Chapter 9

Theory of linear programming, and applications

Reading

There are many books devoted entirely to linear programming, but there is no need to consult such a specialist book for the material of this chapter, since we only cover a small portion of the topic. The following reading is recommended.

Ostaszewski, A. Advanced Mathematical Methods. Chapters 10 and 12.

Simon, C.P. and Blume, L. Mathematics for Economists. Chapter 17, sections 17.1 and 17.2; Chapter 18; Chapter 19, sections 19.1 and 19.2.

Rowcroft, J.E. Mathematical Economics: an Integrated Approach. Chapter 10, Sections 10.1 and 10.2; Chapter 19, Sections 19.1 and 19.2.

Introduction

In this chapter of the guide, we explore linear programming. This is a topic that is often introduced at a very basic level in elementary mathematics courses, since parts of it are very simple. Here, though, we shall look at some of the general theoretical aspects of linear programming. In particular, we shall consider the geometrical aspects of linear programming, and will investigate linear programming duality. We will not be studying many popular computational techniques used for linear programming, such as the simplex algorithm: such topics are covered in other subjects.
Separating and supporting hyperplanes

Before looking at linear programming itself, we shall need some geometry concerning convex sets.

Convex sets

Recall from earlier (Definition 7.1) that a subset $C$ of $\mathbb{R}^n$ is convex if for any $x, y \in C$ and for any $t \in [0, 1]$, 
$$tx + (1 - t)y \in C.$$ 
Alternatively, $C$ is convex if for all $x, y \in C$ and for all $\alpha, \beta \in [0, 1]$ satisfying $\alpha + \beta = 1$, we have $\alpha x + \beta y \in C$. In other words, a set is convex if, for any two points of the set, all points on the line segment between these two points also belong to the set.

For this chapter, we will need a few more observations about convex sets.

First, we need to observe that, given any hyperplane in $\mathbb{R}^n$, the set of points lying on either side of the plane form convex sets. More formally:

**Theorem 9.1** Suppose that a hyperplane is given in $\mathbb{R}^n$ by the equation $(x, p) = c$. Then the open half-spaces 
$$\{x : (x, p) > c\}, \{x : (x, p) < c\}$$ 
and the closed half-spaces 
$$\{x : (x, p) \geq c\}, \{x : (x, p) \leq c\}$$ 
are convex sets.

**Activity 9.1** Prove this theorem.

Next, we need to understand what is meant by a convex combination of points. We already know that a linear combination of $x_1, x_2, \ldots, x_k$ is a point of the form 
$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$ 
for some real numbers $\alpha_i$. The linear combination is said to be a convex combination if

- $\alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_k \geq 0$, and
- $\alpha_1 + \cdots + \alpha_k = 1$.

Thus, for any $x, y \in \mathbb{R}^n$ and any real number $t$ in the interval $[0, 1]$, 
$$tx + (1 - t)y$$ 
is a convex combination of $x$ and $y$. So a set $C$ is convex (by definition) if any convex combination of any two points of $C$ also belongs to $C$. The following more general statement can be proved:\footnote{See Ostaszewski, ‘Advanced Mathematical Methods’, Section 9.4 for an indication of the proof of this result.}

**Theorem 9.2** A set $C \subseteq \mathbb{R}^n$ is convex if and only if for any $x_1, x_2, \ldots, x_k \in C$, all convex combinations of $x_1, \ldots, x_k$ belong to $C$. 

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The intersection of any family of convex sets is convex, so if $S$ is any subset of $\mathbb{R}^n$, we may consider the set $C$ that is the intersection of all convex sets containing $S$. If $C'$ is any convex set containing $S$ then, certainly, $C \subseteq C'$ (since $C'$ is one of the sets that is intersected in order to obtain $C$). So $C$ is, in this sense, the ‘smallest’ convex set containing $S$. We call $C$ the convex hull of $S$ and denote it $\text{conv}(S)$.

The following alternative characterisation of the convex hull of a finite set $S$ is useful:

**Theorem 9.3** For any given finite set of points $S = \{x_1, x_2, \ldots, x_k\}$, the convex hull $\text{conv}(S)$ is precisely the set of all convex combinations of points of $S$,

$$\text{conv}(S) = \{\alpha_1 x_1 + \cdots + \alpha_k x_k : \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0, \alpha_1 + \cdots + \alpha_k = 1\}.$$

Suppose that $C$ is a convex set. Then a point $x \in C$ is an extreme point of $C$ if $x$ cannot be written as $x = ta + (1 - t)b$ for some distinct $a, b \in C$ and some $t$ in the open interval $(0, 1)$: in other words, $x$ is an extreme point if it does not lie on the line segment between any two other points of $C$. Every extreme point of $C$ lies on the boundary of $C$. Since open sets do not contain their boundary points, open convex sets have no extreme points. The only closed convex subsets of $\mathbb{R}^n$ with no extreme points are those that contain the whole of some line in $\mathbb{R}^n$: these are called closed convex cylinders. (Any closed half-space is an example of a closed convex cylinder. Note that the points on the hyperplane are not extreme points: they are merely boundary points of the set.) Extreme points play an important role in linear programming, as we shall see.

**Separating hyperplanes theorem**

An important observation about convex sets is the separating hyperplanes theorem.

Two sets $A, B$ are said to be separated by a hyperplane $H$ given by equation $(x, p) = c$ if

$$A \subseteq \{x : (x, p) \geq c\}, \quad B \subseteq \{x : (x, p) \leq c\},$$

or vice versa. That is, each set is contained in one of the closed half-spaces determined by the hyperplane.

Before presenting the theorem, we state a special case.\(^2\)

**Theorem 9.4** Suppose that $C$ is a non-empty convex set in $\mathbb{R}^n$ and that $a$ is any point of $\mathbb{R}^n$ that does not lie in the interior of $C$. (That is, there is no $\epsilon > 0$ such that $B_\epsilon(a) \subseteq C$.) Then there is a hyperplane $H$ that separates $\{a\}$ and $C$.

For an indication of the proof of this result, see Ostaszewski, ‘Advanced Mathematical Methods’, Section 10.3.

A more general Separating Hyperplanes Theorem is:

**Theorem 9.5 (Separating Hyperplanes Theorem)** Let $A, B$ be two non-empty convex sets with no points in common. Then there is a hyperplane $H$ that separates $A$ and $B$.  

\(^2\) For an indication of the proof of this result and the next one, see Ostaszewski, ‘Advanced Mathematical Methods’, Section 10.3.
Supporting hyperplanes

Suppose that $C$ is a non-empty convex set in $\mathbb{R}^n$ and that $a$ is a boundary point of $H$. Then $a$ is not in the interior of $C$ and so, by Theorem 9.4, there is a hyperplane $H$ such that $C$ lies entirely in one of the two closed half-spaces determined by $H$. Such a hyperplane is called a supporting hyperplane at $a$ and the closed half-space in which $C$ lies is a supporting half-space. (If $H$ is unique with respect to this property, it is the tangential hyperplane at $a$.)

An important property of supporting hyperplanes is the following.

**Theorem 9.6** Suppose that $C$ is a closed convex set that is not a closed convex cyclinder. Then any supporting hyperplane passes through an extreme point of $C$.

**Linear programming in general**

Consider the following maximisation problem:

$$\text{maximise } x_1 + x_2 + 2x_3$$

subject to:

$$2x_1 + 3x_2 + 2x_3 \leq 8$$
$$10x_1 + 5x_2 + 2x_3 \leq 20$$
$$3x_1 + 6x_2 + x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0.$$ 

This is a constrained optimisation problem with three inequality constraints and a full set of non-negativity constraints. It would be possible to use Lagrangian or Kuhn-Tucker methods, but there is something very special about this optimisation problem that enables us to apply a whole range of other ideas and techniques. It is linear, in the sense that the objective function is a linear function of the variables $x_1, x_2, x_3$, and, in the constraints $h_i(x) \leq c$, all the functions $h$ are linear too. This maximisation problem is an example of a linear programming problem. Such problems can be written in matrix form. To do this, we need a convenient notational shorthand: to say, for $x, y \in \mathbb{R}^n$, that $x \leq y$ means that $x_i \leq y_i$ for $i = 1, 2, \ldots, n$ (and $x \geq y$ is defined in the obvious similar way). This given problem can be expressed as

$$\text{maximise } \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

subject to:

$$\begin{pmatrix} 2 & 3 & 2 \\ 10 & 5 & 2 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 20 \\ 12 \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

That is, the problem is:

$$\text{maximise } c^T x$$

subject to $Ax \leq b, \quad x \geq 0,$

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where
\[
\begin{pmatrix}
1 \\
1 \\
2
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 3 & 2 \\
10 & 5 & 2 \\
3 & 6 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
8 \\
20 \\
12
\end{pmatrix}.
\]

The basic linear program

The prototypical linear programming problem, the so-called **primal problem** is:

\[
\text{maximise } c^T x
\quad \text{subject to } A x \leq b, \ x \geq 0.
\]

Many other linear programming problems can be expressed this way. (For example, a minimisation is equivalent to a maximisation problem with the negative of the original objective function.)

Location of optimal solutions

The feasible region of the linear programming problem is the set of all \( x \) satisfying the constraints,

\[
F = \{ x : A x \leq b, \ x \geq 0 \}.
\]

Any point in \( F \) is referred to as a feasible solution. A point \( x^* \in F \) is an optimal solution if \( c^T x^* \) is the maximum value of \( c^T x \) subject to the constraints. Now, suppose that the rows of \( A \) are \( p_1^T, p_2^T, \ldots, p_m^T \). Then

\[
F = \{ x : p_i^T x \leq b_i, \ (i = 1, \ldots, m), \ x \geq 0 \}.
\]

So \( F \) is the intersection of the following \( m + n \) closed half-spaces:

\[
\langle x, p_1 \rangle \leq b_1, \ldots, \langle x, p_m \rangle \leq b_m, x_1 \geq 0, \ldots, x_n \geq 0.
\]

The intersection of any collection of convex sets is convex, and the intersection of a finite collection of closed sets is closed, so \( F \) is a closed convex subset of \( \mathbb{R}^n \). Furthermore, \( F \) is not a closed convex cylinder. (It cannot contain the whole of a line since it lies in the positive orthant \( \{ x : x \geq 0 \} \).) Therefore \( F \) has at least one extreme point. The equation \( \langle c, x \rangle = \lambda \) represents a hyperplane on which the value of the objective function is \( \lambda \). To solve the maximisation problem, we want the largest value \( \lambda^* \) such that \( \langle c, x \rangle = \lambda^* \) intersects \( F \). Assume that such a \( \lambda^* \) exists (for otherwise there is no maximum value of the objective function subject to the constraints). An optimal solution \( x^* \) cannot be in the interior of \( F \). For we would then be able to move the hyperplane \( \langle c, x \rangle = \lambda^* \) passing through \( x^* \) further in the direction of \( c \), and hence increase the objective function \( \langle c, x \rangle \). So the plane \( \langle c, x \rangle = \lambda^* \) must be a supporting hyperplane at \( x^* \) and hence it passes through some extreme point of \( F \). We therefore have the following important result.

**Theorem 9.7** The optimal value of the primal linear programming problem (if there is one) is obtained at some extreme point of the feasible region.

As an intersection of a finite number of closed half-spaces, the feasible region has a finite number of extreme points, so in theory one way to solve the linear program is simply to determine the extreme points and calculate the objective function at each,
checking which gives the largest value. This works well enough for small problems, but is extremely time-consuming for larger problems, and for that reason a number of sophisticated methods have been developed for solving linear programming problems. We shall not discuss these in this subject, but we give a very simple example to show how checking the extreme points in a small problem gives the result.

**Example:** Suppose we want to maximise \(2x + 3y\) subject to the constraints

\[
\begin{align*}
2x + y &\leq 140 \\
x + y &\leq 80 \\
x + 3y &\leq 180 \\
x &\geq 0 \\
y &\geq 0.
\end{align*}
\]

Elementary methods may be used to solve such a two-dimensional (i.e. two-variable) linear programme. A sketch of the feasible region, \(F\), is useful. The following figure shows the feasible region \(F\). It is easiest, and most useful, to draw the lines by determining where they cross the axes. For example, to draw the line \(2x + y = 140\), we observe that it crosses the \(x\)-axis when \(y = 0\), so that \(2x = 140\); that is, \(x = 70\).

Note that the inequalities \(x \geq 0\) and \(y \geq 0\) mean that the region \(F\) lies ‘above’ the \(x\)-axis and to the right of the \(y\)-axis. The extreme points are the ‘corners’ of the feasible region. These have been labelled \(O, A, B, C, D\) in the figure. To find the maximum value of \(2x + 3y\) subject to the given constraints, we can find all the extreme points \(O, A, B, C, D\) and calculate the value of the function at each point. To calculate the extreme points \(O, A, B, C, D\) in our example is easy. Point \(O\) is the origin \((0, 0)\). Point \(A\) is where the line \(x + 3y = 180\) crosses the \(y\)-axis, which is \((0, 60)\). Point \(D\) is where \(2x + y = 140\) crosses the \(x\)-axis, which is \((70, 0)\). Point \(B\) is at the intersection of the two lines

\[
\begin{align*}
x + y &= 80 \\
x + 3y &= 180.
\end{align*}
\]

Subtracting the first equation from the second, we obtain

\[
(x + 3y) - (x + y) = 180 - 80,
\]

\[
x = 70.
\]

Therefore the coordinates of \(B\) are found by solving the simultaneous equations

\[
\begin{align*}
x + y &= 80 \\
x + 3y &= 180.
\end{align*}
\]

Subtracting the first equation from the second, we obtain

\[
(x + 3y) - (x + y) = 180 - 80,
\]

\[
x = 70.
\]

\[
x + y = 80,
\]

\[
x + 3y = 180.
\]
which is $2y = 100$, so $y = 50$. Then, using the first equation, $x + 50 = 80$, so $x = 30$ and $B$ is $(30, 50)$. For point $C$, we observe that it is where the lines $x + y = 80$ and $2x + y = 140$ meet, so we solve the simultaneous equations

\[
\begin{align*}
x + y &= 80 \\
2x + y &= 140.
\end{align*}
\]

Subtracting the first equation from the second gives $x = 60$, so $y = 20$ and $C = (60, 20)$. Now that we have found the exact coordinates of the extreme points of $R$, we calculate the value of the objective function $2x + 3y$ at each of the extreme points.

<table>
<thead>
<tr>
<th>extreme point</th>
<th>$2x + 3y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O = (0, 0)$</td>
<td>0</td>
</tr>
<tr>
<td>$A = (0, 60)$</td>
<td>180</td>
</tr>
<tr>
<td>$B = (30, 50)$</td>
<td>210</td>
</tr>
<tr>
<td>$C = (60, 20)$</td>
<td>180</td>
</tr>
<tr>
<td>$D = (70, 0)$</td>
<td>140</td>
</tr>
</tbody>
</table>

We see that the maximum value is achieved at $B$, and the maximal value is 210.

**Linear programming duality**

Duality theory lies at the heart of the theory of linear programming, and has many applications. We have already seen that the standard primal problem takes the form

\[
\text{maximise } c^T x \\
\text{subject to } Ax \leq b, \ x \geq 0.
\]

The **dual problem** is defined to be the following minimisation problem:

\[
\text{minimise } b^T y \\
\text{subject to } A^T y \geq c, \ y \geq 0.
\]

Note that if the primal problem has $n$ variables and $m$ constraints other than the non-negativity constraints, then the dual has $m$ variables and $n$ constraints other than the non-negativity constraints. The central connection between the primal and dual problems is as follows.

**Theorem 9.8 (Duality theorem)** Suppose that both the primal and dual problems have feasible solutions. Then both have optimal solutions and the optimal value $c^T x^*$ of the primal is equal to the optimal value of the dual; that is,

\[
\max\{c^T x : Ax \leq b, \ x \geq 0\} = \min\{b^T y : A^T y \geq c, \ y \geq 0\}.
\]

Thus, a primal linear program may be solved by solving the dual problem and, conversely, a dual problem may be solved by solving the corresponding primal problem. It’s important to understand that dualisation works both ways: the dual of the primal problem is the dual problem, and the dual of the dual problem is the primal problem.

**Example:** Let us determine the dual of the following linear programme.

Minimise $M(u, v, w) = 140u + 180v + 80w$
subject to:

\[ 2u + v + w \geq 2 \]
\[ u + 3v + w \geq 3 \]
\[ u, v, w \geq 0. \]

We shall find the minimum value of \( M(u, v, w) \) by solving the dual of the program.

As a first step, we write the problem in matrix form. The problem is to minimise

\[
\begin{pmatrix}
140 & 180 & 80
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\]

subject to

\[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 3 & 1
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\geq
\begin{pmatrix}2 \\ 3
\end{pmatrix},
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\geq
\begin{pmatrix}0 \\ 0
\end{pmatrix}.
\]

The dual is:

maximise \( (2, 3) \begin{pmatrix}x \\ y
\end{pmatrix} \)

subject to:

\[
\begin{pmatrix}
2 & 1 \\
1 & 3 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\leq
\begin{pmatrix}140 \\ 180 \\ 80
\end{pmatrix},
\begin{pmatrix}
x \\
y
\end{pmatrix}
\geq
\begin{pmatrix}0 \\ 0
\end{pmatrix}.
\]

That is, the dual is: maximise \( 2x + 3y \) subject to the constraints

\[
2x + y \leq 140 \\
x + y \leq 80 \\
x + 3y \leq 180 \\
x \geq 0 \\
y \geq 0.
\]

We solved this problem above, and we found that the maximum is 210. So the solution of the original problem (that is, the required minimum value) is also 210.

Games

Zero-sum games

Game Theory is a very large area of interest to economists and mathematicians. We shall only be able to touch on it in this subject.\(^3\) Our main interest will be finite matrix games, but we start with some generalities about \textbf{two-person, zero-sum games}. For a game of this type, there are two players, I and II, each of whom has a \textbf{strategy set} of possible strategies. Player I chooses a strategy \( x \) from his set \( X \) and player II chooses a strategy \( y \) from her set \( Y \). Then, player I receives a reward or \textbf{pay-off} \( K(x, y) \), where \( K \) is called the \textbf{pay-off function}. Player II’s pay-off is \(-K(x, y)\). (The pay-offs to the players sum to zero, and for this reason it is called a zero-sum game: player I’s gain is player II’s loss.) A pair \( (x^*, y^*) \) of strategies (where \( x^* \in X \) and \( y^* \in Y \)) is called an \textbf{equilibrium point} if for all \( x \in X \) and \( y \in Y \),

\[
K(x, y^*) \leq K(x^*, y^*) \leq K(x^*, y).
\]

\(^3\) See Ostaszewski, ‘Advanced Mathematical Methods’, Sections 12.5 and 12.6 for more details.
If an equilibrium point exists, then neither player can do better than to play their equilibrium strategy. For example, by playing \( x^* \), Player I ensures a pay-off of at least \( K(x^*, y^*) \), whatever Player II does. On the other hand, Player II can hold the pay-off to \( K(x^*, y^*) \) by playing \( y^* \), so no move of Player I can improve on \( x^* \). The pay-off \( K(x^*, y^*) \) of the optimal strategies is known as the value of the game.

Matrix games

The preceding discussion was fairly general. We now focus on matrix games. Such a game is defined by an \( m \times n \) pay-off matrix \( A = (a_{ij}) \). We assume that each entry of the matrix is non-negative. The strategy sets in such a game are defined in terms of fundamental pure strategies. Player I has \( m \) pure strategies and Player II has \( n \) pure strategies, and if I plays his \( i \)th pure strategy and II her \( j \)th, then the pay-off to player I is \( a_{ij} \) and the pay-off to player II is \( -a_{ij} \) (this being a zero-sum game). The strategy set for player I is the set of all convex combinations of his \( m \) pure strategies: each strategy is represented by a vector \( p = (p_1, \ldots, p_m)^T \) where \( p_i \geq 0 \) and \( p_1 + \cdots + p_m = 1 \). (The strategy \( p \) may be thought of as describing the strategy in which I plays his \( i \)th pure strategy with probability \( p_i \).) Similarly, the strategies for player II consist of all vectors \( q = (q_1, \ldots, q_n)^T \) where \( q_i \geq 0 \) and \( q_1 + \cdots + q_n = 1 \).

The pay-off to player I when he plays \( p \) and II plays \( q \) is defined to be

\[
K(p, q) = p^T A q = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} p_i q_j,
\]

the pay-off to II being the negative of this. (If we interpret the \( p_i \) and \( q_i \) as probabilities of playing respective pure strategies, this may be seen as the expected pay-off to player I when the game is repeatedly played and the players play their pure strategies with the probabilities described by \( p \) and \( q \).

Existence of optimal mixed strategies

If the pair of mixed strategies \((p, q)\) forms an equilibrium for the matrix game, we say that \( p \) is an optimal mixed strategy for player I and that \( q \) is an optimal mixed strategy for player II. We can use linear programming duality to show that optimal mixed strategies exist (that is, that the matrix game has an equilibrium).

First, we note that Player I wants to maximise pay-off, whereas Player II wants to minimise it (since her loss is I’s pay-off, this being a zero-sum game). Player I can ensure (no matter what strategy II plays) that his pay-off will be at least \( \lambda \) if there is a strategy \( p \) such that

\[
A^T p \geq \lambda e, \quad e^T p = 1, \quad p \geq 0,
\]

where \( e \) is the all-1 vector. To see this, note that the last two of these three conditions simply say that \( p \) is a mixed strategy (as defined earlier), and that if \( p \) satisfies the first condition then for any mixed strategy \( q \) for II, noting that \( e^T q = 1 \), we have

\[
K(p, q) = p^T A q = (A^T p)^T q \geq (\lambda e)^T q = \lambda e^T q = \lambda.
\]

The fact that the matrix \( A \) has non-negative entries means that there is a \( \lambda > 0 \) for which these conditions hold. An alternative formulation is (taking \( y = p/\lambda \)) that I wants to solve the following linear program:\(^4\)

\[
\text{minimise } e^T y, \quad \text{subject to } A^T y \geq e, \quad y \geq 0. \tag{9.1}
\]

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See Ostaszewski, ‘Advanced Mathematical Methods, Section 12.6, for details.
One can argue similarly that Player II wants to minimise $\mu$ subject to

$$Aq \leq \mu e, \quad e^T q = 1, \quad q \geq 0,$$

and again (setting $x = q/\mu$) this has an alternative formulation:

$$\text{maximise } e^T x, \text{ subject to } Ax \leq e, \quad x \geq 0. \quad (9.2)$$

Now, linear programs (9.1) and (9.2) are dual, and both are feasible, so the duality theorem establishes that both have optimal solutions $x^*$ and $y^*$ and that if $\lambda^*$ and $\mu^*$ are the optimal values of $\lambda, \mu$, then

$$\frac{1}{\lambda^*} = e^T x^* = e^T y^* = \frac{1}{\mu^*}.$$

This means that the strategies $p = \lambda^* y^*$ and $q = \mu^* x^*$ together give an equilibrium of the game. (An alternative proof of the existence of an equilibrium can be developed using separating hyperplanes theory.5)

Finding the optimal mixed strategies geometrically

Player I wants to maximise $\lambda$ such that there is $p$ as follows:

$$A^T p \geq \lambda e, \quad e^T p = 1, \quad p \geq 0.$$

Let

$$U = \{A^T p : p \geq 0, \; e^T p = 1\}.$$

Note that $U$ is precisely the set of all convex combinations of the rows of $A$; that is, $U$ is the convex hull of the rows of $A$. Then I wants to maximise $\lambda$ such that

$$K(\lambda) \cap U \neq \emptyset,$$

where

$$K(\lambda) = \{x : x \geq \lambda e\} = \lambda e + \{z : z \geq 0\}$$

is the translate by $(\lambda, \lambda, \ldots, \lambda)$ of the non-negative orthant $\{z : z \geq 0\}$. These observations help us solve small matrix games geometrically. We give an example.

**Example:** We find the value and optimal mixed strategies of the matrix game with pay-off matrix

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}.$$

The set $U = \{A^T p : p \geq 0, \; e^T p = 1\}$ is the convex hull of the rows of $A$, so it is the line segment between $(2, 3)$ and $(5, 1)$. This is pictured below.
If $\lambda^*$ is the value of the game, then it is the maximum value of $\lambda$ for which $K(\lambda)$ intersects $U$. As $\lambda$ is increased, the region $K(\lambda)$ moves north-east, so the optimal $\lambda^*$ will be such that $K(\lambda^*)$ just touches the line segment. (See the figure.) The point of contact must be the ‘corner’ $(\lambda^*, \lambda^*)$ of $K(\lambda^*)$. So we need to find the value of $\lambda$ such that $(\lambda, \lambda)$ lies on the line segment. The equation of the line segment is easily calculated to be $y = -(2/3)x + 13/3$, so we should have $\lambda^* = -(2/3)\lambda^* + 13/3$, so $\lambda^* = 13/5$. The theory above ensures that the optimal value $\mu^*$ is the same. The value of the game is therefore $13/5$. To find the optimal mixed strategies $p^*, q^*$, we have to solve the equations

$$A^T p^* = \lambda^* e, \quad Aq^* = \mu^* e.$$ 

The first of these is

$$\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} p^* = \frac{13}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

Taking the inverse of the matrix, we obtain

$$p^* = \frac{13}{5} \begin{pmatrix} -1 \\ -3 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 1/5 \end{pmatrix}.$$ 

Similarly, we obtain

$$q^* = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}.$$ 

Note that $p^*, q^*$ are indeed mixed strategies: they each have non-negative entries summing to 1. (The result can be interpreted as saying that I should play row 1 with probability $4/5$ and row 2 with probability $1/5$, and that II should play column 1 with probability $2/5$ and column 2 with probability $3/5$.

Sometimes the optimal mixed strategies are in fact pure strategies, and it is often easy to spot when this is the case. A pure strategy for I corresponds to choosing a row of the matrix. If it happens that some row dominates all others in the sense that its $j$th entry is at least as large as the $j$th entry of every other row, then I will certainly find it optimal to play the pure strategy of choosing that row. For example, suppose that the matrix describing the game is

$$\begin{pmatrix} 2 & 0 & 1 \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{pmatrix}.$$
Row 2 dominates the other two rows. No matter what strategy (mixed or pure) II would play, the pay-off to I is largest if he plays the pure strategy \((0, 1, 0)^T\): that is, if he chooses the second row. Given that he will do so, what then will II do? Recall that II wants to minimise the pay-off to I (since this is equal to the loss she incurs), so given that row 2 is played, II will play the pure strategy of playing the third column (since this gives the smallest entry of the second row). So the optimal strategies are the pure strategies \(p = (0, 1, 0)^T\) and \(q = (0, 0, 1)^T\).

**Learning outcomes**

At the end of this chapter and the relevant reading, you should be able to:

- explain what is meant by convex combination and convex hull
- use facts about convex sets
- describe the separating hyperplanes theorem and what is meant by a supporting hyperplane
- explain what is meant by the primal and dual linear programming problems
- state and use the duality theorem of linear programming
- explain what is meant by optimal strategies for, and the value of, a two-person, zero-sum game
- solve matrix games.

**Sample examination questions**

The following are typical exam questions, or parts of questions.

**Question 9.1** If \(C\) is any convex subset of \(\mathbb{R}^n\) and \(T\) is any linear transformation from \(\mathbb{R}^n\) to \(\mathbb{R}^n\), let \(T(C) = \{T(c) : c \in C\}\). Prove that \(T(C) \cap C\) is convex.

**Question 9.2** Suppose we have the linear programming problem of minimising

\[
M(x_1, x_2, x_3) = 36x_1 + 30x_2 + 40x_3
\]

subject to the constraints \(x_1, x_2, x_3 \geq 0\) and

\[
\begin{align*}
6x_1 + 3x_2 + 2x_3 & \geq 5 \\
2x_1 + 5x_2 + 8x_3 & \geq 4.
\end{align*}
\]

Find the dual of the problem. By solving the dual problem, find the minimum value of \(M(x_1, x_2, x_3)\) subject to the constraints of the original linear program.

**Question 9.3** Consider the problem of maximising \(c^Tz\) subject to the equality constraint \(Bz = d\) and the constraint \(z \geq 0\). Express this in the standard form

\[
\text{maximise } c^Tx \text{ subject to } Ax \leq b, \ x \geq 0.
\]
for suitable $A, b$. Show that the dual is equivalent to the following problem:

$$\text{minimise } d^T y \text{ subject to } B^T y \geq c.$$ 

**Question 9.4** Find the value and optimal mixed strategies of the matrix game with pay-off matrix

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}.$$

**Question 9.5** Find the value of and optimal mixed strategies for the game with pay-off matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 3 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix}$$

**Sketch answers or comments on selected questions**

**Question 9.1** We first show that $T(C)$ is convex. Suppose $a, b \in T(C)$ and $t \in [0, 1]$. There are $a_1, b_1 \in C$ such that $a = T(a_1)$ and $b = T(b_1)$. Then, because $T$ is linear,

$$tT(a_1) + (1 - t)T(b_1) = T(ta_1 + (1 - t)b_1).$$

But $c = ta_1 + (1 - t)b_1 \in C$ since $C$ is convex (and $a_1, b_1 \in C$). So $T(C)$ is convex. Since the intersection of any two convex sets is again convex, it follows that $T(C) \cap C$ is convex.

**Question 9.2** In matrix form, the program is:

$$\text{minimise } (36, 30, 40)x$$

subject to

$$\begin{pmatrix} 6 & 3 & 2 \\ 2 & 5 & 8 \end{pmatrix} x \geq \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad x \geq 0.$$

The dual of this is:

$$\text{maximise } (5, 4)y$$

subject to

$$\begin{pmatrix} 6 & 2 \\ 3 & 5 \\ 2 & 8 \end{pmatrix} y \leq \begin{pmatrix} 36 \\ 30 \\ 40 \end{pmatrix}, \quad y \geq 0.$$

In other words, the dual is:

$$\text{maximise } 5y_1 + 4y_2$$

subject to

$$\begin{align*}
6y_1 + 2y_2 &\leq 36 \\
3y_1 + 5y_2 &\leq 30 \\
2y_1 + 8y_2 &\leq 40 \\
y_1, y_2 &\geq 0.
\end{align*}$$

This is a two-variable problem, which can be solved, as in the example in this chapter, by sketching the feasible region, determining the extreme points and calculating the objective at each. The details are omitted, but you should find that the maximum value is 37 (obtained at the point $(5, 3)$). Thus the answer to the original minimisation problem is also 37.
Question 9.3 The single constraint $Bx = d$ is equivalent to the two inequality constraints $Bx \leq d$ and $Bx \geq d$, which may be written as

$$Bx \leq d, \quad (\neg B)x \leq -d$$

or

$$\begin{pmatrix} B \\ -B \end{pmatrix} x \leq \begin{pmatrix} d \\ -d \end{pmatrix}.$$ 

So the problem is equivalent to:

$$\text{maximise } c^T x \text{ subject to } Ax \leq b, \ x \geq 0,$$

where

$$A = \begin{pmatrix} B \\ -B \end{pmatrix}, \quad b = \begin{pmatrix} d \\ -d \end{pmatrix}.$$ 

The dual of this is:

$$\text{minimise } (d^T, -d^T) Y \text{ subject to } (B^T - B^T) Y \geq c, Y \geq 0.$$ 

Suppose that $B$ has $m$ rows and write

$$Y = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $u, v$ are $m$-vectors. Then this problem is:

$$\text{minimise } d^T (u - v) \text{ subject to } B^T (u - v) \geq c, u, v \geq 0.$$ 

Now, any vector $y$ can be written in the form $y = u - v$ for some $u, v \geq 0$, so the problem is entirely equivalent to:

$$\text{minimise } d^T y \text{ subject to } B^T y \geq c.$$ 

(There is no constraint of the form $y \geq 0$.)

Question 9.4 The value of the game is $3/5$, and the optimal mixed strategies are $p = (1/5, 4/5)^T$ and $q = (2/5, 3/5)^T$.

Question 9.5 The third row dominates the other two, so player I’s optimal strategy is $(0, 0, 1)^T$. Given that I plays the (pure) strategy of choosing the second row, Player II will then choose to play the second column, so II’s optimal mixed strategy is $(0, 1, 0)^T$, and the value of the game is 1.