π OPTIONS*

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Abstract

We consider a discretionary stopping problem that arises in the context of pricing a class of perpetual American-type call options, which include the perpetual American, Russian and lookback-American call options as special cases. We solve this genuinely two-dimensional optimal stopping problem by means of an explicit construction of its value function. In particular, we fully characterise the free-boundary that provides the optimal strategy, and which involves the analysis of a highly non-linear ordinary differential equation (ODE). In accordance with other optimal stopping problems involving a running maximum process that have been studied in the literature, it turns out that the associated variational inequality has an uncountable set of solutions that satisfy the so-called principle of smooth fit.

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1 Introduction

We denote by X the geometric Brownian motion given by

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x > 0, \tag{1}$$

for some constants μ and σ, where W is a standard one-dimensional Brownian motion. Also, given a point $s \geq x$, we denote by S the running maximum process defined by

$$S_t = \max \left\{ s, \max_{0 \leq u \leq t} X_u \right\}. \tag{2}$$

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In this context, we consider the discretionary stopping problem whose value function is defined by

$$v(x,s) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}\left(X_\tau^a S_\tau^b - K\right)^+ \mathbf{1}_{\{\tau<\infty\}}\right], \tag{3}$$

for some constants $r > 0$ and $a, b, K \geq 0$, where $\mathcal{T}$ is the set of all stopping times.

Special cases of the discretionary stopping problem defined by (1)–(3) include the optimal stopping problems arising in the context of pricing the well-known perpetual American call option ($a = 1$, $b = 0$ and $K > 0$), which was solved by McKean [McK65] (in fact, he solved the perpetual American put option), the Russian option introduced by Shepp and Shiryaev [SS93, SS94] ($a = 0$, $b = 1$ and $K = 0$), and the lookback-American option studied by Pedersen [P00] and Guo and Shepp [GS01] ($a = 0$, $b = 1$ and $K > 0$). Our analysis focuses on the generic case in which $a, b, K > 0$ because, at the parameter limits giving rise to the special cases mentioned above, the continuation and the stopping regions that characterise the optimal strategy take qualitatively different forms. However, we use analytic arguments to show that the continuation and the stopping regions of the problem we focus on do transform continuously to the corresponding regions arising in the context of these special cases when the parameters a, b and K tend to the corresponding limits.

The problem (1)–(3) is also related with optimisation problems involving the so-called percentage or maximum drawdown. The percentage drawdown XS^{-1} is associated with risk measures that are useful for the quantification of portfolio performance by fund managers. Relevant contributions include Cvitanic and Karatzas [CK94], Magdon-Ismail and Atiya [MA04], Chekhlov, Urgasev and Zabarankin [CUZ05], Carr [C06], Vecer [V06], and references therein.

Optimal stopping has a well-developed body of theory, which has been documented, e.g., in Shiryaev [S08], Krylov [K80], El Karoui [EK81], Bensoussan and Lions [BL82], and Peskir and Shiryaev [PS06]. The vast majority of the problems that admit closed form analytic solutions consists of problems that can be associated with one-dimensional variational inequalities. This paper provides a new addition to the class of genuinely two-dimensional explicitly solvable problems that involve the running maximum process of a one-dimensional diffusion; see Jacka [J91], Dubins, Shepp and Shiryaev [DSS93], Graversen and Peskir [GP98], Peskir [P98], Pedersen [P00], Guo and Shepp [GS01], Peskir and Shiryaev [PS06, Section 13], Hobson [H07], Obloj [O07], and Cox, Hobson and Obloj [CHO08] for relevant references.

An interesting feature of the problem that we solve is that its associated variational inequality has uncountably many solutions that satisfy the so-called principle of smooth fit (see Lemma 4 below). In fact, this feature has been observed in the solution of other optimal stopping problems involving the running maximum process. To address this situation, our analysis relies in a highly non-trivial way on the use of the so-called transversality condition. In particular (see Lemma 2 below), we identify the value function v with the unique solution w of the associated variational inequality that satisfies an appropriate growth condition and

$$\liminf_{T\to\infty} \mathbb{E}\left[e^{-rT} w(X_T, S_T)\right] = 0. \tag{4}$$

In general, the transversality condition can be viewed as a growth condition that can be used as an appropriate boundary condition for the Hamilton-Jacobi-Bellman equations of infinite time horizon problems with unbounded state space domains. In the literature, it appears in various forms, similar to the one in (4), and it figures among the assumptions of standard verification theorems concerned with such problems (e.g., see Fleming and Soner [FS93, Theorems IV.5.1, VII.4.1], and Øksendal and Sulem [ØS07, Theorems 2.2, 3.1, 4.2, 5.2, 6.2, 8.1] for stochastic control problems, and Øksendal [Ø03, Theorem 10.4.1] for optimal stopping problems).

In the context of a class of optimal stopping problems involving the running maximum of a one-dimensional Itô diffusion, Peskir [P98] identified uniquely the value function by means of a technique that he termed as the maximality principle. Also, Pedersen [P00] verified that Peskir's maximality principle can be used to solve the special case of the optimal stopping problem (1)–(3) that arises when $a = 0$, $b = 1$ and $K > 0$. Although the general problem (1)–(3) does not belong to the above class of problems, it turns out that its solution exhibits the pattern suggested by Peskir's maximality principle. Indeed, a subset of the free-boundary that characterises the solution of (1)–(3) is obtained by solving a given ODE that is parametrised by appropriate choices of its initial condition. In accordance with the maximality principle, the correct initial condition can be identified as the one that corresponds to the maximal solution of the ODE that does not hit the diagonal of $\mathbb{R}_+^2$ (see Figure 2 and Lemma 3.(I)). In the context of the problem that we solve here, our analysis goes a step further. Indeed, we establish precise asymptotics for all solutions of the relevant ODE that do not hit the diagonal of $\mathbb{R}_+^2$.

The paper is organised as follows. Section 2 is concerned with the problem formulation and some preliminary issues. In Section 3, we prove a verification theorem, the assumptions of which are tailored to fit the solution of our problem rather than aspire to maximal generality. We solve the discretionary problem in Section 4. Finally, we consider a number of limiting cases that arise in the context of pricing perpetual American, Russian and lookback-American call options in Section 5.

2 The optimal stopping problem

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by the $\mathbb{P}$-negligible sets, and carrying a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{T}$ the set of all $(\mathcal{F}_t)$-stopping times.

Our objective is to solve the optimal stopping problem defined by (1)–(3) in the introduction in the presence of the following assumption.

Assumption 1 $\sigma, r, a, b, K > 0$. □

The solution of this problem involves the general solution of the ODE

$$\frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - ru(x) = 0, \tag{5}$$

which is given by

$$u(x) = Ax^n + Bx^m, \tag{6}$$

for $A, B \in \mathbb{R}$, where the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\frac{1}{2}\sigma^2 k^2 + \left(\mu - \frac{1}{2}\sigma^2\right) k - r = 0, \tag{7}$$

given by

$$m, n = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{8}$$

The following result shows that the value function is identically equal to ∞ if $a + b > n$. We are going to prove later that the same is true if $a + b = n$ (see Theorem 5 below).

Lemma 1 *Consider the optimal stopping problem defined by (1)–(3) and suppose that its data is such that $a + b > n$. Then $v \equiv \infty$.*

Proof. Recalling that n is the positive solution of the quadratic equation (7), we note that, if $a + b > n$, then

$$\frac{1}{2}\sigma^2 (a+b)^2 + \left(\mu - \frac{1}{2}\sigma^2\right)(a+b) - r > 0.$$

In view of this inequality, we can see that

$$\begin{aligned} v(x, s) &\geq \sup_{t \geq 0} \mathbb{E}\left[e^{-rt}\left(X_t^a S_t^b - K\right)^+\right] \\ &\geq \sup_{t \geq 0} \mathbb{E}\left[e^{-rt} X_t^{a+b}\right] - K \\ &= \sup_{t \geq 0} \mathbb{E}\left[\exp\left(-rt + \left(\mu - \frac{1}{2}\sigma^2\right)(a+b)t + \sigma(a+b)W_t\right)\right] - K \\ &= \sup_{t \geq 0} \exp\left(\left[\frac{1}{2}\sigma^2(a+b)^2 + \left(\mu - \frac{1}{2}\sigma^2\right)(a+b) - r\right] t\right) - K \\ &= \infty, \end{aligned}$$

which establishes the result. □

3 A verification theorem

With reference to the general theory of optimal stopping (e.g., see Shiryaev [S08], Krylov [K80], Bensoussan and Lions [BL82], and Peskir and Shiryaev [PS06]) and optimal stopping problems involving a running maximum process that are related to the one we solve here (e.g., see Graversen and Peskir [GP98], Peskir [P98], and Peskir and Shiryaev [PS06, Section 13]), we expect that the value function v of our optimal stopping problem should identify with an appropriate positive solution w of the variational inequality

$$\max\left\{\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}(x,s) + \mu x \frac{\partial w}{\partial x}(x,s) - rw(x,s),\ x^a s^b - K - w(x,s)\right\} = 0, \tag{9}$$

with boundary condition

$$\frac{\partial w}{\partial s}(s,s) = 0. \tag{10}$$

Rather than going for maximal generality, we prove here a verification theorem that is tailored to the requirements of the problem defined by (1)–(3). To this end, we assume that there exists a point $s_* > 0$, two C^1 functions $G, H : [s_*, \infty[\rightarrow \mathbb{R}_+$ such that

$$0 < G(s) < H(s) < s \quad \text{for all } s > s_*,$$

and a solution $w : \{(x,s) \in \mathbb{R}_+^2 \mid \ 0 < x \leq s\} \rightarrow [0, \infty[$ of (9)–(10) such that

$$(x,s) \mapsto w(x,s) \text{ is } C^2 \text{ outside } \{(G(s),s), (H(s),s) \mid \ s \geq s_*\} \tag{11}$$

and

$$x \mapsto w(x,s) \text{ is } C^1 \text{ at } G(s) \text{ and } H(s) \text{ for all } s \geq s_*. \tag{12}$$

The function w satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 w}{\partial x^2}(x,s) + \mu x \frac{\partial w}{\partial x}(x,s) - rw(x,s) \leq 0 \tag{13}$$

inside the set $\big\{(x,s) \in \mathbb{R}_+^2 \mid \ 0 < x < s\big\} \setminus \{(G(s),s), (H(s),s) \mid \ s \geq s_*\}$ and

$$\big(x^a s^b - K\big)^+ \leq w(x,s) \quad \text{for all } (x,s). \tag{14}$$

Furthermore, (9) is true for all $(x,s) \notin \{(G(s),s), (H(s),s) \mid \ s \geq s_*\}$ and (10) holds for all $s > 0$.

Lemma 2 *Consider the optimal stopping problem defined by (1)–(3), and suppose that the variational inequality (9) with boundary condition (10) has a solution w as described above and such that*

$$w(x,s) \leq C\,(1+s^\gamma) \quad \text{for all } \ 0<x\leq s, \tag{15}$$

for some constants $C, \gamma > 0$. Also, consider the stochastic processes Z and R defined by

$$Z_t = e^{-rt} w(X_t, S_t) \quad \text{and} \quad R_t = e^{-rt}(X_t^a S_t^b - K)^+. \tag{16}$$

The following statements hold true:
(I) *Z is an $(\mathcal{F}_t)$-supermartingale majorising the reward process R, and*

$$v(x,s) \leq w(x,s) \quad \text{for all } 0<x\leq s. \tag{17}$$

(II) *If w satisfies the transversality condition*

$$\liminf_{T\to\infty} e^{-rT}\mathbb{E}\left[w(X_T,S_T)\right] \equiv \liminf_{T\to\infty} \mathbb{E}\left[Z_T\right] = 0, \tag{18}$$

then Z is the least *$(\mathcal{F}_t)$-supermartingale that majorises the reward process R, i.e., Z is the Snell envelop of R,*

$$v(x,s) = w(x,s) \quad \text{for all } 0<x\leq s, \tag{19}$$

and the first hitting time $\tau_{\mathcal{S}}$ of the stopping region $\mathcal{S} = \left\{(x,s)\in\mathbb{R}_+^2 \mid \ w(x,s) = (x^a s^b - K)^+\right\}$, defined by

$$\tau_{\mathcal{S}} = \inf\left\{t\geq 0 \mid \ (X_t,S_t)\in\mathcal{S}\right\}, \tag{20}$$

is an optimal stopping time.

Proof. In view of (11)–(12), w possesses enough regularity for an application of Itô-Tanaka-Meyer's formula (see Graversen and Peskir [GP98, Remark 4.2]), which yields

$$\begin{aligned}
e^{-rT}w(X_T,S_T) &= w(x,s) + \int_0^T e^{-rt}\frac{\partial w}{\partial s}(S_t,S_t)\,dS_t + M_T \\
&\quad + \int_0^T e^{-rt}\left[\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 w}{\partial x^2}(X_t,S_t) + \mu X_t \frac{\partial w}{\partial x}(X_t,S_t) - rw(X_t,S_t)\right] dt \\
&= w(x,s) + M_T \\
&\quad + \int_0^T e^{-rt}\left[\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 w}{\partial x^2}(X_t,S_t) + \mu X_t \frac{\partial w}{\partial x}(X_t,S_t) - rw(X_t,S_t)\right] dt,
\end{aligned} \tag{21}$$

where

$$M_T = \int_0^T e^{-rt} \frac{\partial w}{\partial x}(X_t, S_t)\, dW_t,$$

and the second identity follows from the boundary condition (10) and the fact that S increases only on the set $\{X_t = S_t\}$. The growth assumption (15) and the fact that $\mathbb{E}\,[S_T^\gamma] < \infty$ imply that

$$\mathbb{E}\,[Z_T] = \mathbb{E}\left[e^{-rT} w(X_T, S_T)\right] < \infty \quad \text{for all } T \geq 0.$$

Also, if (T_n) is any localising sequence of $(\mathcal{F}_t)$-stopping times for the local martingale M, then (21) and the inequality (13) imply that

$$\mathbb{E}\left[e^{-r(T\wedge T_n)} w(X_{T\wedge T_n}, S_{T\wedge T_n}) \mid \mathcal{F}_t\right] \leq e^{-r(t\wedge T_n)} w(X_{t\wedge T_n}, S_{t\wedge T_n}) \quad \text{for all } t \leq T \text{ and } n \geq 1. \tag{22}$$

To prove part (I) of the lemma, we use the dominated convergence theorem, Fatou's lemma and (22) to calculate

$$\begin{aligned}
\mathbb{E}\,[Z_T \mid \mathcal{F}_t] &= \mathbb{E}\left[e^{-rT} w(X_T, S_T) \mid \mathcal{F}_t\right] \\
&\leq \lim_{n\to\infty} \mathbb{E}\left[e^{-rT} w(X_T, S_T) \mathbf{1}_{\{T \leq T_n\}} \mid \mathcal{F}_t\right] + \liminf_{n\to\infty} \mathbb{E}\left[e^{-rT_n} w(X_{T_n}, S_{T_n}) \mathbf{1}_{\{T_n < T\}} \mid \mathcal{F}_t\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[e^{-r(T\wedge T_n)} w(X_{T\wedge T_n}, S_{T\wedge T_n}) \mid \mathcal{F}_t\right] \\
&\leq \lim_{n\to\infty} e^{-r(t\wedge T_n)} w(X_{t\wedge T_n}, S_{t\wedge T_n}) \\
&= Z_t \qquad \text{for all } t \leq T,
\end{aligned}$$

which proves that Z is an $(\mathcal{F}_t)$-supermartingale. Furthermore, the claim that Z majorises R follows immediately from inequality (14), while (17) follows from the fact that Z is a positive supermartingale majorising R and the definition (3) of the value function v.

To establish part (II) of the theorem, we observe that, if $\tau_{\mathcal{S}} \in \mathcal{T}$ is the stopping time defined by (20), then we can see that (21) and the fact that w satisfies the variational inequality (9) imply that

$$\begin{aligned}
& e^{-r\tau_{\mathcal{S}}} \left(X_{\tau_{\mathcal{S}}}^a S_{\tau_{\mathcal{S}}}^b - K\right)^+ \mathbf{1}_{\{\tau_{\mathcal{S}} \leq T\wedge T_n\}} \\
&\quad = w(x, s) + M_{T\wedge T_n\wedge\tau_{\mathcal{S}}} \\
&\qquad + e^{-r\tau_{\mathcal{S}}} \left[\left(X_{\tau_{\mathcal{S}}}^a S_{\tau_{\mathcal{S}}}^b - K\right)^+ - w(X_{\tau_{\mathcal{S}}}, S_{\tau_{\mathcal{S}}})\right] \mathbf{1}_{\{\tau_{\mathcal{S}} \leq T\wedge T_n\}} \\
&\qquad - e^{-r(T\wedge T_n)} w(X_{T\wedge T_n}, S_{T\wedge T_n}) \mathbf{1}_{\{\tau_{\mathcal{S}} > T\wedge T_n\}} \\
&\qquad + \int_0^{T\wedge T_n\wedge\tau_{\mathcal{S}}} e^{-rt} \left[\frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 w}{\partial x^2}(X_t, S_t) + \mu X_t \frac{\partial w}{\partial x}(X_t, S_t) - rw(X_t, S_t)\right] dt. \\
&\quad = w(x, s) + M_{T\wedge T_n\wedge\tau_{\mathcal{S}}} - e^{-r(T\wedge T_n)} w(X_{T\wedge T_n}, S_{T\wedge T_n}) \mathbf{1}_{\{\tau_{\mathcal{S}} > T\wedge T_n\}}.
\end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau_{\mathcal{S}}}\left(X^a_{\tau_{\mathcal{S}}}S^b_{\tau_{\mathcal{S}}}-K\right)^+\mathbf{1}_{\{\tau_{\mathcal{S}}\leq T\wedge T_n\}}\right]\\ = w(x,s)-\mathbb{E}\left[e^{-r(T\wedge T_n)}w(X_{T\wedge T_n},S_{T\wedge T_n})\mathbf{1}_{\{\tau_{\mathcal{S}}>T\wedge T_n\}}\right].\end{aligned} \tag{23}$$

The growth condition (15) and the fact that S is an increasing process imply that

$$0\leq e^{-r(T\wedge T_n)}w(X_{T\wedge T_n},S_{T\wedge T_n})\mathbf{1}_{\{\tau_{\mathcal{S}}>T\wedge T_n\}}\leq C\left(1+S^\gamma_T\right).$$

Since $\mathbb{E}\left[S^\gamma_T\right]<\infty$ for all $T>0$, these inequalities and the dominated convergence theorem imply that

$$\lim_{n\to\infty}\mathbb{E}\left[e^{-r(T\wedge T_n)}w(X_{T\wedge T_n},S_{T\wedge T_n})\mathbf{1}_{\{\tau_{\mathcal{S}}>T\wedge T_n\}}\right]=\mathbb{E}\left[e^{-rT}w(X_T,S_T)\mathbf{1}_{\{\tau_{\mathcal{S}}>T\}}\right].$$

Furthermore, the transversality condition (18) implies that

$$0\leq\liminf_{T\to\infty}\mathbb{E}\left[e^{-rT}w(X_T,S_T)\mathbf{1}_{\{\tau_{\mathcal{S}}>T\}}\right]\leq\liminf_{T\to\infty}\mathbb{E}\left[e^{-rT}w(X_T,S_T)\right]=0,$$

while the monotone convergence theorem implies that

$$\lim_{T\to\infty}\lim_{n\to\infty}\mathbb{E}\left[e^{-r\tau_{\mathcal{S}}}\left(X^a_{\tau_{\mathcal{S}}}S^b_{\tau_{\mathcal{S}}}-K\right)^+\mathbf{1}_{\{\tau_{\mathcal{S}}\leq T\wedge T_n\}}\right]=\mathbb{E}\left[e^{-r\tau_{\mathcal{S}}}\left(X^a_{\tau_{\mathcal{S}}}S^b_{\tau_{\mathcal{S}}}-K\right)^+\mathbf{1}_{\{\tau_{\mathcal{S}}<\infty\}}\right].$$

In view of these limits, we can see that (23) yields

$$\mathbb{E}\left[e^{-r\tau_{\mathcal{S}}}\left(X^a_{\tau_{\mathcal{S}}}S^b_{\tau_{\mathcal{S}}}-K\right)^+\mathbf{1}_{\{\tau_{\mathcal{S}}<\infty\}}\right]=w(x,s),$$

which, combined with (17), implies that $v(x,s)=w(x,s)$ and that $\tau_{\mathcal{S}}$ is optimal. Finally, the assertion that Z is the Snell envelop of R follows from the results that we have established thus far and the general theory of optimal stopping (e.g., see El Karoui [EK81, 2.51–2.76], and Peskir and Shiryaev [PS06, Section 2]). □

4 The solution of the optimal stopping problem

We now solve the optimal stopping problem (1)–(3) by constructing a solution w of the variational inequality (9) with boundary condition (10) that satisfies the requirements of the verification Lemma 2 when $a+b<n$ (see also Lemma 1). To this end, we first note that, by considering simple sub-optimal stopping times, such as the first hitting time of the set $\{(x,s)\mid\ x^as^b-K\geq1\}$, which is finite, $\mathbb{P}$-a.s., we can see that the value function is strictly positive in the domain $\{(x,s)\mid 0<x\leq s\}$. It follows that the set

$$\left\{(x,s)\in\mathbb{R}^2_+\mid\ (x^as^b-K)^+=0\right\}=\left\{(x,s)\in\mathbb{R}^2_+\mid\ x\leq\left(\frac{K}{s^b}\right)^{1/a}\right\}$$

must be a subset of the continuation region $\mathcal{W}$. Furthermore, we observe that, since

$$\frac{\partial}{\partial s}\left(x^a s^b - K\right)\bigg|_{x=s} = b s^{a+b-1} > 0 \quad \text{for all } s > 0,$$

the line $\{(x,s) \in \mathbb{R}_+^2 \mid \; x = s > (K/s^b)^{1/a}\}$, which is part of the state space's boundary, must also be a subset of the continuation region $\mathcal{W}$ because the boundary condition (10) cannot hold otherwise.

In view of these observations, we conjecture that there exists a point $s_* > 0$, a strictly decreasing function $G : [s_*, \infty[\to \mathbb{R}$ and a strictly increasing function $H : [s_*, \infty[\to \mathbb{R}$ with

$$G(s_*) = H(s_*) \quad \text{and} \quad 0 < \left(\frac{K}{s^b}\right)^{1/a} < G(s) < H(s) < s \text{ for all } s > s_*,$$

such that the stopping region $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{(x,s) \in \mathbb{R}_+^2 \mid \; s_* \leq s \text{ and } G(s) \leq x \leq H(s)\right\}. \tag{24}$$

Accordingly the continuation region $\mathcal{W}$ is given by $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$, where

$$\mathcal{W}_1 = \left\{(x,s) \in \mathbb{R}_+^2 \mid \; s_* \leq s \text{ and } 0 < x < G(s)\right\}, \tag{25}$$

$$\mathcal{W}_2 = \left\{(x,s) \in \mathbb{R}_+^2 \mid \; s_* \leq s \text{ and } H(s) < x \leq s\right\}, \tag{26}$$

$$\mathcal{W}_3 = \left\{(x,s) \in \mathbb{R}_+^2 \mid \; 0 < s < s_* \text{ and } 0 < x \leq s\right\}. \tag{27}$$

This conjecture is depicted by Figure 1.

To proceed further, we recall the fact that the functions $w(\cdot, s)$ should satisfy the ODE (5) in the interior of the waiting region $\mathcal{W}$. Since the general solution of (5) is given by (6), we therefore look for functions A_j and B_j such that

$$w(x,s) = A_j(s)x^n + B_j(s)x^m, \quad \text{if } (x,s) \in \mathcal{W}_j, \quad \text{for } j = 1, 2, 3.$$

To determine these functions and the free-boundaries G and H, we consider each of the cases associated with $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ separately.

In the region $\mathcal{W}_1$, we must have $B_1 \equiv 0$, otherwise, the transversality condition (18) of the verification Lemma 2 cannot be satisfied. Furthermore, by appealing to the so-called principle of smooth fit, which we have incorporated into the requirements of Lemma 2, we look for C^1 continuity along the free-boundary function G, which yields the system of algebraic equations

$$A_1(s)G^n(s) = G^a(s)s^b - K,$$

$$nA_1(s)G^n(s) = aG^a(s)s^b,$$

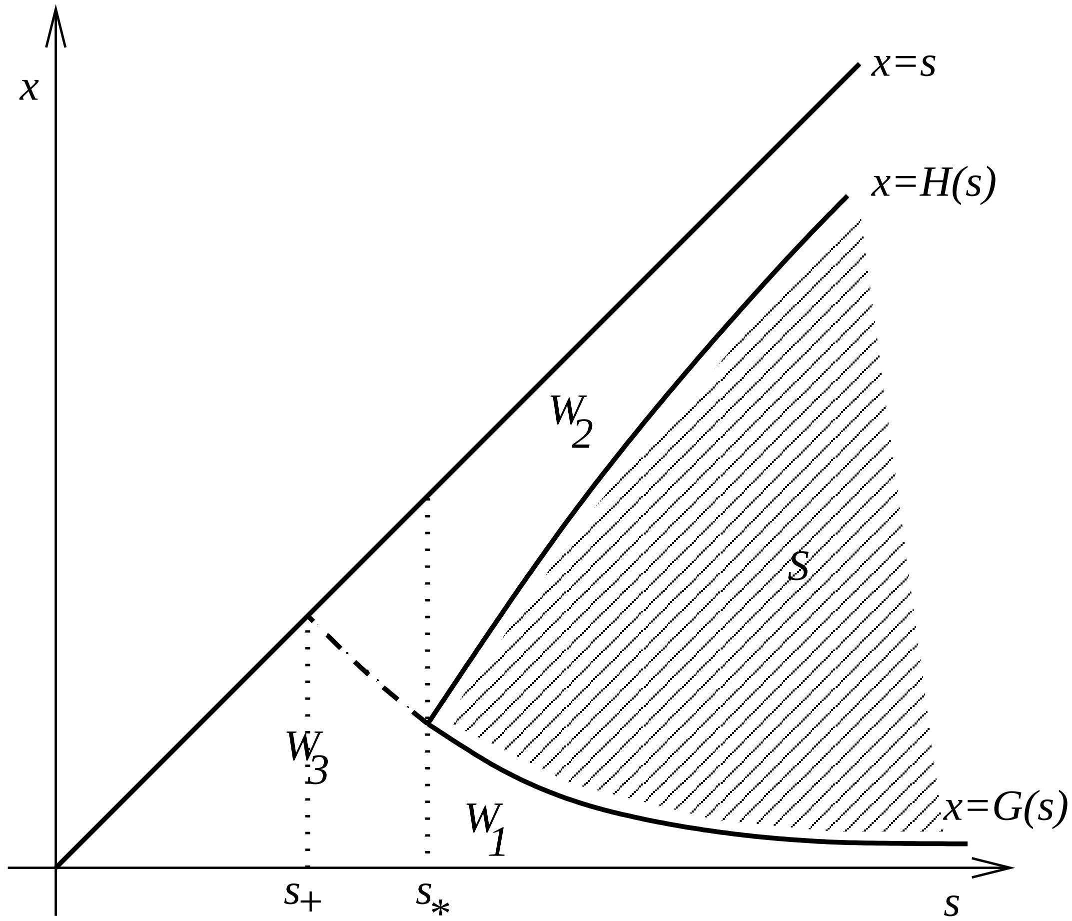

Figure 1: The continuation region $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3$ and the stopping region $\mathcal{S}$ of the discretionary stopping problem (1)–(3) when $a + b = n$

for $s \geq s^*$. This system is straightforward to solve, and we are faced with the expressions

$$A_1(s) = \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s^{bn/a} > 0, \qquad B_1(s) = 0, \tag{28}$$

and

$$G(s) = \left(\frac{nK}{n-a} \right)^{1/a} s^{-b/a} > \left(\frac{K}{s^b} \right)^{1/a}. \tag{29}$$

Plainly, the function G given by (29) is strictly decreasing. Also, the equivalence

$$\left(\frac{nK}{n-a} \right)^{1/a} s^{-b/a} < s \quad \Leftrightarrow \quad s > \left(\frac{nK}{n-a} \right)^{1/(a+b)}$$

implies that we must have

$$s_* > \left(\frac{nK}{n-a}\right)^{1/(a+b)} =: s_\dagger \tag{30}$$

for our construction to make sense (see also Figure 1).

In the region $\mathcal{W}_2$, the boundary condition (10) becomes relevant and yields the expression

$$\dot{A}_2(s)s^n + \dot{B}_2(s)s^m = 0. \tag{31}$$

Also, C^1 continuity along the free-boundary function H is associated with the system of equations

$$\begin{aligned} A_2(s)H^n(s) + B_2(s)H^m(s) &= H^a(s)s^b - K, \\ nA_2(s)H^n(s) + mB_2(s)H^m(s) &= aH^a(s)s^b, \end{aligned}$$

for $s \geq s_*$, which is equivalent to

$$A_2(s) = \frac{(a-m)H^a(s)s^b + mK}{n-m} H^{-n}(s), \tag{32}$$

$$B_2(s) = \frac{(n-a)H^a(s)s^b - nK}{n-m} H^{-m}(s). \tag{33}$$

Differentiating these expressions with respect to s and substituting the results for $\dot{A}$ and $\dot{B}$ in (31), we can see that H should satisfy the ODE

$$\dot{H}(s) = \mathcal{H}\left(H(s), s\right), \tag{34}$$

where

$$\mathcal{H}\left(\bar{H}, s\right) = \frac{b\left[(a-m)\left(\frac{s}{\bar{H}}\right)^n + (n-a)\left(\frac{s}{\bar{H}}\right)^m\right]\bar{H}^{a+1}s^{b-1}}{\left[(a-m)(n-a)\bar{H}^a s^b + mnK\right]\left[\left(\frac{s}{\bar{H}}\right)^n - \left(\frac{s}{\bar{H}}\right)^m\right]}. \tag{35}$$

We need to solve this ODE with initial condition

$$H(s_*) = G(s_*) \equiv \left(\frac{nK}{n-a}\right)^{1/a} s_*^{-b/a}, \tag{36}$$

for some appropriate $s_* > s_\dagger$ inside the domain

$$\mathcal{D}_H = \left\{(\bar{H}, s) \in \mathbb{R}_+^2 \mid \; s_\dagger < s \text{ and } \left(\frac{nK}{n-a}\right)^{1/a} s^{-b/a} \leq \bar{H} < s\right\}. \tag{37}$$

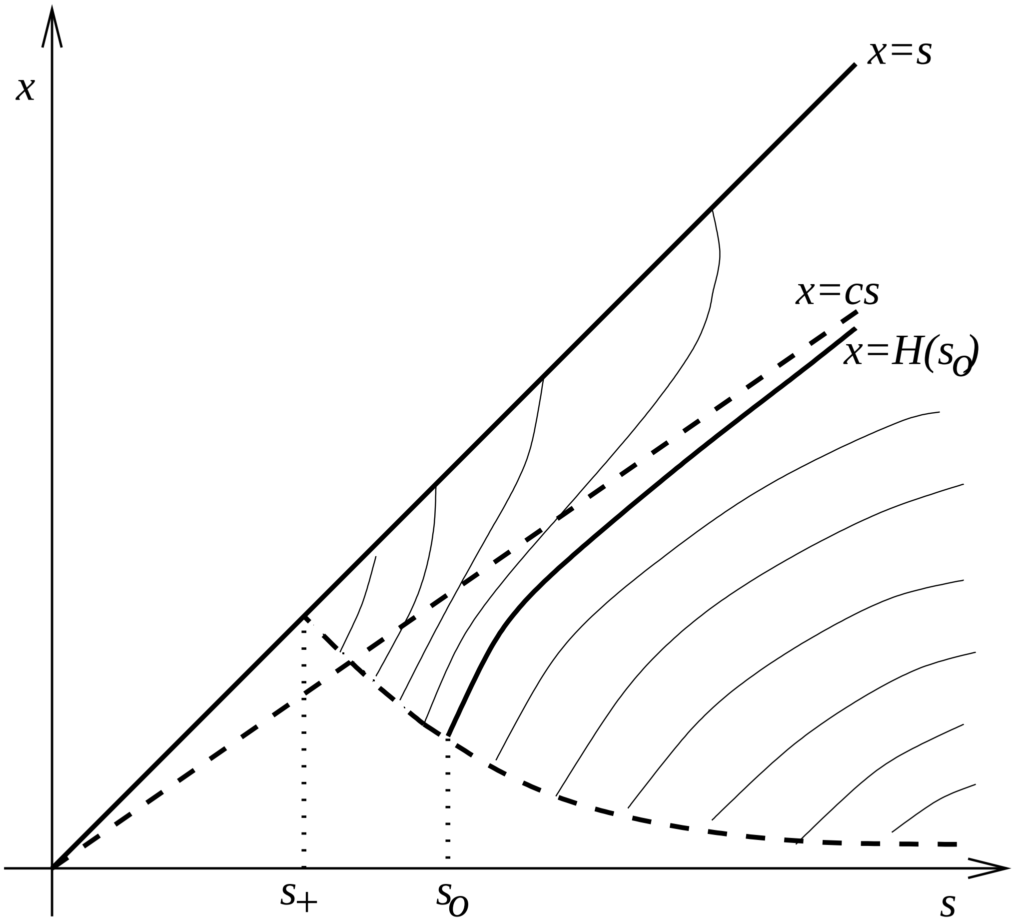

Figure 2: The solution of (34)–(36) for different values of $s_* > s_\dagger$

For future reference, we also note that, if (34) has the required solution, then (32), (33) and (36) imply that

$$\lim_{s \downarrow s_*} A_2(s) = \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_*^{bn/a} = \lim_{s \downarrow s_*} A_1(s) \quad \text{and} \quad \lim_{s \downarrow s_*} B_2(s) = 0. \tag{38}$$

The following result, which we prove in the Appendix, and which can be illustrated by Figure 2, is mainly concerned with the solvability of (34)–(36).

Lemma 3 *Suppose that the problem data satisfy Assumption 1 and $a + b < n$. There exist points $s_\circ$ and s° satisfying*

$$s_\dagger < \left(\frac{nK}{n-a} \right)^{1/(a+b)} \left[\frac{(n-a)(a+b-m)}{(a-m)(n-a-b)} \right]^{a/[(a+b)(n-m)]} < s_\circ \le s^\circ < \infty, \tag{39}$$

where $s_\dagger$ is given by (30), such that the following statements hold true:

(I) *Given any* $s_* \in]s_\dagger, s_\circ[$, *the ODE (34) with initial condition (36) has a unique solution* $H(\cdot) \equiv H(\cdot; s_*)$ *in* $\mathcal{D}_H$ *that is defined up to an "explosion" point* $\hat{s} = \hat{s}(s_*) < \infty$. *In particular,* $(H(s), s) \in \mathcal{D}_H$ *for all* $s \in [s_*, \hat{s}[$, $\lim_{s \uparrow \hat{s}} H(s) = \hat{s}$, *and* H *is strictly increasing in its domain* $[s_*, \hat{s}[$.
(II) *Given any* $s_* \in [s_\circ, s^\circ]$, *the ODE (34) with initial condition (36) has a unique solution* $H(\cdot)$ *in* $\mathcal{D}_H$. *This solution is a strictly increasing function such that*

$$H(s) < cs \text{ for all } s \geq s_*, \quad \text{and} \quad \lim_{s \to \infty} \frac{H(s)}{s} = c, \tag{40}$$

where

$$c = \left[\frac{(a-m)(n-a-b)}{(n-a)(a+b-m)} \right]^{1/(n-m)} \in]0, 1[. \tag{41}$$

(III) *Given any* $s_* > s^\circ$, *the ODE (34) with initial condition (36) has a unique solution* $H(\cdot) \equiv H(\cdot; s_*)$ *in* $\mathcal{D}_H$. *This solution is a strictly increasing function such that*

$$s_*^{-b/(n-a)} H(s_*) s^{b/(n-a)} \leq H(s) \leq \min \left\{ cs, C s^{b/(n-a)} \right\} \quad \text{for all } s \geq s_*, \tag{42}$$

for some constant $C = C(s_*) > 0$, *and for* c *given by (41).*
(IV) *In each of the three cases above,* $A_2(s), B_2(s) > 0$ *for all* $s > s_*$ *in the domain of* H, *where* $A_2(s)$ *and* $B_2(s)$ *are given by (32) and (33).*

This lemma suggests that we must have $s_* \geq s_\circ$ because, otherwise, the candidate for the value function that we construct does not satisfy the boundary condition (10) (see the discussion at the beginning of the section).

Now, in the region $\mathcal{W}_3$, we must have $B_3 \equiv 0$ for the requirement (18) of the verification Lemma 2 to be satisfied. In this context, the boundary condition (10) implies that $\dot{A}_3 = 0$. Combining these observations with (28), (38) and the requirement that w should be continuous, we are faced with the expressions

$$A_3(s) = \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_*^{bn/a} > 0 \quad \text{and} \quad B_3(s) = 0 \tag{43}$$

for all $s \leq s_*$.

Summarising the analysis thus far, we conjecture that the value function v of our optimal stopping problem identifies with the function w given by

$$
\begin{aligned}
w(x,s) &\equiv w(x,s;s_*) \\
&= \begin{cases} x^a s^b - K, & x \in \mathcal{S}, \\ A_1(s)x^n, & x \in \mathcal{W}_1, \\ A_2(s)x^n + B_2(s)x^m, & x \in \mathcal{W}_2, \\ A_3(s)x^n, & x \in \mathcal{W}_3, \end{cases} \\
&= \begin{cases} x^a s^b - K, & x \in \mathcal{S}, \\ \frac{a}{n}\left(\frac{nK}{n-a}\right)^{-(n-a)/a} s^{bn/a} x^n, & x \in \mathcal{W}_1, \\ \frac{(a-m)H^a(s)s^b+mK}{n-m}\left(\frac{x}{H(s)}\right)^n + \frac{(n-a)H^a(s)s^b-nK}{n-m}\left(\frac{x}{H(s)}\right)^m, & x \in \mathcal{W}_2, \\ \frac{a}{n}\left(\frac{nK}{n-a}\right)^{-(n-a)/a} s_*^{bn/a} x^n, & x \in \mathcal{W}_3, \end{cases}
\end{aligned}
\tag{44}
$$

for some $s_* \geq s_\circ$. The following result, which we prove in the Appendix, establishes that each of these functions, which are parametrised by $s_* \geq s_\circ$, is a solution of the variational inequality (9)–(10).

Lemma 4 *Suppose that the problem data satisfy Assumption 1 and $a+b<n$. Also, fix any $s_* \geq s_\circ$, where $s_\circ$ is as in Lemma 3. The function $w(\cdot) \equiv w(\cdot;s_*)$ defined by (44), where G is given by (29), $H(\cdot) \equiv H(\cdot;s_*)$ is the associated solution of (34)–(36), and $\mathcal{S}$, $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ are defined by (24)–(27), is a solution of the variational inequality (9) with boundary condition (10) that has the properties (11)–(14).*

We can now prove the main result of the paper.

Theorem 5 *Consider the optimal stopping problem defined by (1)–(3), and suppose that the problem data satisfy Assumption 1.*
(I) *If $a+b<n$, then $s_\circ = s^\circ$, where $s_\circ$, s° are as in Lemma 3, and $v = w(\cdot;s_\circ)$, where $w(\cdot;s_\circ)$ is defined by (44) with $s_* = s_\circ$. Furthermore, the first hitting time $\tau_{\mathcal{S}}$ of the stopping region $\mathcal{S}$, which is defined as in (20) in the verification Lemma 2, is optimal.*
(II) *If $a+b \geq n$, then $v \equiv \infty$.*

Proof. Given a constant $\lambda \in]0,n[$, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\frac{1}{2}\sigma^2\lambda^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\lambda - (r-\varepsilon_1) = -\varepsilon_2\sigma\lambda.$$

For such a choice of constants fixed, we can see that, given any initial condition $0 < x \leq s$,

$$
\begin{aligned}
e^{-rT}\mathbb{E}\left[S_T^\lambda\right] &= e^{-rT}\mathbb{E}\left[\max\left\{s^\lambda, \max_{0\leq t\leq T} X_t^\lambda\right\}\right] \\
&\leq s^\lambda e^{-rT} + x^\lambda e^{-\varepsilon_1 T}\mathbb{E}\left[\exp\left(\max_{0\leq t\leq T}\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\lambda t - (r-\varepsilon_1)t + \sigma\lambda W_t\right\}\right)\right] \\
&\leq s^\lambda e^{-rT} + x^\lambda e^{-\varepsilon_1 T}\mathbb{E}\left[\exp\left(\sigma\lambda \sup_{0\leq t}\left\{-\left(\frac{1}{2}\sigma\lambda + \varepsilon_2\right)t + W_t\right\}\right)\right] \\
&= s^\lambda e^{-rT} + x^\lambda \frac{\sigma\lambda + 2\varepsilon_2}{2\varepsilon_2} e^{-\varepsilon_1 T},
\end{aligned}
$$

the last equality following because the maximum of a Brownian motion with drift $-\nu < 0$ is an exponentially distributed random variable with parameter 2ν (e.g., see Borodin and Salminen [BS96, 2.1.1.4]). It follows that

$$
\lim_{T\to\infty} e^{-rT}\mathbb{E}\left[S_T^\lambda\right] = 0 \quad \text{for all } \lambda \in]0, n[. \tag{45}
$$

To proceed further, we consider part (I) of the theorem, and we assume that $a + b < n$. Given any $s_* \geq s_\circ$, we note that (40) and the left-hand side of (42) in Lemma 3 imply the inequalities

$$
x^a s^b - K \leq s^{a+b} \quad \text{for all } (x, s) \in \mathcal{S}, \tag{46}
$$

$$
A_1(s)x^n \leq A_1(s)G^n(s) = \frac{aK}{n-a} \quad \text{for all } (x, s) \in \mathcal{W}_1, \tag{47}
$$

$$
\begin{aligned}
\frac{-mK}{n-m}\left(\frac{x}{H(s)}\right)^n &\leq \frac{-mK}{n-m} s^n H^{-n}(s) \\
&\leq \frac{-mK s_*^{bn/(n-a)}}{(n-m)H^n(s_*)} s^{n(n-a-b)/(n-a)} \quad \text{for all } (x, s) \in \mathcal{W}_2,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
B_2(s)x^m &\leq \frac{n-a}{n-m} H^a(s) s^b \left(\frac{x}{H(s)}\right)^m \\
&\leq \frac{n-a}{n-m} H^a(s) s^b \\
&\leq \frac{n-a}{n-m} s^{a+b} \quad \text{for all } (x, s) \in \mathcal{W}_2,
\end{aligned} \tag{49}
$$

and

$$
A_3(s)x^n \leq \frac{a}{n}\left(\frac{nK}{n-a}\right)^{-(n-a)/a} s_*^{(a+b)n/a} \quad \text{for all } (x, s) \in \mathcal{W}_3. \tag{50}
$$

If we write

$$
w(x, s) = \frac{a-m}{n-m} H^{-(n-a)}(s) s^b x^n \mathbf{1}_{[s_*, \infty[}(s) + \tilde{w}(x, s), \tag{51}
$$

then an inspection of (44) reveals that $|\tilde{w}|$ is bounded by the sum of the strictly positive terms on the left-hand sides of inequalities (46)–(50). In view of this observation, we can see that there exists a constant $C_1 = C_1(s_*)$ such that

$$|\tilde{w}(x,s)| \leq C_1 \left[1 + s^{a+b} + s^{n(n-a-b)/(n-a)}\right] \quad \text{for all } 0 < x \leq s. \tag{52}$$

Combining this estimate with (45), we can see that

$$\lim_{T\to\infty} e^{-rT}\mathbb{E}\left[|\tilde{w}(X_T, S_T)|\right] = 0, \tag{53}$$

thanks to the assumption that the constants $a, b > 0$ satisfy $a + b < n$.

Given any $s_* > s^\circ$, the estimate given by the right-hand side of (42) implies that

$$\begin{aligned} H^{-(n-a)}(s) s^b x^n \mathbf{1}_{[s_*,\infty[}(s) &\geq C^{-(n-a)} x^n - C^{-(n-a)} x^n \mathbf{1}_{]0,s_*[}(s) \\ &\geq C^{-(n-a)} x^n - C^{-(n-a)} s_*^n \quad \text{for all } 0 < x \leq s. \end{aligned}$$

In view of (53) and the fact that the process $(e^{-rt}X_t^n,\ t \geq 0)$ is a martingale, it follows that

$$\begin{aligned} \liminf_{T\to\infty} e^{-rT}\mathbb{E}\left[w(X_T, S_T)\right] &\geq \frac{a-m}{(n-m)C^{n-a}} \lim_{T\to\infty} e^{-rT}\left(\mathbb{E}\left[X_T^n\right] - s_*^n\right) \\ &\quad - \lim_{T\to\infty} e^{-rT}\mathbb{E}\left[|\tilde{w}(X_T, S_T)|\right] \\ &= \frac{a-m}{(n-m)C^{n-a}} x^n \\ &> 0, \end{aligned}$$

which proves that, when $s_* > s^\circ$, $w(\cdot; s_*)$ does not satisfy the transversality condition (18) of the verification Lemma 2 and cannot be identified with the value function v. On the other hand, for $s_* \in [s_\circ, s^\circ]$, (40) implies that there exist constants $\varepsilon \in]0, c[$ and $s_\varepsilon \geq s_*$ such that $H(s) \geq (c - \varepsilon)s$ for all $s \geq s_\varepsilon$. For such a choice of constants, we can see that

$$\begin{aligned} H^{-(n-a)}(s) s^b x^n \mathbf{1}_{[s_*,\infty[}(s) &\leq (c-\varepsilon)^{-(n-a)} s^{-(n-a-b)} x^n \mathbf{1}_{]s_\varepsilon,\infty[}(s) + H^{-(n-a)}(s) s^b x^n \mathbf{1}_{[s_*,s_\varepsilon]}(s) \\ &\leq (c-\varepsilon)^{-(n-a)} s^{a+b} + H^{-(n-a)}(s_*) s_\varepsilon^{b+n} \quad \text{for all } 0 < x \leq s. \end{aligned} \tag{54}$$

Combining these inequalities with (45) and (53), we can see that

$$\begin{aligned} &\lim_{T\to\infty} e^{-rT}\mathbb{E}\left[w(X_T, S_T)\right] \\ &\leq \frac{a-m}{(n-m)(c-\varepsilon)^{(n-a)}} \lim_{T\to\infty} e^{-rT}\left(\mathbb{E}\left[S_T^{a+b}\right] + (c-\varepsilon)^{(n-a)} H^{-(n-a)}(s_*) s_\varepsilon^{b+n}\right) \\ &\quad + \lim_{T\to\infty} e^{-rT}\mathbb{E}\left[|\tilde{w}(X_T, S_T)|\right] \\ &= 0, \end{aligned}$$

which proves that $w(\cdot; s^*)$ satisfies the transversality condition (18) when $s_* \in [s_\circ, s^\circ]$. Since $w(\cdot; s_*)$ satisfies all of the requirements of the verification Lemma 2 when $s_* \in [s_\circ, s^\circ]$, it follows that $v = w(\cdot; s_*)$ for all $s_* \in [s_\circ, s^\circ]$, which establishes part (I) of the theorem. In particular, the identity $s_\circ = s^\circ$ follows from the uniqueness of the value function v.

In view of Lemma 1, we will prove part (II) of the theorem if we show that $v \equiv \infty$ if $a+b = n$. To this end, we fix the rest of the problem data, we parametrise the value function v by b, and we note that

$$v(x, s; n-a) \geq v(x, s; b) \quad \text{for all } b \in]0, n-a[\text{ and } 0 < x \leq s,$$

by the definition (3) of the value function v. In light of this observation, the fact that $v(\cdot; b) = w(\cdot; b, s_\circ(b))$ for all $b \in]0, n-a[$, which we have established above, the fact that $\lim_{b \uparrow n-a} s_\circ(b) = \infty$, which follows from (39), and the expression (44) of w, we can see that

$$v(x, s; n-a) \geq \lim_{b \uparrow n-a} \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_\circ^{bn/a}(b) x^n = \infty \quad \text{for all } 0 < x \leq s,$$

and the proof is complete. □

5 Limiting cases

We now study the robustness of our optimal strategy by considering the form that it takes as certain of the parameters a, b and K tend to 0 in a fashion that gives rise to problems studied in the literature. To this end, we denote by $G(\cdot; a, b, K)$ and $H(\cdot; a, b, K)$ the free-boundaries that characterise our optimal solution, by $c(a, b)$ the constant defined by (41), and so on, to stress the dependence of such objects on the data a, b and K. Also, we assume that the parameters a and b always satisfy the inequality $a + b < n$. It is worth noting that we focus on showing that the continuation and the stopping regions of the problem we have solved transform continuously to the corresponding regions of the limit problems that we consider. The fact that the limit regions indeed provide the optimal stopping strategies of the limit problems has been proved in the references we list; it can also be established using the verification Lemma 2.

The payoff structure of the well-known perpetual American call option, essentially solved by McKean [McK65], arises formally when $a = 1$, $K > 0$ and $b \downarrow 0$. In this case, we can check that

$$\lim_{b \downarrow 0} G(s; 1, b, K) = \frac{nK}{n-1} \quad \text{for all } s > 0,$$

and that

$$\lim_{b \downarrow 0} \lim_{s \to \infty} \frac{H(s; 1, b, K)}{s} = \lim_{b \downarrow 0} c(1, b) = 1. \tag{55}$$

Noting that $\lim_{b\downarrow 0} \mathcal{H}(\bar{H}, s; 1, b, K) = 0$ for all $(\bar{H}, s) \in \mathcal{D}_H$, we can see that (55) is satisfied if and only if

$$\lim_{b\downarrow 0} s_*(1, b, K) = \lim_{b\downarrow 0} s_\dagger(1, b, K) = \frac{nK}{n-1}$$

and the strictly increasing functions $H(\cdot; 1, b, K)$ converge pointwise to the function $H(\cdot; 1, 0, K)$ given by $H(s; 1, 0, K) = s$, for $s \geq nK/(n-1)$. (Note that the graph of $H(\cdot; 1, 0, K)$ lies on the part of the boundary of the domain $\mathcal{D}_H$ where the ODE (34) becomes singular.) In particular, at the limit $b \downarrow 0$, the stopping region is given by

$$\mathcal{S} = \left\{ (x, s) \in \mathbb{R}_+^2 \mid \frac{nK}{n-1} \leq x \leq s \right\}$$

and the continuation region is given by

$$\mathcal{W}_1 \cup \mathcal{W}_3 = \left\{ (x, s) \in \mathbb{R}_+^2 \mid 0 < x \leq \min \left\{ s, \frac{nK}{n-1} \right\} \right\},$$

while the value function is given by

$$w(x, s) = \begin{cases} x - K, & \text{for } (x, s) \in \mathcal{S}, \\ \frac{1}{n} \left(\frac{nK}{n-1} \right)^{-1/(n-1)} x^n, & \text{for } (x, s) \in \mathcal{W}_1 \cup \mathcal{W}_3, \end{cases}$$

as expected (see also Figure 3).

The payoff structure of the lookback American option studied by Pedersen [P00] and Guo and Shepp [GS01] arises formally when $b = 1$, $K > 0$ and $a \downarrow 0$. In this case, we can check that

$$\lim_{a\downarrow 0} G(s; a, 1, K) = 0 \quad \text{for all } s > \lim_{a\downarrow 0} s_\dagger(a, 1, K) = K,$$
$$\lim_{a\downarrow 0} c(a, 1) = \left(\frac{-m(n-1)}{n(1-m)} \right)^{1/(n-m)}, \tag{56}$$

and that

$$\lim_{a\downarrow 0} \mathcal{H}(\bar{H}, s; a, 1, K) = \frac{\left[-m \left(\frac{s}{\bar{H}} \right)^n + n \left(\frac{s}{\bar{H}} \right)^m \right] \bar{H}}{-mn(s-K) \left[\left(\frac{s}{\bar{H}} \right)^n - \left(\frac{s}{\bar{H}} \right)^m \right]} =: \mathcal{H}(\bar{H}, s; 0, 1, K). \tag{57}$$

It follows that, at the limit $a \downarrow 0$, the stopping region is given by

$$\mathcal{S} = \left\{ (x, s) \in \mathbb{R}_+^2 \mid s_* < s \text{ and } 0 < x \leq H(s) \right\},$$

while the continuation region is given by

$$\mathcal{W}_2 \cup \mathcal{W}_3 = \left\{ (x, s) \in \mathbb{R}_+^2 \mid \text{either } s_* \leq s \text{ and } H(s) < x \leq s \text{ or } 0 < x \leq s < s_* \right\},$$

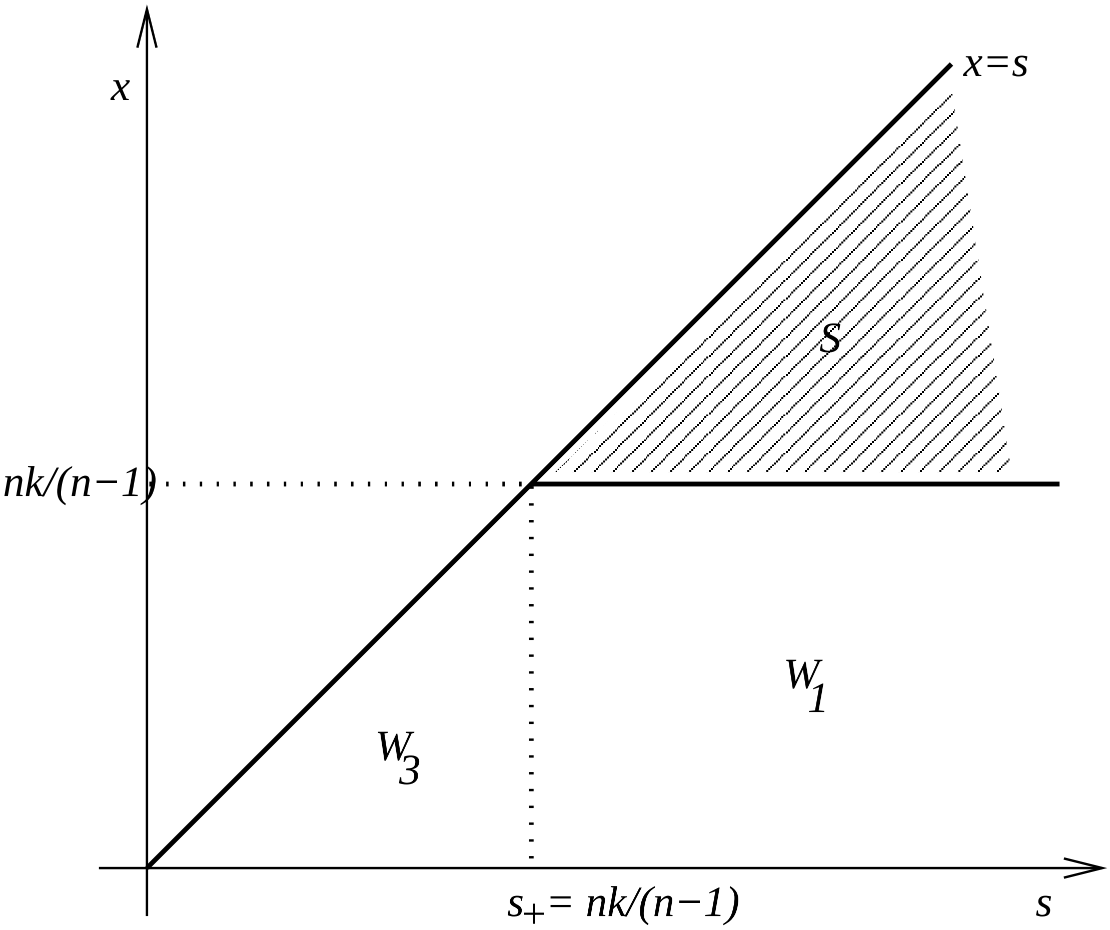

Figure 3: The continuation region $\mathcal{W}_1 \cup \mathcal{W}_3$ and the stopping region $\mathcal{S}$ of the perpetual American option

where H is the solution of the ODE (34) with $\mathcal{H}$ given by (57) that tends asymptotically to the line with slope given by the constant in (56), which is in agreement with the results of Pedersen [P00] and Guo and Shepp [GS01] (see also Figure 4).

The payoff structure of the Russian option introduced by Shepp and Shiryaev [SS93] arises formally when $b = 1$, $a \downarrow 0$ and $K \downarrow 0$. In this case, we can check that

$$\lim_{a,K\downarrow 0} G(s; a, 1, K) = 0 \quad \text{for all } s > \lim_{a,K\downarrow 0} s_\dagger(a, 1, K) = 0,$$

and that

$$\lim_{a,K\downarrow 0} \mathcal{H}(\bar{H}, s; a, 1, K) = \frac{-m \left(\frac{s}{\bar{H}}\right)^{n-1} + n \left(\frac{s}{\bar{H}}\right)^{m-1}}{-mn \left[\left(\frac{s}{\bar{H}}\right)^{n} - \left(\frac{s}{\bar{H}}\right)^{m}\right]} =: \mathcal{H}(\bar{H}, s; 0, 1, 0). \tag{58}$$

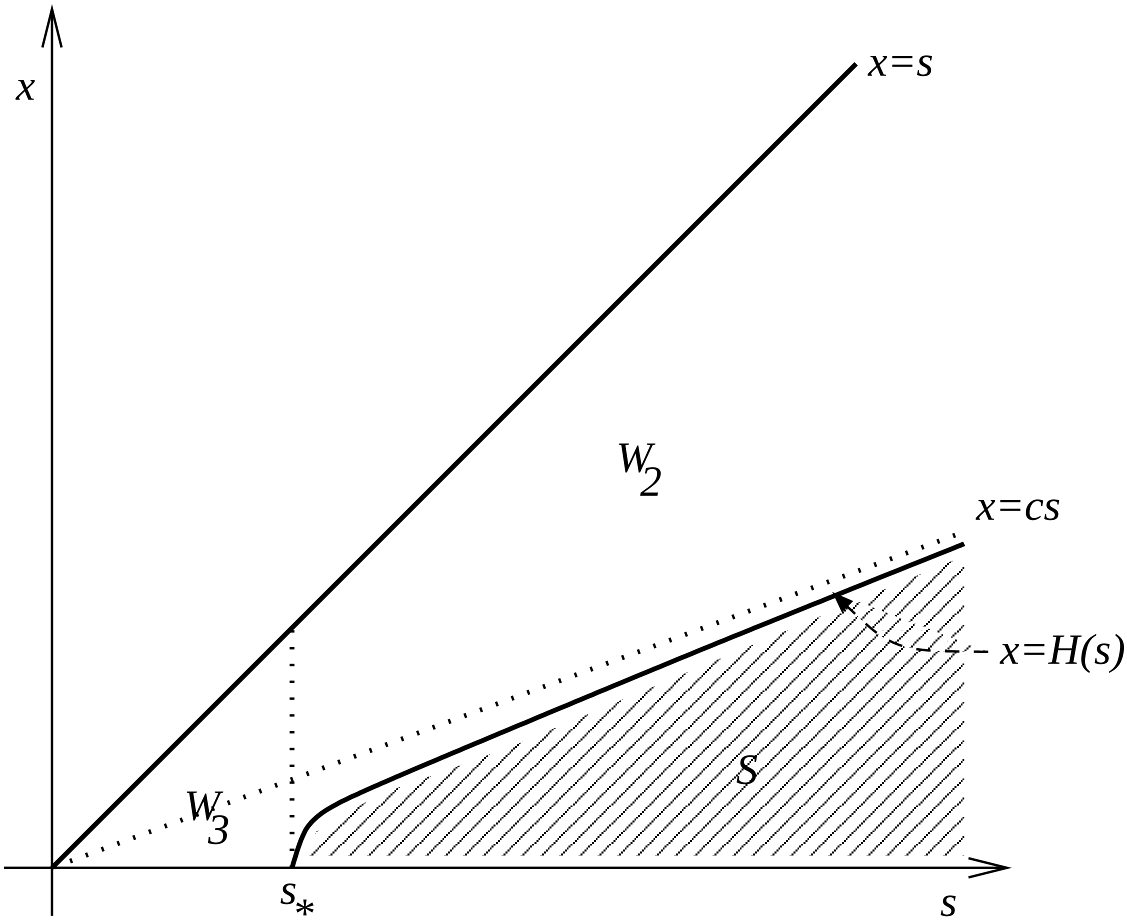

Figure 4: The continuation region $\mathcal{W}_2 \cup \mathcal{W}_3$ and the stopping region $\mathcal{S}$ of the perpetual lookback American option

For $H(s) = cs$, with c given by (56), we can see that

$$\mathcal{H}(H(s), s; a, 1, K) = \frac{-m + nc^{n-m}}{-mn(1 - c^{n-m})} c = c.$$

This calculation shows that the function $s \mapsto cs$, which plainly has the required asymptotic behaviour, satisfies the ODE (34) with $\mathcal{H}$ given by (58). It follows that, at the limit $a, K \downarrow 0$, $s_* = 0$, the stopping region is given by

$$\mathcal{S} = \left\{ (x, s) \in \mathbb{R}_+^2 \mid \ 0 < x \le cs \right\},$$

and the continuation region is given by

$$\mathcal{W}_2 = \left\{ (x, s) \in \mathbb{R}_+^2 \mid \ cs < x \le s \right\},$$

which is in agreement with the results of Shepp and Shiryaev [SS93] (see also Figure 5).

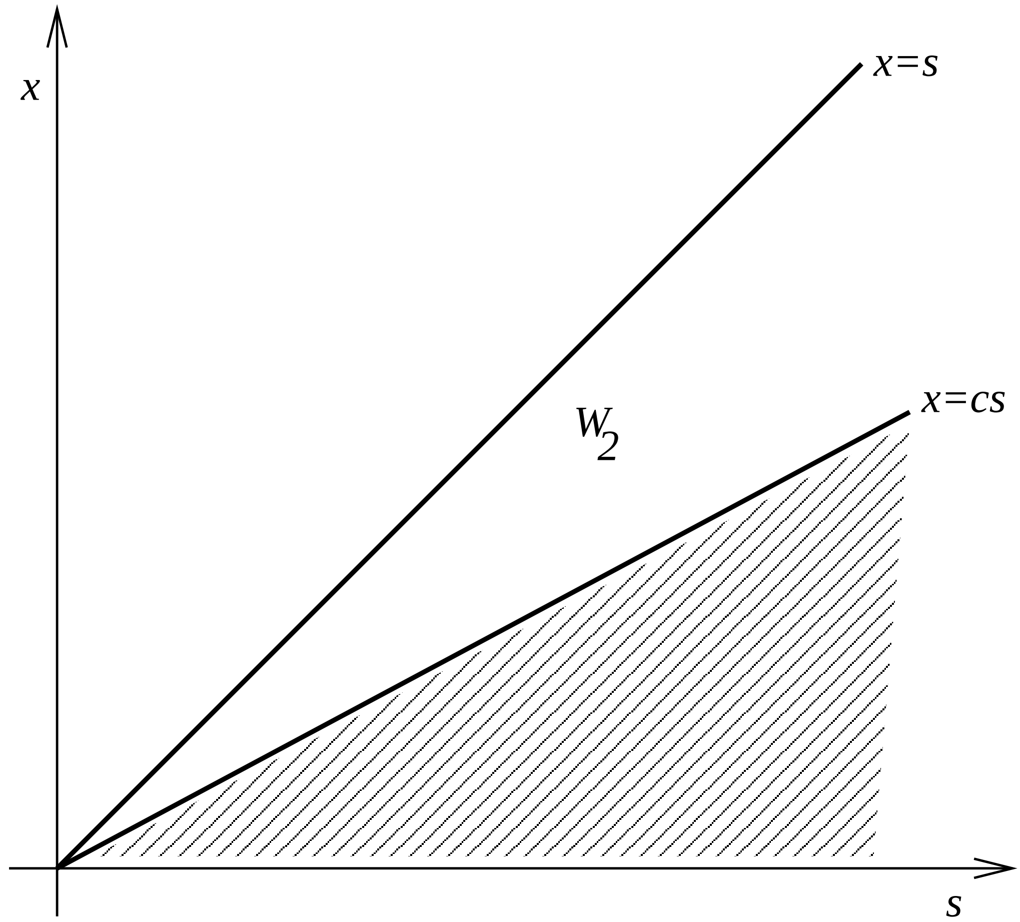

Figure 5: The continuation region $\mathcal{W}_2$ and the stopping region $\mathcal{S}$ of the Russian option

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Appendix: Proof of results in Section 4

Proof of Lemma 3. Recalling the definition (37) of $\mathcal{D}_H$, we can see that the calculation

$$\begin{aligned}(a-m)(n-a)\bar{H}^a s^b + mnK &\geq (a-m)(n-a)\left[\left(\frac{nK}{n-a}\right)^{1/a} s^{-b/a}\right]^a s^b + mnK \\ &= anK \\ &> 0 \quad \text{ for all } (\bar{H}, s) \in \mathcal{D}_H, \end{aligned} \tag{59}$$

implies that the function $(\bar{H}, s) \mapsto 1/\left[(a-m)(n-a)\bar{H}^a s^b + mnK\right]$ is strictly positive and Lipschitz continuous in the closure of $\mathcal{D}_H$, and that the function $\mathcal{H}$ defined by (35) is strictly positive and locally Lipschitz in $\mathcal{D}_H$. In light of these observations, we can see that, given any $s_* > s_\dagger$, the ODE (34) with initial condition (36) has a unique, strictly increasing solution $H(\cdot) \equiv H(\cdot; s_*)$ in $\mathcal{D}_H$ up to a possible "explosion" point $\hat{s}(s_*)$ at which this solution hits the boundary of $\mathcal{D}_H$ that coincides with the line defined by $\bar{H} = s$. Furthermore, uniqueness implies that

$$s_*^1 < s_*^2 \quad \Leftrightarrow \quad H(s; s_*^1) < H(s; s_*^2) \text{ for all } s \in \left[s_*^2, \hat{s}(s_*^1)\right[, \tag{60}$$

where we set $\hat{s}(s_*^1) = \infty$ if $H(s; s_*^1) \in \mathcal{D}_H$ for all $s \geq s_*^1$, and we adopt the convention $[\gamma, \gamma[= \emptyset$ for $\gamma \in \mathbb{R}$.

We now fix any initial condition $s_* > s_\dagger$ and we consider the associated solution $H(\cdot) \equiv H(\cdot; s_*)$ of (34)–(36). We define $h(s) = H(s)/s$, and we calculate

$$\begin{aligned}\dot{h}(s) = &\frac{\left[-(a-m)(n-a-b) + (n-a)(a+b-m)h^{n-m}(s)\right] h^a(s) s^{a+b}}{(a-m)(n-a)\left[1-h^{n-m}(s)\right] h^a(s) s^{a+b} + mnK\left[1-h^{n-m}(s)\right]} \frac{h(s)}{s} \\ &+ \frac{-mnK\left[1-h^{n-m}(s)\right]}{(a-m)(n-a)\left[1-h^{n-m}(s)\right] h^a(s) s^{a+b} + mnK\left[1-h^{n-m}(s)\right]} \frac{h(s)}{s}\end{aligned} \tag{61}$$

and

$$h(s_*) = \left(\frac{nK}{n-a}\right)^{1/a} s_*^{-(a+b)/a} =: g(s^*). \tag{62}$$

Also, we note that $(\bar{H}, s) \in \mathcal{D}_H$ if and only if $(\bar{H}/s, s) \in \mathcal{D}_h$, where the domain $\mathcal{D}_h$ is defined by

$$\mathcal{D}_h = \left\{(\bar{h}, s) \in \mathbb{R}_+^2 \;\middle|\; s_\dagger < s \text{ and } \left(\frac{nK}{n-a}\right)^{1/a} s^{-(a+b)/a} \leq \bar{h} < 1\right\},$$

that (59) implies that

$$(a-m)(n-a)\bar{h}^a s^{a+b} + mnK \geq anK > 0 \quad \text{for all } (\bar{h}, s) \in \mathcal{D}_h, \tag{63}$$

and that (60) implies trivially the equivalence

$$s_*^1 < s_*^2 \quad \Leftrightarrow \quad h(s; s_*^1) < h(s; s_*^2) \text{ for all } s \in \left[s_*^2, \hat{s}(s_*^1)\right[. \tag{64}$$

If there exists $\tilde{s} \geq s_*$ such that $h(\tilde{s}) = c$, where c is defined by (41), then (61) and (63) imply that

$$\dot{h}(s) \geq \frac{-mnK}{(a-m)(n-a)h^a(s)s^{a+b} + mnK} \frac{h(s)}{s} > 0, \quad \text{for } s \geq \tilde{s}. \tag{65}$$

In this case, there exists $\hat{s} = \hat{s}(s_*) < \infty$ such that

$$h(s) \equiv \frac{H(s)}{s} < 1 \text{ for all } s \in [s_*, \hat{s}[, \quad \text{and} \quad \lim_{s \uparrow \hat{s}} h(s) \equiv \lim_{s \uparrow \hat{s}} \frac{H(s)}{s} = 1. \tag{66}$$

To see this claim, we argue by contradiction, and we assume that $h(s) < 1$ for all $s \geq s_*$. Since h is strictly increasing in $[\tilde{s}, \infty[$ (see (65)) and $h(\tilde{s}) = c$ satisfies

$$-(a-m)(n-a-b) + (n-a)(a+b-m)h^{n-m}(\tilde{s}) = 0,$$

there exist $\varepsilon > 0$ and $s_\varepsilon > \tilde{s}$ such that

$$-(a-m)(n-a-b) + (n-a)(a+b-m)h^{n-m}(s) > \varepsilon \quad \text{for all } s \geq s_\varepsilon. \tag{67}$$

For such parameter values, we can use the fact that $h(s) \in]c, 1[$ for all $s > s_\varepsilon$, to calculate

$$\begin{aligned}
\ln h(s) - {} & \ln h(s_\varepsilon) \\
&= \int_{s_\varepsilon}^{s} \frac{\dot{h}(u)}{h(u)}\, du \\
&\geq \int_{s_\varepsilon}^{s} \frac{1 - h^{n-m}(u)}{h(u)} \dot{h}(u)\, du \\
&\overset{(61)}{=} \int_{s_\varepsilon}^{s} \frac{\left[-(a-m)(n-a-b) + (n-a)(a+b-m)h^{n-m}(u)\right] h^a(u)u^{a+b}}{(a-m)(n-a)h^a(u)u^{a+b} + mnK} u^{-1}\, du \\
&\quad + \int_{s_\varepsilon}^{s} \frac{-mnK\left[1 - h^{n-m}(u)\right]}{(a-m)(n-a)h^a(u)u^{a+b} + mnK} u^{-1}\, du \\
&\overset{(63),(65),(67)}{>} \int_{s_\varepsilon}^{s} \frac{\varepsilon h^a(u)u^{a+b}}{(a-m)(n-a)h^a(u)u^{a+b}} u^{-1}\, du \\
&= \frac{\varepsilon}{(a-m)(n-a)} \left[\ln s - \ln s_\varepsilon\right],
\end{aligned}$$

which implies that

$$h(s) > h(s_\varepsilon) \left(\frac{s}{s_\varepsilon}\right)^{\varepsilon/[(a-m)(n-a)]} \qquad \text{for all } s \geq s_\varepsilon.$$

However, this inequality contradicts the assumption that $h(s) < 1$ for all $s \geq s_*$. It follows that, for all $s_* > s_\dagger$ such that $h(s; s_*) \geq c$, for some $s \geq s_*$, (66) holds true for some $\hat{s} = \hat{s}(s_*) < \infty$. Furthermore, noting that $h(s_*) > c$ for all $s_* \in]s_\dagger, c^{-a/(a+b)} s_\dagger[$, we can see that this conclusion and (64) establish part (I) of the lemma, provided that we define

$$\begin{aligned} s_\circ &= \sup \{ s_* > s_\dagger \mid \ h(s; s_*) \geq c \text{ for some } s \geq s_* \} \\ &> c^{-a/(a+b)} s_\dagger \\ &= \left(\frac{nK}{n-a} \right)^{1/(a+b)} \left[\frac{(n-a)(a+b-m)}{(a-m)(n-a-b)} \right]^{a/[(a+b)(n-m)]} . \end{aligned} \tag{68}$$

To proceed further, we define

$$s^\circ = \inf \left\{ s_* > s_\dagger \mid \ \sup_{s \geq s_*} h(s; s_*) < c \right\} \geq s_\circ, \tag{69}$$

and we note that part (II) of the lemma will be established if we show that

$$s^\circ < \infty, \qquad h(s; s_\circ) < c \quad \text{for all } s \geq s_\circ, \tag{70}$$

and

$$\lim_{s \to \infty} h(s; s_*) = c \quad \text{for all } s_* \in [s_\circ, s^\circ]. \tag{71}$$

To this end, we observe that the second inequality in (70) follows immediately from the analysis above, (64) and a straightforward contradiction argument. To show that $s^\circ < \infty$, we note that (61) and (63) imply that $\dot{h}(s) < 0$ if and only if

$$-mnK \left[1 - h^{n-m}(s) \right] < - \left[-(a-m)(n-a-b) + (n-a)(a+b-m) h^{n-m}(s) \right] h^a(s) s^{a+b}.$$

In view of the definition (41) of c, it follows that

$$\dot{h}(s) < 0 \quad \Leftrightarrow \quad (h(s), s) \in \mathcal{D}_h^- = \left\{ (\bar{h}, s) \in \mathcal{D}_h \mid \ \bar{h} < c \text{ and } \ s > \check{s}(\bar{h}) \right\}, \tag{72}$$

where the function $\check{s} :]0, c[\to \mathbb{R}_+$ is defined by

$$\check{s}(\bar{h}) = \left(\frac{mnK \left[1 - \bar{h}^{n-m} \right]}{\left[-(a-m)(n-a-b) + (n-a)(a+b-m) \bar{h}^{n-m} \right] \bar{h}^a} \right)^{1/(a+b)} .$$

To appreciate the structure of the domain $\mathcal{D}_h^-$, we note that

$$\lim_{\bar{h} \downarrow 0} \check{s}(\bar{h}) = \lim_{\bar{h} \uparrow c} \check{s}(\bar{h}) = \infty. \tag{73}$$

Also, we can calculate

$$\operatorname{sgn}\left(\frac{d\check{s}(\bar{h})}{d\bar{h}}\right) = \operatorname{sgn}\left(Q(\bar{h})\right), \tag{74}$$

where Q is the quadratic in $\bar{h}^{n-m}$ defined by

$$\begin{aligned} Q(\bar{h}) = (n-m)\left[-(a-m)(n-a-b) + (n-a)(a+b-m)\bar{h}^{n-m}\right]\bar{h}^{n-m} \\ + \left[-a(a-m)(n-a-b) + (n-a)(a+b-m)(n-m+a)\bar{h}^{n-m}\right]\left[1-\bar{h}^{n-m}\right]. \end{aligned}$$

Plainly,

$$Q(0) = -a(a-m)(n-a-b) < 0.$$

Also, in view of the definition (41) of c, we can calculate

$$Q(c) = (n-m)(a-m)(n-a-b)\left[1-c^{n-m}\right] > 0.$$

These inequalities, the fact that Q is a quadratic in $\bar{h}^{n-m}$ and (74) imply that there exists a point $\bar{h}^\star \in]0, c[$ such that

$$\frac{d\check{s}(\bar{h})}{d\bar{h}} \begin{cases} < 0 & \text{for all } \bar{h} \in]0, \bar{h}^\star[, \\ > 0 & \text{for all } \bar{h} \in]\bar{h}^\star, c[. \end{cases}$$

It follows that $\check{s}(\bar{h})$ strictly decreases from ∞ to $\check{s}(\bar{h}^\star) > 0$ as $\bar{h}$ increases from 0 to $\bar{h}^\star$ and then strictly increases from $\check{s}(\bar{h}^\star)$ to ∞ as $\bar{h}$ increases from $\bar{h}^\star$ to c. Combining this observation with the fact that the function g defined by (62) strictly decreases to 0 as s increases to ∞, we can see that there exists a point $\bar{h}^\dagger \geq \bar{h}^\star$ such that

$$\check{s}(\bar{h}) > g(s) \quad \text{for all } \bar{h} \in]\bar{h}^\dagger, c[\text{ and } s > g^{[-1]}(\bar{h}^\dagger),$$

where $g^{[-1]}$ is the inverse function of g. This inequality, (72) and a straightforward contradiction argument imply that

$$\sup_{s \geq s_*} h(s; s_*) < c \quad \text{for all } s_* \geq \check{s}(\bar{h}^\dagger),$$

which establishes the claim that $s^\circ < \infty$.

To prove (71), we first note that

$$\{(h(s; s^\circ), s) \mid \ s \geq s^\circ\} \cap \mathcal{D}_h^- = \emptyset. \tag{75}$$

To see this claim, we argue by contradictions, and we assume that there exists $s^\ddagger \geq s^\circ$ such that $h(s^\ddagger; s^\circ) \in \mathcal{D}_h^-$. In this case, $h(s; s^\circ) < h(s^\ddagger; s^\circ)$ for all $s > s^\ddagger$ thanks to (72). Since $s_\circ \leq s^\circ$, this observation, (64) and (70) imply that $\max_{s \geq s^\circ} h(s; s^\circ) < c$. In view of (64) and the continuity of the vector field associated with the ODE (61) that h satisfies, we can see that this conclusion contradicts the definition (69) of s°, and (75) has been established.

Combining (75) with (72), we can see that $h(\cdot; s^\circ)$ is increasing. Furthermore, the continuity of the vector field associated with the ODE (61) that h satisfies, the definition (69) of s°, (70) and (72) imply that

$$c > h(s; s^\circ) \geq \sup\left\{\bar{h} \mid \ (\bar{h}, s) \in \mathcal{D}_h^-\right\} \quad \text{for all } s \geq \check{s}(\bar{h}^\star).$$

Combining this observation with (64) and (70), we can see that $h(\cdot; s_*)$ is increasing and $\lim_{s\to\infty} h(s; s_*)$ exists for all $s_* \in [s_\circ, s^\circ]$. In particular, (71) is true thanks to the second limit in (73).

To establish (42) in part (III) of the lemma, we fix any $s_* > s_\circ$, we note that the associated solution h of the ODE (61) with initial condition (62) satisfies $\sup_{s\geq s_*} h(s) < c$, and we fix any $\varepsilon > 0$ such that

$$h^{n-m}(s) \leq c^{n-m} - \varepsilon \quad \text{for all } s \geq s_*. \tag{76}$$

Recalling the definition (41) of $c \in]0, 1[$, we can see that

$$\begin{aligned} -(a-m)(n-a-b) + (n-a)(a+b-m)h^{n-m}(s) &\leq -\varepsilon(n-a)(a+b-m) \\ &< 0 \qquad \text{for all } s \geq s_*. \end{aligned} \tag{77}$$

Furthermore, we can use the fact that $s \mapsto H(s) \equiv sh(s)$ is strictly increasing to obtain

$$\begin{aligned} (a-m)(n-a)h^a(s)s^a + mnKs^{-b} &> (a-m)(n-a)H^a(s_*) + mnKs_*^{-b} \\ &\overset{(36),(59)}{=} anKs_*^{-b} \\ &> 0 \quad \text{for all } s \geq s_*, \end{aligned} \tag{78}$$

which implies that

$$0 < \frac{-mnK}{(a-m)(n-a)h^a(s)s^{a+b} + mnK} < \frac{-ms_*^b}{as^b} \quad \text{for all } s \geq s_*. \tag{79}$$

Now, (78) and the fact that $h(s) \in]0, 1[$ for all $s \geq s_*$, imply that

$$\begin{aligned} 0 &< (a-m)(n-a)\left[1 - h^{n-m}(s)\right] h^a(s)s^{a+b} + mnK\left[1 - h^{n-m}(s)\right] \\ &< (a-m)(n-a)h^a(s)s^{a+b} \qquad \text{for all } s \geq s_*. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{1}{(a-m)(n-a)\left[1 - h^{n-m}(s)\right] h^a(s)s^{a+b} + mnK\left[1 - h^{n-m}(s)\right]} \\ &\qquad > \frac{1}{(a-m)(n-a)h^a(s)s^{a+b}}, \\ &\qquad > 0 \qquad \text{for all } s \geq s_*. \end{aligned} \tag{80}$$

These inequalities and (77) imply that

$$\begin{aligned}
&\frac{\left[-(a-m)(n-a-b)+(n-a)(a+b-m)h^{n-m}(s)\right]h^{a}(s)s^{a+b}}{(a-m)(n-a)\left[1-h^{n-m}(s)\right]h^{a}(s)s^{a+b}+mnK\left[1-h^{n-m}(s)\right]}\\
&\qquad\qquad\leq \frac{-(a-m)(n-a-b)+(n-a)(a+b-m)h^{n-m}(s)}{(a-m)(n-a)}\\
&\qquad\qquad\leq -\frac{\varepsilon(a+b-m)}{a-m}. \qquad (81)
\end{aligned}$$

In view of this calculation and (79), we can see that (61) implies that

$$\frac{d\ln h(s)}{ds} \leq -\frac{\varepsilon(a+b-m)}{a-m}s^{-1} + \frac{-ms_*^b}{a}s^{-(b+1)}.$$

If we define $C_\varepsilon = \varepsilon(a+b-m)/(a-m)$, then we can see that this inequality implies that

$$\begin{aligned}
\ln h(s) - \ln h(s_*) &\leq -C_\varepsilon \int_{s_*}^{s} u^{-1}\,du + \frac{-ms_*^b}{a}\int_{s_*}^{s} u^{-(b+1)}\,du\\
&= \ln s^{-C_\varepsilon} + \ln s^{C_\varepsilon} - \frac{-ms_*^b}{ab}s^{-b} + \frac{-m}{ab}\\
&\leq \ln s^{-C_\varepsilon} + \ln s^{C_\varepsilon} + \frac{-m}{ab} \qquad \text{for all } s \geq s_*.
\end{aligned}$$

Therefore,

$$h(s) \leq \Gamma_\varepsilon s^{-C_\varepsilon} \quad \text{for all } s \geq s_*,$$

where $\Gamma_\varepsilon = \Gamma_\varepsilon(s_*) = e^{-m/(ab)}s_*^{C_\varepsilon}h(s_*)$. Using this estimate, (79) and the first inequality in (81), we can see that (61) implies that

$$\begin{aligned}
\frac{d\ln h(s)}{ds} &\leq \frac{-(a-m)(n-a-b)+(n-a)(a+b-m)h^{n-m}(s)}{(a-m)(n-a)}s^{-1} + \frac{-ms_*^b}{a}s^{-(b+1)}\\
&\leq -\frac{n-a-b}{n-a}s^{-1} + \frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{a-m}s^{-(C_\varepsilon(n-m)+1)} + \frac{-ms_*^b}{a}s^{-(b+1)}
\end{aligned}$$

for all $s \geq s_*$. It follows that

$$\begin{aligned}
\ln h(s) - \ln h(s_*) \leq & -\frac{n-a-b}{n-a}\int_{s_*}^{s} u^{-1}\,du + \frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{a-m}\int_{s_*}^{s} u^{-(C_\varepsilon(n-m)+1)}\,du \\
& + \frac{-ms_*^b}{a}\int_{s_*}^{s} u^{-(b+1)}\,du \\
= & \ln s^{-(n-a-b)/(n-a)} + \ln s_*^{(n-a-b)/(n-a)} \\
& + \frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{(a-m)(n-m)C_\varepsilon}\left[-s^{-C_\varepsilon(n-m} + s_*^{-C_\varepsilon(n-m)}\right] + \frac{-ms_*^b}{ab}\left[-s^{-b} + s_*^{-b}\right] \\
\leq & \ln s^{-(n-a-b)/(n-a)} + \ln s_*^{(n-a-b)/(n-a)} \\
& + \frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{(a-m)(n-m)C_\varepsilon s_*^{C_\varepsilon(n-m}} + \frac{-m}{ab} \qquad \text{for all } s \geq s_*.
\end{aligned}$$

Therefore,

$$h(s) \leq s_*^{(n-a-b)/(n-a)} h(s_*) \exp\left(\frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{(a-m)(n-m)C_\varepsilon s_*^{C_\varepsilon(n-m)}} + \frac{-m}{ab}\right) s^{-(n-a-b)/(n-a)}$$

for all $s \geq s_*$. This inequality, (76) and the identity $H(s) = sh(s)$ imply immediately the right-hand side of (42) if we choose any

$$C \geq s_*^{(n-a-b)/(n-a)} h(s_*) \exp\left(\frac{(a+b-m)\Gamma_\varepsilon^{n-m}}{(a-m)(n-m)C_\varepsilon s_*^{C_\varepsilon(n-m)}} + \frac{-m}{ab}\right).$$

The left-hand side of (42) follows immediately from the calculation

$$\begin{aligned}
\frac{d\ln H(s)}{ds} &= \frac{b\left[(a-m)+(n-a)h^{n-m}(s)\right]H^a(s)s^b}{\left[(a-m)(n-a)H^a(s)s^b + mnK\right]\left[1-h^{n-m}(s)\right]}s^{-1} \\
&> \frac{b\left[(a-m)+(n-a)h^{n-m}(s)\right]H^a(s)s^b}{\left[(a-m)(n-a)H^a(s)s^b + mnK\right]}s^{-1} \\
&> \frac{b}{n-a}s^{-1},
\end{aligned}$$

where we have used (59) and the fact that $1-h^{n-m}(s) \in]0,1[$ for all $s \geq s_*$, to establish the first inequality.

Finally, (36) and the fact that H is strictly increasing imply the inequalities

$$(a-m)H^a(s)s^b + mK > (a-m)H^a(s_*)s_*^b + mK = \frac{n-m}{n-a}aK > 0$$

and

$$(n-a)H^a(s)s^b - nK > (n-a)H^a(s_*)s_*^b - nK = 0 \tag{82}$$

for all $s > s_*$ in the domain of H, which establish part (IV) of the lemma. □

Proof of Lemma 4. We fix any $s_* \geq s_\circ$, and we consider the associated function $w(\cdot) \equiv w(\cdot; s_*)$ defined by (44). By construction, we will prove that the positive function w is a solution of the variational inequality (9) with boundary condition (10) that satisfies (11)–(14) if we show that

$$\begin{aligned} f(x,s) &:= \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\left(x^a s^b - K\right) + \mu x \frac{\partial}{\partial x}\left(x^a s^b - K\right) - r\left(x^a s^b - K\right) \\ &\equiv \left[\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right) a - r\right] x^a s^b + rK \\ &\leq 0 \qquad \text{for all } (x,s) \in \mathcal{S}, \end{aligned} \tag{83}$$

and

$$g(x,s) := w(x,s) - x^a s^b + K \geq 0 \quad \text{for all } (x,s) \in \mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3. \tag{84}$$

In view of Assumption 1, $a \in]m, n[$, where $m < 0 < n$ are the solutions of the quadratic equation (7). Therefore,

$$\frac{\partial f(x,s)}{\partial x} = a\left[\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right) a - r\right] x^{a-1} s^b < 0.$$

This observation and the calculation

$$\begin{aligned} f(G(s),s) &= \left[\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right) a - r\right] \left[\left(\frac{nK}{n-a}\right)^{1/a} s^{-b/a}\right]^a s^b + rK \\ &= \frac{aK}{n-a}\left[\frac{1}{2}\sigma^2 n^2 + \left(\mu - \frac{1}{2}\sigma^2\right) n - r - \frac{1}{2}\sigma^2 n(n-a)\right] \\ &= -\frac{1}{2}\sigma^2 anK \\ &< 0 \end{aligned}$$

imply that

$$f(x,s) < 0 \quad \text{for all } s \geq s_* \text{ and } x \in [G(s), s]. \tag{85}$$

In particular, (83) holds true.

Taking note of the identity

$$\left(\frac{nK}{n-a}\right)^{-1} G^a(s) s^b = 1,$$

which follows from the definition (29) of G, we can see that, for $(x,s) \in \mathcal{W}_1$,

$$\begin{aligned}\frac{\partial g(x,s)}{\partial x} &= a x^{a-1} s^b \left[\left[\left(\frac{nK}{n-a} \right)^{-1} x^a s^b \right]^{(n-a)/a} - 1 \right] \\ &< a x^{a-1} s^b \left[\left[\left(\frac{nK}{n-a} \right)^{-1} G^a(s) s^b \right]^{(n-a)/a} - 1 \right] \\ &= 0.\end{aligned}$$

This calculation implies that the function $x \mapsto g(x,s)$ is strictly decreasing in $\mathcal{W}_1$. Combining this observation with the fact that $w(G(s),s) - G^a(s)s^b + K = 0$, which follows from the C^1-continuity of w at $G(s)$, we can see that the inequality (84) holds true for all $(x,s) \in \mathcal{W}_1$.

For $(x,s) \in \mathcal{W}_3$, $g(x,s) = \overline{g}(x,s)$, where $\overline{g}$ is defined by

$$\overline{g}(x,s) = \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_*^{bn/a} x^n - x^a s^b + K, \quad \text{for } (x,s) \in \mathbb{R}_+^2.$$

In view of the calculation

$$\frac{\partial \overline{g}(x,s)}{\partial x} = a x^{a-1} \left[\left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_*^{bn/a} x^{n-a} - s^b \right],$$

we can see that the function $x \mapsto \overline{g}(x,s)$ has a unique minimum at

$$x = \left[\frac{nK}{n-a} s_*^{-bn/(n-a)} \right]^{1/a} s^{b/(n-a)}.$$

It follows that, given any $(x,s) \in \mathcal{W}_3$,

$$\begin{aligned}g(x,s) \geq{}& \frac{a}{n} \left(\frac{nK}{n-a} \right)^{-(n-a)/a} s_*^{bn/a} \left[\frac{nK}{n-a} s_*^{-bn/(n-a)} \right]^{n/a} s^{bn/(n-a)} \\ & - \frac{nK}{n-a} s_*^{-bn/(n-a)} s^{ab/(n-a)} s^b + K \\ ={}& -K s_*^{-bn/(n-a)} s^{bn/(n-a)} + K \\ >{}& 0, \end{aligned} \tag{86}$$

with the last inequality following because $s < s_*$. These arguments establish that (84) holds true for all $(x,s) \in \mathcal{W}_3$.

To proceed further, we note that

$$\begin{aligned}\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g(x,s)}{\partial x^2} + \mu x \frac{\partial g(x,s)}{\partial x} - r g(x,s) &= - \left[\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2 \right) a - r \right] x^a s^b - rK \\ &> 0 \qquad \text{for all } s \geq s_* \text{ and } x \in]H(s), s[.\end{aligned}$$

Here, the equality follows because the function $w(\cdot, s)$ satisfies the ODE (5) in the waiting region $\mathcal{W}_2$, and the inequality follows from (85) and the definition of f in (83). This inequality and the maximum principle imply that, given any $s \geq s_*$,

$$\text{the function } g(\cdot, s) \text{ has no positive maximum in the set }]H(s), s[. \tag{87}$$

Now, given (x, s) in the interior of $\mathcal{W}_2$, we can use the identity

$$\begin{aligned}\frac{\partial^2 g(x,s)}{\partial x^2} &= n(n-1)\frac{(a-m)H^a(s)s^b + mK}{n-m}H^{-n}(s)x^{n-2} \\ &\quad + m(m-1)\frac{(n-a)H^a(s)s^b - nK}{n-m}H^{-m}(s)x^{m-2} - a(a-1)s^b x^{a-2}\end{aligned}$$

to calculate

$$\begin{aligned}\lim_{x \downarrow H(s)} \frac{\partial^2 g(x,s)}{\partial x^2} &= -[a^2 - a(m+n) + mn]H^{a-2}(s)s^b + mnKH^{-2}(s) \\ &\overset{(8)}{=} -\frac{2}{\sigma^2}\left\{\left[\frac{1}{2}\sigma^2 a^2 + \left(\mu - \frac{1}{2}\sigma^2\right)a - r\right]H^a(s)s^b + rK\right\}H^{-2}(s) \\ &> 0,\end{aligned} \tag{88}$$

the inequality following thanks to (85) and the definition of f in (83). Combining this calculation with the fact that $g(H(s), s) = \partial g(H(s), s)/\partial x = 0$, which follows from the C^1-continuity of $w(\cdot, s)$ at $H(s)$, we can see that $\partial g(x, s)/\partial x > 0$ and $g(x, s) > 0$ for all x sufficiently close to $H(s)$. These observations and (87) imply that, given any $s \geq s_*$, the function $g(\cdot, s)$ is increasing and positive in $[H(s), s]$, which establishes (84) for $(x, s) \in \mathcal{W}_2$. □

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