE

$$v_t(t, s) + rsv_s(t, s) + \frac{1}{2}\sigma(t, s)^2 s^2 v_{s,s}(t, s) - rv(t, s) = 0,$$

$$v(T, s) = h(s).$$

## Example: Constant Elasticity of Variance (CEV) model

- Important example:

$$\sigma(t, s) = \delta s^\beta$$

for  $\delta > 0$  and usually  $\beta \leq 0$ .

- Case  $\beta > 0$  needs care (strict local martingality).
- $\beta = -1/2$ : Cox-Ingersoll-Ross / square-root process (well-known from modeling interest rates).
- $\beta < 0$  yields *leverage effect*: spot volatility increases as asset price declines.
- Generally, be cautious concerning possibly positive probability of hitting zero.
- Process is analytically tractable (including analytic formulas for barrier and lookback options).
- Can be extended (by making  $\delta$  stochastic) to SABR model.

## Another example: Quadratic Normal Volatility model

- Risk-neutral dynamics described by

$$dS_t = (aS_t^2 + bS_t + c)dB_t.$$

- Here,

$$\sigma(t, s) = as + b + \frac{c}{s}.$$

- Used to price Foreign Exchange options.
- Strict local martingality is again an issue,
- but process is analytically tractable.

## Calibration

- We shall assume that there is a liquid market for “vanilla” calls / puts.
- Then, these contingent claims are marked to market,
- and could be used as hedging instruments to price exotic contingent claims.
- E.g., vanillas written on major indices (SP500, SP100, DJ, DAX, FTSE, ...), large stocks, currencies.
- Throughout this section, we shall try to find a local volatility function  $\sigma$ , such that the model prices (expectations under the risk-neutral measure) agree with the observed market prices (*calibration*).
- In other words, we are trying to choose a certain distribution among a class of distributions.





## Kolmogorov equations

- Assume from now on that  $S$  is given by

$$dS_t = rS_t dt + \sigma(t, S_t)S_t dB_t,$$

in particular,  $S$  is Markovian, and (assume that it) has a density  $p(t, s, T, \cdot)$  for  $S_T$ , conditioned on  $S_t = s$ .

- Then  $p(t, s, T, y)$  satisfies “BS PDE”

$$-p_t(t, s, T, y) = r s p_s(t, s, T, y) + \frac{1}{2} \sigma(t, s)^2 s^2 p_{s,s}(t, s, T, y),$$

$$p(T, s, T, y) = \delta(s - y)$$

where  $\delta(\cdot)$  is Dirac delta function, satisfying

$$\int_{-\infty}^{\infty} h(y) \delta(x - y) dy = h(x).$$

- This PDE is also called *Kolmogorov backward equation*.

## Kolmogorov equations II

- Observed that  $T, y$  are constant in Kolmogorov backward equation.
- By a simple argument based on integrating by parts (see lecture notes), one can derive the following PDE for  $p(t, s, T, y)$ :

$$p_T(t, s, T, y) = -r \frac{\partial}{\partial y} (yp(t, s, T, y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} (\sigma(T, y)^2 y^2 p(t, s, T, y)),$$

$$p(t, s, t, y) = \delta(s - y)$$

- This PDE is called *Kolmogorov forward equation* or *Fokker-Planck equation*.
- Now,  $t, s$  are constant.
- This PDE is quite useful as its solution corresponds to the whole surface  $p(t, s, \cdot, \cdot)$ .



## Dupire's formula II

- Do not confuse Dupire's equation

$$\frac{\partial}{\partial T} C + ry \frac{\partial}{\partial y} C - \frac{1}{2} \sigma(T, y)^2 y^2 \frac{\partial^2}{\partial y^2} C = 0$$

with the BS equation

$$\frac{\partial}{\partial t} C + rs \frac{\partial}{\partial s} C + \frac{1}{2} \sigma(t, s)^2 s^2 \frac{\partial^2}{\partial s^2} C - rC = 0.$$

- Main advantage of Dupire's equation is that it treats call price as a function of strike and maturity.
- Dupire's formula can be used to calibrate a local volatility model to call prices.

## Dupire's formula III

Uniqueness: Given a continuum of arbitrage-free call prices there exists at most one local vol surface which calibrates them:

$$\sigma(T, K) = \sqrt{2 \frac{\frac{\partial}{\partial T} C_t(s, T, K) + rK \frac{\partial}{\partial K} C_t(s, T, K)}{K^2 \frac{\partial^2}{\partial K^2} C_t(s, T, K)}}.$$

Existence: Not obvious as the SDE

$$dS_t = rS_t dt + \sigma(t, S_t) S_t dB_t$$

needs to have a solution.

## Dupire's formula IV

- Neither Dupire nor Derman-Kani (who developed a discrete-time version) thought of local volatility as a realistic model for the evolution of actual volatility.
- Local volatility can be interpreted as a “code-book”, a translation of a call price surface, due to the one-to-one mapping of a (arbitrage-free) call price surface and a local volatility surface.
- Why do we need more complicated models? Local volatility models basically capture all marginal distributions.

## Dupire's formula and implied volatility

- Remember: Implied volatility  $\Sigma_t(T, K)$  defined via

$$C_t^{\text{BS}}(\Sigma_t(T, K)^2, s, T, K) = C_t(s, T, K).$$

- Reparameterize (dimensionless variables):

$$w(T, x) = \Sigma_t(T, se^{r(T-t)}e^x)^2(T-t).$$

- Then,

$$\sigma(T, se^{r(T-t)}e^x)^2 = \frac{w_T}{1 - \frac{x}{w}w_x + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{x^2}{w^2}\right)w_x^2 + \frac{1}{2}w_{x,x}}.$$

- For details, see for example Gatheral, The Volatility Surface (Wiley).

## Dupire's formula: drawbacks

- Requires continuum of observed call prices.
- Prices are not exactly observed due to bid-ask spreads.
- Differentiation is numerically unstable.
- Interpolations are difficult as no-arbitrage conditions have to be guaranteed.
- More generally, local volatility models do not capture the correct dynamics. (E.g., calibrate local volatility model at times  $t_0$  and  $t_1$  and observe that parameters usually change completely.)



## Inverse problems

- A problem is called an *inverse problem* if it is defined as a inverse of some other, “more explicitly” stated problem.
- E.g., instead of going from a model (here described through the local volatility  $\sigma(\cdot, \cdot)$ ) to option prices, going the other way around, from option prices to model parameters.
- A problem is called *well-posed* if
  1. a solution exists,
  2. the solution is unique, and
  3. the solution depends “continuously” on the data (input).
- Otherwise, the problem is called *ill-posed*.





## Tikhonov regularization

- Consider an abstract ill-posed inverse problem

$$F(x) = y, x \in X, y \in Y.$$

- Solution may not exist, may not be unique, or may be unstable.
- Remedy: Solve a *regularized* optimization problem instead:

$$\min_{x \in X} \|F(x) - y\|_Y^2 + \alpha G(x),$$

where  $G$  is some (convex) penalty function, e.g.,

$G(x) = \|x - x_0\|_X^2$  with  $x_0$  a “prior” guess.

- *Regularization factor*  $\alpha$  needs to be determined by cross-validation.



# Tikhonov reg. applied to local vol. calibration II

## Advantages:

- No need to have continuum of observed strikes and maturities.
- No need to interpolate market prices.
- Convex penalization leads to numerical stability.
- Calibrated surface is smooth due to choice of penalization norm.

## Disadvantages:

- Computationally demanding.
- Penalization criterion does not include weights to take into account distribution of  $S_t$ ; thus, criterion overweights values with small probability of occurrence.

# Problems of local volatility models

- Local volatility models can perfectly fit marginals (European-style path-independent options),
- but have problems with pricing path-dependent options, and
- their dynamics are not realistic.

## Formal description: stochastic volatility models

- Allow now  $\sigma_t$  to be a stochastic process:

$$dS_t = \mu(t, S_t, Y_t)S_t dt + \sigma(t, S_t, Y_t)S_t dB_t, \quad (2)$$

$$dY_t = a(t, S_t, Y_t)dt + b(t, S_t, Y_t)dW_t, \quad (3)$$

$$d\langle B, W \rangle_t = \rho dt.$$

- We can write  $W_t = \rho B_t + \sqrt{1 - \rho^2} \widehat{B}_t$ , where  $\widehat{B}$  is a BM independent of  $B$ .
- If the underlying asset (with price process  $S_t$ ) is the only hedging instrument, then the market is *incomplete* since, in general, contingent claims written on  $Y$  cannot be replicated by trading in  $S$  only.
- $\rho < 0$  corresponds to the *leverage effect*.



## Risk-neutral measures II

- Denote the class of equivalent local martingale measures by  $\mathcal{M}$ .
- Any  $\mathbb{Q} \in \mathcal{M}$  is characterized by its *stochastic discount factor* / *Radon-Nikodym derivative* with respect to  $\mathbb{P}$ :

$$Z_t^{\mathbb{Q}} := \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathcal{E}(-\lambda \cdot B - \psi \cdot \widehat{B})_t,$$

where  $\mathcal{E}$  is the *Doléans-Dade (stochastic) exponential*

$$\mathcal{E}(X)_t := e^{X_t - X_0 - \frac{1}{2}[X]_t},$$

and

$$\lambda_t := \frac{\mu(t, S_t, Y_t) - r}{\sigma(t, S_t, Y_t)},$$

and  $\psi$  is progressively measurable.

## Risk-neutral measures III

- As we have seen,  $\lambda$  is fixed by the model (representing the *market price of risk*), but  $\psi$  is an (almost) arbitrary process.
- For any  $\mathbb{Q} \in \mathcal{M}$ , we have  $\mathbb{E}[Z_T^{\mathbb{Q}}] = 1$ .
- Thus, this yields a necessary condition on  $\psi$ . For this condition to hold, Novikov's condition is sufficient:

$$\mathbb{E} \left[ e^{\frac{1}{2} \int_0^T (\lambda_t^2 + \psi_t^2) dt} \right] < \infty.$$

## Risk-neutral measures IV

- Under  $\mathbb{Q}$ , we obtain that  $B^{\mathbb{Q}}, \widehat{B}^{\mathbb{Q}}$  are independent BM with

$$B_t^{\mathbb{Q}} = B_t + \int_0^t \lambda(u, S_u, Y_u) du,$$

$$\widehat{B}_t^{\mathbb{Q}} = \widehat{B}_t + \int_0^t \psi_u du.$$

- Furthermore, we obtain the dynamics

$$dS_t = rS_t dt + \sigma(t, S_t, Y_t) S_t dB_t^{\mathbb{Q}},$$

$$dY_t = \left( a(t, S_t, Y_t) - b(t, S_t, Y_t) \left( \rho \lambda(t, S_t, Y_t) + \sqrt{1 - \rho^2} \psi_t \right) \right) dt \\ + b(t, S_t, Y_t) dW_t^{\mathbb{Q}},$$

$$d\langle B^{\mathbb{Q}}, W^{\mathbb{Q}} \rangle_t = \rho dt.$$

## Arbitrage-free prices

- Consider a claim that pays  $h(S_T, Y_T)$  at time  $T$ .
- Denote

$$C_t^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}} \left[ e^{-r(T-t)} h(S_T, Y_T) | \mathcal{F}_t \right]$$

- If  $(S, Y)$  is Markovian under  $\mathbb{Q}$ , e.g. if  $\psi_t = \psi(t, S_t, Y_t)$ , we have  $C_t^{\mathbb{Q}} = v^{\mathbb{Q}}(t, S_t, Y_t)$  for

$$v^{\mathbb{Q}}(t, s, y) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-r(T-t)} h(S_T, Y_T) | S_t = s, Y_t = y \right].$$

- Clearly,  $C^{\mathbb{Q}}$  and  $v^{\mathbb{Q}}$  depend on the choice of risk-neutral measure  $\mathbb{Q}$ .
- All possible, arbitrage-free prices of the claim with payoff  $h(S_T, Y_T)$ , are given by the interval

$$\left[ \inf_{\mathbb{Q} \in \mathcal{M}} C_t^{\mathbb{Q}}, \sup_{\mathbb{Q} \in \mathcal{M}} C_t^{\mathbb{Q}} \right].$$

## Choice of risk-free measure

- If  $\inf_{\mathbb{Q} \in \mathcal{M}} C_t^{\mathbb{Q}} = \sup_{\mathbb{Q} \in \mathcal{M}} C_t^{\mathbb{Q}}$ , then there is a unique price and the claim can be perfectly replicated, by the general theory. (e.g., usually [but not always]  $h(S_T, Y_T) = S_T$ ).
- Otherwise, distinguish two cases:
  1. Only  $S$  is a liquidly traded asset, and there are no other hedging instruments available.
  2. There is another hedging instrument available (e.g., a call).
- In case 1, we have a problem. Possible approaches:
  - Reconsider, whether the model should be changed.
  - Find a hedging strategy that minimizes risk (e.g., VaR), quadratic hedging, ...
  - Take an ad-hoc measure: *Minimal Martingale (Entropy) Measure*, ...
  - Choose the superreplicating price:  $\sup_{\mathbb{Q} \in \mathcal{M}} C_t^{\mathbb{Q}}$ . This often is a very conservative approach.

## Portfolio dynamics and lack of replication

- If an investor holds  $\Delta_t$  units of  $S$  at each time  $t$  and keeps / borrows the remainder in / from the bank account (self-financing strategy), then her wealth process  $X := \{X_t\}_{t \in [0, T]}$  satisfies

$$\begin{aligned} dX_t &= rX_t dt + \Delta_t S_t \sigma(t, S_t, Y_t) (\lambda(t, S_t, Y_t) dt + dB_t) \\ &= rX_t dt + \Delta_t S_t \sigma(t, S_t, Y_t) dB_t^{\mathbb{Q}}. \end{aligned}$$

- Price of a claim  $C_t^{\mathbb{Q}} = v^{\mathbb{Q}}(t, S_t, Y_t)$  satisfies:

$$\begin{aligned} dC_t^{\mathbb{Q}} &= (v_t^{\mathbb{Q}} + \mathcal{A}v^{\mathbb{Q}}) dt + (\sigma S_t v_s^{\mathbb{Q}} + \rho b v_y^{\mathbb{Q}}) dB_t \\ &\quad + \sqrt{1 - \rho^2} b v_y^{\mathbb{Q}} d\widehat{B}_t, \end{aligned}$$

where

$$\mathcal{A}v = \frac{1}{2} \sigma^2 s^2 v_{s,s} + \frac{1}{2} b^2 v_{y,y} + \rho \sigma s b v_{s,y} + \mu s v_s + a v_y.$$

- It is simple to see that, usually, perfect hedging with  $S$  only is impossible.

## Market completion

- Introduce another traded asset, for example, a contingent claim (that cannot be replicated by trading in  $S$  only) with payoff  $g(S_{\tilde{T}}, Y_{\tilde{T}})$  at time  $\tilde{T} \geq T$ . Denote its price at time  $t$  by  $O_t = u(t, S_t, Y_t)$ .
- Now,

$$\begin{aligned} dO_t &= (u_t + \mathcal{A}u)dt + (\sigma S_t u_s + \rho b u_y)dB_t \\ &\quad + \sqrt{1 - \rho^2} b u_y d\hat{B}_t, \\ &= (u_t + \mathcal{A}u - \lambda(\sigma S_t u_s + \rho b u_y) - \psi \sqrt{1 - \rho^2} b u_y)dt \\ &\quad + (\sigma S_t u_s + \rho b u_y)dB_t^{\mathbb{Q}} + \sqrt{1 - \rho^2} b u_y d\hat{B}_t^{\mathbb{Q}}. \end{aligned}$$

- Thus, there exists exactly one  $\psi$  (assuming  $u_y \neq 0$ ) such that the drift term equals  $ru$  under the corresponding  $\mathbb{Q}$ .
- Thus, with  $S$  and  $O$  as traded assets, there exists only one equivalent local martingale measure and thus, the market is complete.

## Market completion II

- An investor can hold  $\Delta_t$  units of  $S$ ,  $N_t$  units of  $O$ , and the bank account. Her wealth process  $X$  satisfies

$$\begin{aligned} dX_t &= \Delta_t dS_t + N_t dO_t + r(X_t - \Delta_t S_t - N_t O_t) dt \\ &= (r(X_t - N_t u) + \Delta_t \sigma \lambda S_t + N_t (u_t + \mathcal{A}u)) dt \\ &\quad + (\Delta_t \sigma S_t + N_t (\sigma S_t u_s + \rho b u_y)) dB_t + N_t \sqrt{1 - \rho^2} b u_y d\widehat{B}_t. \end{aligned}$$

- In order to hedge a contingent claim perfectly, we need a wealth process with  $X_T = h(S_T, Y_T)$ .
- Recall the dynamics of  $C^{\mathbb{Q}}$ :

$$\begin{aligned} dC_t^{\mathbb{Q}} &= (v_t^{\mathbb{Q}} + \mathcal{A}v^{\mathbb{Q}}) dt + (\sigma S_t v_s^{\mathbb{Q}} + \rho b v_y^{\mathbb{Q}}) dB_t \\ &\quad + \sqrt{1 - \rho^2} b v_y^{\mathbb{Q}} d\widehat{B}_t, \end{aligned}$$

## Market completion III

- Equating the  $d\widehat{B}$ -terms yields

$$N_t u_y = v_y^Q.$$

This component of the hedging strategy is called “Vega hedging”.

- Now, equating the  $dB$ -terms yields

$$\Delta_t = v_s^Q - \frac{v_y^Q}{u_y} u_s.$$

This component of the hedging strategy is called “Delta hedging”.

- Thus, the “volatility risk” of the claim is offset by the vega hedge and the delta hedge gets adjusted by the delta provided through the vega hedge.

## Market completion IV

- Equating now the  $dt$ -terms yields

$$\frac{u_t + \mathcal{A}u - \sigma\lambda S_t u_s - ru}{u_y} = \frac{v_t^{\mathbb{Q}} + \mathcal{A}v^{\mathbb{Q}} - \sigma\lambda S_t v_s^{\mathbb{Q}} - rv^{\mathbb{Q}}}{v_y^{\mathbb{Q}}}.$$

- Observe that the left-hand side equals

$$(\rho\lambda + \sqrt{1 - \rho^2}\psi)b.$$

- This comes from setting the drift equal to  $ru$  in

$$dO_t = (u_t + \mathcal{A}u - \lambda(\sigma S_t u_s + \rho b u_y) - \psi\sqrt{1 - \rho^2} b u_y)dt + (\sigma S_t u_s + \rho b u_y)dB_t^{\mathbb{Q}} + \sqrt{1 - \rho^2} b u_y d\hat{B}_t^{\mathbb{Q}}.$$



## Special case: zero correlation

- Assume for the moment that  $\rho = 0$ .
- Assume moreover that

$$dS_t = rS_t dt + \sigma(t, Y_t)S_t dB_t^{\mathbb{Q}},$$

$$dY_t = (a(t, Y_t) - b(t, Y_t)\psi_t) dt + b(t, Y_t)dW_t^{\mathbb{Q}},$$

$$d\langle B^{\mathbb{Q}}, W^{\mathbb{Q}} \rangle_t = 0 \times dt.$$

- A call with payoff  $h(S_T) = (S_T - K)^+$  has a price

$$\begin{aligned} v(t, s, y) &= \mathbb{E}^{\mathbb{Q}} \left[ e^{-r(T-t)} h(S_T) | S_t = s, Y_t = y \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \mathbb{E}^{\mathbb{Q}} [e^{-r(T-t)} h(S_T) | \mathcal{F}_t, \{Y_t\}_{t \geq 0}] | S_t = s, Y_t = y \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ C_t^{\text{BS}}(\bar{\sigma}_t^2, s, T, K) | Y_t = y \right], \end{aligned}$$

where

$$\bar{\sigma}_t^2 := \frac{1}{T-t} \int_t^T \sigma^2(s, Y_s) ds.$$

## Examples of stochastic volatility models

- Hull-White model:  $\rho = 0$ ,  $\mu(\cdot, \cdot, \cdot) = \mu$ ,  $\sigma(\cdot, \cdot, y) = \sqrt{y}$  and  $Y$  GBM:

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dB_t, \\dY_t &= a Y_t dt + b Y_t dW_t, \\d\langle B, W \rangle_t &= 0 \times dt.\end{aligned}$$

- Heston model:  $\mu(\cdot, \cdot, \cdot) = \mu$ ,  $\sigma(\cdot, \cdot, y) = \sqrt{y}$  and  $Y$  a square-root / Cox-Ingersoll-Ross process (mean-reverting!):

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{Y_t} S_t dB_t, \\dY_t &= a(b - Y_t) dt + b\sqrt{Y_t} dW_t, \\d\langle B, W \rangle_t &= \rho dt.\end{aligned}$$

- Standard models are mainly chosen due to their analytic tractability.



## Hedging: stochastic volatility model vs. BS

- In theory, stochastic volatility models can help in hedging the “volatility risk”, which is an improvement in comparison with the BS model.
- However, hedging ratios depend strongly upon the parameters, and are sensitive with respect to changes in parameters (recalibration).
- This is another example of an ill-posed problem.
- Often, a simple model does better since its parameters can be calibrated more efficiently, are more robust, and hedging errors can get “averaged out”.