On the submartingale / supermartingale property of diffusions in natural scale

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November 13, 2014

Abstract

Kotani [5] has characterised the martingale property of a one-dimensional diffusion in natural scale in terms of the classification of its boundaries. We complement this result by establishing a necessary and sufficient condition for a one-dimensional diffusion in natural scale to be a submartingale or a supermartingale. Furthermore, we study the asymptotic behaviour of the diffusion's expected state at time t as $t \to \infty$. We illustrate our results by means of several examples.

1 Introduction and the main result

We consider a one-dimensional conservative regular continuous strong Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X_t;\ t \geq 0,\ x \in \mathcal{I})$ with values in an interval $\mathcal{I} \subseteq [-\infty, \infty]$ with endpoints $\alpha < \beta$ that is open, closed or semi-open. We recall that a Markov process is called conservative if there is no killing and a one-dimensional continuous strong Markov process with state space $\mathcal{I}$ is called regular if

$$\mathbb{P}_x(T_y < \infty) > 0 \quad \text{for all } x \in (\alpha, \beta) \text{ and } y \in \mathcal{I}.$$

Throughout the paper we denote

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad T_{\leq y} = \inf\{t \geq 0 \mid \ X_t \leq y\} \quad \text{and} \quad T_{\geq y} = \inf\{t \geq 0 \mid \ X_t \geq y\},$$

for $y \in [\alpha, \beta]$, with the usual convention that $\inf \emptyset = \infty$. Also, we denote by p and m the scale function and the speed measure of X. Given a measure ξ on $\big(\mathcal{I}, \mathcal{B}(\mathcal{I})\big)$ and an interval $J \subseteq \mathcal{I}$, we denote by $\xi|_J$ the restriction of ξ in $\big(\mathcal{I} \cap J, \mathcal{B}(\mathcal{I} \cap J)\big)$.

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Given a probability measure ν on $\big(\mathcal{I}, \mathcal{B}(\mathcal{I})\big)$, we define the probability measure $\mathbb{P}_\nu(\cdot) = \int_{\mathcal{I}} \mathbb{P}_x(\cdot)\, \nu(dx)$ on $(\Omega, \mathcal{F})$. To avoid trivialities, we make the following assumption.

Assumption 1. $\nu \neq \lambda\delta_\alpha + (1-\lambda)\delta_\beta$ for all $\lambda \in [0,1]$. □

The process

$$Y_t = p(X_{t\wedge T_\alpha \wedge T_\beta}), \quad t \geq 0, \tag{1}$$

is a continuous $\mathbb{P}_\nu$-local martingale for all probability measures ν on $\big(\mathcal{I}, \mathcal{B}(\mathcal{I})\big)$ in the sense of Definition IV.1.5 in Revuz and Yor [7]:

Definition 1. A continuous $(\mathcal{F}_t)$-adapted process (Z_t) defined on $(\Omega, \mathcal{F}, \mathcal{F}_t)$ is a $\mathbb{P}_\nu$-local martingale if there exists a sequence (τ_n) of $(\mathcal{F}_t)$-stopping times such that $\tau_n \nearrow \infty$, $\mathbb{P}_\nu$-a.s., and the process $(Z_{t\wedge\tau_n} \mathbf{1}_{\{\tau_n > 0\}})$ is a $\mathbb{P}_\nu$-martingale for all $n \geq 1$. □

The reason for considering this more general definition is because we allow for the initial value $Y_0 = p(X_0)$ to be non-integrable, namely, we allow for the possibility that $\mathbb{E}_\nu\big[|Y_0|\big] = \int_{\mathcal{I}} \big|p(x)\big|\, \nu(dx) = \infty$. In Remark 1 in the next section, we discuss the fact that (Y_t) is indeed a $\mathbb{P}_\nu$-local martingale for all ν.

In this note we establish conditions under which (Y_t) is a $\mathbb{P}_\nu$-submartingale, a $\mathbb{P}_\nu$-supermartingale or a $\mathbb{P}_\nu$-martingale. To this end, we recall Definition II.1.1 in Revuz and Yor [7]:

Definition 2. The process (Y_t) is a $\mathbb{P}_\nu$-supermartingale if
(i) $\mathbb{E}_\nu[Y_t^-] < \infty$ for all $t \geq 0$, and
(ii) $\mathbb{E}_\nu[Y_t \mid \mathcal{F}_s] \leq Y_s$ for all $s < t$.
The process (Y_t) is a $\mathbb{P}_\nu$-submartingale if $(-Y_t)$ is a $\mathbb{P}_\nu$-supermartingale, while, it is a $\mathbb{P}_\nu$-martingale if it is both a $\mathbb{P}_\nu$-supermartingale and a $\mathbb{P}_\nu$-submartingale. □

Kotani [5] has proved the following result.

Theorem 1. *The following statements are equivalent:*
(a) (Y_t) *is a* $\mathbb{P}_\nu$*-martingale.*
(b) $\int_{\mathcal{I}} \big|p(u)\big|\, \nu(du) < \infty$ *and neither* α *nor* β *is an entrance boundary point*[1] *for* X.

It is worth noting that the condition $\int_{\mathcal{I}} \big|p(u)\big|\, \nu(du) < \infty$ is equivalent to the condition $\mathbb{E}_\nu\big[|Y_t|\big] < \infty$ for all $t \geq 0$, which is requirement (i) of Definition 2 for the case of a martingale (see Corollary 4 in the next section). On the other hand, the property that neither α nor β is an entrance boundary point for X, which does not depend on the choice of ν, is equivalent

[1] The terminology in the boundary classification we adopt here is consistent with the one in Karlin and Taylor [4, Table 15.6.2], Revuz and Yor [7, Section VII.3], Rogers and Williams [8, Section V.51] and Urusov and Zervos [9, Section 3]. However, it is different from the terminology in Itô and McKean [3, Sections 4.1 and 4.6] and Borodin and Salminen [1, Section II.1]. In particular, what is called "entrance boundary" in the latter references is different from what we call "entrance boundary".

to the property that one of the following conditions (i)–(iv) holds. (Here, $c \in (\alpha, \beta)$ is an arbitrary point.)

(i) $p(\alpha) > -\infty$ and $p(\beta) < \infty$.

(ii) $p(\alpha) > -\infty$, $p(\beta) = \infty$ and $\int_{(c,\beta)} p(u)\, m(du) = \infty$.

(iii) $p(\alpha) = -\infty$, $p(\beta) < \infty$ and $\int_{(\alpha,c)} \big|p(u)\big|\, m(du) = \infty$.

(iv) $p(\alpha) = -\infty$, $p(\beta) = \infty$, $\int_{(\alpha,c)} \big|p(u)\big|\, m(du) = \infty$ and $\int_{(c,\beta)} p(u)\, m(du) = \infty$.

The equivalence stated here follows from the criteria for a boundary point of X to be an entrance one (see e.g. Urusov and Zervos [9, Section 3]). In fact, Kotani [5] stated Theorem 1 in terms of conditions (i)–(iv) rather than in terms of (b). At this point, it is worth noting that Delbaen and Shirakawa [2] had earlier proved this result in a special case. In Table 1, we provide additional information on the $\mathbb{P}_\nu$-martingale (Y_t) for each of the cases identified by (i)–(iv) above.

Case number	Properties of the martingale (Y_t) under $\mathbb{P}_\nu$
(i)	U.I. martingale (bounded martingale)
(ii)	Not a U.I. martingale ($\lim_{t\to\infty} Y_t = p(\alpha)$, $\mathbb{P}_\nu$-a.s.); $\mathbb{E}_\nu \left[\sup_{t\geq 0} Y_t\right] = \infty$
(iii)	Not a U.I. martingale ($\lim_{t\to\infty} Y_t = p(\beta)$, $\mathbb{P}_\nu$-a.s.); $\mathbb{E}_\nu \left[\inf_{t\geq 0} Y_t\right] = -\infty$
(iv)	Not a U.I. martingale ($\limsup_{t\to\infty} Y_t = -\liminf_{t\to\infty} Y_t = \infty$, $\mathbb{P}_\nu$-a.s.)

Table 1. Properties of the martingale (Y_t) under $\mathbb{P}_\nu$ for each of the cases identified by (i)–(iv) above.

Our main result is the next theorem, which provides necessary and sufficient conditions for (Y_t) to be a $\mathbb{P}_\nu$-supermartingale. The criterion for (Y_t) to be a $\mathbb{P}_\nu$-submartingale is symmetric.

Theorem 2. *The following statements are equivalent:*

(A) (Y_t) *is a* $\mathbb{P}_\nu$*-supermartingale.*

(B) $\int_{\mathcal{I}} p(u)^-\, \nu(du) < \infty$ *and* α *is not an entrance boundary point for* X*.*

Once again, we note that the condition $\int_{\mathcal{I}} p(u)^-\, \nu(du) < \infty$ is equivalent to requirement (i) of Definition 2 (see Corollary 4). Also, we repeat that the property that α is not an entrance boundary point for X, which does not depend on the choice of ν, is equivalent to the property that one of the following conditions (I)–(II) holds, where $c \in (\alpha, \beta)$ is an arbitrary point.

(I) $p(\alpha) > -\infty$.

(II) $p(\alpha) = -\infty$ and $\int_{(\alpha,c)} \big|p(u)\big|\, m(du) = \infty$.

Clearly, Theorem 2 and its submartingale counterpart imply Theorem 1. On the other hand, Theorem 2 can be proved in exactly the same way as Theorem 1. Here, we make use of Theorem 1 and Corollary 4 that have already been established in Kotani [5] to derive a slightly shorter proof of our main result. In Section 3, we present a couple of examples illustrating the scope of application of Theorem 2.

Beyond the submartingale / supermartingale property of Y, we study the asymptotic behaviour of the function $t \mapsto \mathbb{E}_x[Y_t]$ as $t \to \infty$ for the most important class of diffusions that arises when X identifies with the solution to a SDE (see Example 1 in Section 3 for the precise context). Our main result in this direction, Theorem 5 in Section 4, establishes that, if $p(\alpha) > -\infty$, then Y is a strict $\mathbb{P}_x$-supermartingale if and only if

$$\lim_{t\to\infty} \mathbb{E}_x[Y_t] = \lim_{t\to\infty} \mathbb{E}_x\big[p(X_{t\wedge T_\alpha})\big] = p(\alpha).$$

Apart from its independent interest, this result is instrumental in the study of Example 4 in Section 5, which has the interesting property that the function $t \mapsto \mathbb{E}_x\big[p(X_t)\big]$ is monotone and switches sign as the initial point $x \in \mathbb{R}$ switches sign.

2 Proof of Theorem 2

Before addressing the proof of Theorem 2, we make three remarks that clarify material presented in the previous section and make observations we need in the proof.

Remark 1. To see that the process (Y_t) defined by (1) is indeed a $\mathbb{P}_\nu$-local martingale for all probability measures ν on $\big(\mathcal{I}, \mathcal{B}(\mathcal{I})\big)$, we first recall that (Y_t) is a $\mathbb{P}_x$-local martingale for all $x \in \mathcal{I}$ (see Revuz and Yor [7, Proposition VII.3.5]). In particular, the process $(Y_{t\wedge T_{\le a}\wedge T_{\ge b}})$ is a $\mathbb{P}_x$-martingale for all x and $a < b$ in $\mathcal{I}$. Note that α (resp., β) belongs to $\mathcal{I}$, i.e., it is accessible, only if $p(\alpha) > -\infty$ (resp., $p(\beta) < \infty$). In view of this observation and the definition (1) of (Y_t), we can see that, if we define

$$\tau_n^\alpha = \begin{Bmatrix} \infty, & \text{if } \alpha \text{ is accessible} \\ T_{\le a_n}, & \text{if } \alpha \text{ is inaccessible} \end{Bmatrix} \quad \text{and} \quad \tau_n^\beta = \begin{Bmatrix} \infty, & \text{if } \beta \text{ is accessible} \\ T_{\ge b_n}, & \text{if } \beta \text{ is inaccessible} \end{Bmatrix},$$

for some monotone sequences (a_n) and (b_n) in (α, β) such that $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$, then $(Y_{t\wedge\tau_n^\alpha\wedge\tau_n^\beta})$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$ and $n \ge 1$. Furthermore, $(Y_{t\wedge\tau_n^\alpha\wedge\tau_n^\beta}\mathbf{1}_{\{\tau_n^\alpha\wedge\tau_n^\beta>0\}})$ is a $\mathbb{P}_x$-martingale for all $x \in \mathcal{I}$ and $n \ge 1$ because the event $\{\tau_n^\alpha \wedge \tau_n^\beta > 0\}$ belongs to $\mathcal{F}_0$. It follows that $(Y_{t\wedge\tau_n^\alpha\wedge\tau_n^\beta}\mathbf{1}_{\{\tau_n^\alpha\wedge\tau_n^\beta>0\}})$ is a $\mathbb{P}_\nu$-martingale for all ν, which implies that (Y_t) is a $\mathbb{P}_\nu$-local martingale for all ν because $\tau_n^\alpha \wedge \tau_n^\beta \nearrow \infty$, $\mathbb{P}_\nu$-a.s. □

Remark 2. If the process (Y_t) is a $\mathbb{P}_\nu$-supermartingale, then the process $(Y_t\mathbf{1}_{\{\tau>0\}})$ is a $\mathbb{P}_\nu$-supermartingale for every $(\mathcal{F}_t)$-stopping time τ. Plainly, property (i) of Definition 2 remains valid. To verify property (ii) of Definition 2, we note that the $\mathbb{P}_\nu$-supermartingale property of (Y_t) implies that

$$\mathbb{E}_\nu\big[Y_t \mathbf{1}_A\big] \le \mathbb{E}_\nu\big[Y_s \mathbf{1}_A\big] \quad \text{for all } s < t \text{ and } A \in \mathcal{F}_s.$$

Therefore, given any $(\mathcal{F}_t)$-stopping time τ,

$$\mathbb{E}_\nu\big[Y_t \mathbf{1}_{\{\tau>0\}} \mathbf{1}_A\big] \le \mathbb{E}_\nu\big[Y_s \mathbf{1}_{\{\tau>0\}} \mathbf{1}_A\big] \quad \text{for all } s < t \text{ and } A \in \mathcal{F}_s$$

because $\{\tau > 0\} \in \mathcal{F}_s$ for all $s \ge 0$, and the $\mathbb{P}_\nu$-supermartingale property of $(Y_t \mathbf{1}_{\{\tau>0\}})$ follows. □

Remark 3. Consider a point $b \in (\alpha, \beta)$ and the diffusion $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}}_x, \tilde{X}_t)$ with state space $\mathcal{I} \cap (-\infty, b]$, and scale function $\tilde{p}$ and speed measure $\tilde{m}$ given by

$$\tilde{p}(x) = p(x) \text{ for all } x \in \mathcal{I} \cap (-\infty, b], \quad \tilde{m}|_{\mathcal{I}\cap(-\infty,b)} = m|_{\mathcal{I}\cap(-\infty,b)} \quad \text{and} \quad \tilde{m}\big(\{b\}\big) = \infty.$$

The diffusion $\tilde{X}$ behaves in the same way as X inside $\mathcal{I} \cap (-\infty, b)$ but is absorbed at b. In particular, the process $(\tilde{Y}_t) = \big(p(\tilde{X}_{t\wedge T_\alpha})\big)$ under the measure $\tilde{\mathbb{P}}_x$ has the same distribution as the process $(Y_{t\wedge T_b})$ under the measure $\mathbb{P}_x$ for all $x \in \mathcal{I} \cap (-\infty, b)$. □

We will need the following result. The case corresponding to $f(x) = |x|$ is Lemma 1 in Kotani [5]. The other cases can be proved in the same way.

Lemma 3. *For each of the functions defined by $f(x) = |x|$, $f(x) = x^+$ and $f(x) = x^-$, there exist constants $c_1, c_2 > 0$ such that*

$$\mathbb{E}_x\big[f(Y_t)\big] \le c_1 + f\big(p(x)\big) + c_2 t \quad \textit{for all } x \in \mathcal{I} \textit{ and } t \ge 0.$$

The following is an immediate consequence of this lemma.

Corollary 4. *For each of the functions defined by $f(x) = |x|$, $f(x) = x^+$ and $f(x) = x^-$,*

$$\mathbb{E}_\nu\big[f(Y_0)\big] \equiv \int_{\mathcal{I}} f\big(p(x)\big)\, \nu(dx) < \infty \quad \Leftrightarrow \quad \mathbb{E}_\nu\big[f(Y_t)\big] < \infty \textit{ for all } t \ge 0.$$

Proof of Theorem 2. (A) ⇒ (B) Suppose that the $\mathbb{P}_\nu$-local martingale (Y_t) is a $\mathbb{P}_\nu$-supermartingale. Definition 2.(i) implies that $\int_{\mathcal{I}} p(x)^- \,\nu(dx) = \mathbb{E}_\nu[Y_0^-] < \infty$. To show that α is not an entrance boundary point, we first note that, given any $b \in (\alpha, \beta)$, the process $(Y_t \mathbf{1}_{\{T_{\ge b}>0\}})$ is a $\mathbb{P}_\nu$-supermartingale (see Remark 2). Combining this observation with Doob's optional stopping theorem, we can see that $(Y_{t\wedge T_{\ge b}} \mathbf{1}_{\{T_{\ge b}>0\}})$ is a $\mathbb{P}_\nu$-supermartingale. Therefore, $(Y_{t\wedge T_{\ge b}} \mathbf{1}_{\{T_{\ge b}>0\}})$, which is both a $\mathbb{P}_\nu$-local martingale and a $\mathbb{P}_\nu$-supermartingale, is a $\mathbb{P}_\nu$-martingale because a local martingale that is bounded from above is a submartingale.

Now, we choose any $b \in (\alpha, \beta)$ such that $\nu\big((\alpha, b)\big) > 0$ (such a point b exists thanks to Assumption 1), and we note that, given any $x \in (\alpha, b)$,

$$\text{there exists } T \in (0, \infty) \text{ such that } \mathbb{P}_\nu(T_x < T \wedge T_{\ge b}) > 0 \tag{2}$$

because

$$\mathbb{P}_\nu\big(T_x < T_{\geq b}\big) = \int_{(\alpha,b)} \mathbb{P}_z\big(T_x < T_b\big)\, \nu(dz) > 0.$$

In view of the fact that $(Y_{t\wedge T_{\geq b}} \mathbf{1}_{\{T_{\geq b}>0\}})$ is a $\mathbb{P}_\nu$-martingale, we can use Doob's optional stopping theorem to obtain

$$\begin{aligned}
&\mathbb{E}_\nu\big[Y_{(T_x\wedge T+t)\wedge T_{\geq b}} \mathbf{1}_{\{T_{\geq b}>0\}} \mid \mathcal{F}_{T_x\wedge T\wedge T_{\geq b}}\big] \\
&\quad = Y_{T_x\wedge T\wedge T_{\geq b}} \mathbf{1}_{\{T_{\geq b}>0\}} = p(x)\mathbf{1}_{\{T_x<T\wedge T_{\geq b}\}} + Y_{T\wedge T_{\geq b}} \mathbf{1}_{\{T\wedge T_{\geq b}\leq T_x\}\cap\{T_{\geq b}>0\}} \quad \text{for all } t \geq 0.
\end{aligned}$$

On the other hand, we use the strong Markov property of (Y_t) to calculate

$$\begin{aligned}
&\mathbb{E}_\nu\big[Y_{(T_x\wedge T+t)\wedge T_{\geq b}} \mathbf{1}_{\{T_{\geq b}>0\}} \mid \mathcal{F}_{T_x\wedge T\wedge T_{\geq b}}\big] \\
&\quad = \mathbb{E}_{Y_{T_x\wedge T\wedge T_{\geq b}}}\big[Y_{t\wedge T_{\geq b}}\big] \mathbf{1}_{\{T_{\geq b}>0\}} \\
&\quad = \mathbb{E}_x\big[Y_{t\wedge T_{\geq b}}\big] \mathbf{1}_{\{T_x<T\wedge T_{\geq b}\}} + \mathbb{E}_{Y_{T\wedge T_{\geq b}}}\big[Y_{t\wedge T_{\geq b}}\big] \mathbf{1}_{\{T\wedge T_{\geq b}\leq T_x\}\cap\{T_{\geq b}>0\}} \quad \text{for all } t \geq 0.
\end{aligned}$$

These identities, together with (2), imply that

$$\mathbb{E}_x\big[Y_{t\wedge T_b}\big] = \mathbb{E}_x\big[Y_{t\wedge T_{\geq b}}\big] = p(x) \quad \text{for all } t \geq 0.$$

Combining this conclusion with the observation that $(Y_{t\wedge T_b})$ is a $\mathbb{P}_x$-submartingale, which follows from the fact that it is a $\mathbb{P}_x$-local martingale that is bounded from above, we can see that $(Y_{t\wedge T_b})$ is a $\mathbb{P}_x$-martingale. The required conclusion now follows from Theorem 1.

(B) ⇒ (A) If $p(\beta) < \infty$, then the conditions in (b) of Theorem 1 are satisfied. Hence, (Y_t) is actually a $\mathbb{P}_\nu$-martingale (compare (I)–(II) after the statement of Theorem 2 with (i) and (iii) after the statement of Theorem 1). We therefore assume that $p(\beta) = \infty$ in what follows. (Note that this assumption implies that β is inaccessible.)

Suppose first that $\nu = \delta_x$, for some $x \in (\alpha, \beta)$, and fix any $b \in (x, \beta)$. In view of Lemma 3, we can see that $\mathbb{E}_x\big[|Y_{t\wedge T_b}|\big] < \infty$ for all $t \geq 0$. The $\mathbb{P}_x$-local martingale $(Y_{t\wedge T_b})$ is an integrable $\mathbb{P}_x$-submartingale because a local martingale that is bounded from above is a submartingale. On the other hand, the process $(\tilde{Y}_t)$ introduced by Remark 3 is a $\tilde{\mathbb{P}}_x$-martingale because the diffusion $\tilde{X}$ satisfies the requirements of Theorem 1.(b). In particular, the function $t \mapsto \tilde{\mathbb{E}}_x[\tilde{Y}_t]$ is constant and finite. Since $(\tilde{Y}_t)$ has the same distribution under $\tilde{\mathbb{P}}_x$ as the process $(Y_{t\wedge T_b})$ under the measure $\mathbb{P}_x$, the function $t \mapsto \mathbb{E}_x[Y_t]$ is constant and finite. Therefore, the integrable $\mathbb{P}_x$-submartingale $(Y_{t\wedge T_b})$ is a $\mathbb{P}_x$-martingale, which implies that

$$\mathbb{E}_x\big[Y_t \mathbf{1}_{\{T_b>t\}}\big] \leq \mathbb{E}_x[Y_{t\wedge T_b}] = p(x)$$

for all $b \in (x, \beta)$ such that $p(b) \geq 0$. In view of Lemma 3, we can pass to the limit $b \nearrow \beta$ using the dominated convergence theorem to obtain

$$\mathbb{E}_x[Y_t] \leq p(x) \quad \text{for all } x \in (\alpha, \beta) \text{ and } t \geq 0.$$

Using this inequality and the strong Markov property, we can see that, given any $t > s \geq 0$,

$$\mathbb{E}_x[Y_t \mid \mathcal{F}_s] = \mathbf{1}_{\{T_\alpha \wedge T_\beta \leq s\}} Y_s + \mathbf{1}_{\{T_\alpha \wedge T_\beta > s\}} \mathbb{E}_{X_s}[Y_{t-s}] \leq p(X_{s \wedge T_\alpha \wedge T_\beta}) = Y_s.$$

It follows that (Y_t) is a $\mathbb{P}_x$-supermartingale.

To complete the proof, we consider any measure ν satisfying the integrability condition $\int_{\mathcal{I}} p(x)^- \, \nu(dx) < \infty$. This condition and Corollary 4 imply that (i) in Definition 2 holds true. On the other hand, given any $s < t$ and any event $A \in \mathcal{F}_s$, we use the fact that (Y_t) is a $\mathbb{P}_x$-supermartingale and the definition of $\mathbb{P}_\nu$ to calculate

$$\mathbb{E}_\nu\big[Y_t \mathbf{1}_A\big] = \int_{\mathcal{I}} \mathbb{E}_x\big[Y_t \mathbf{1}_A\big] \, \nu(dx) \leq \int_{\mathcal{I}} \mathbb{E}_x\big[Y_s \mathbf{1}_A\big] \, \nu(dx) = \mathbb{E}_\nu\big[Y_s \mathbf{1}_A\big],$$

which proves that property (ii) of Definition 2 is satisfied. □

3 Examples illustrating Theorem 2

To illustrate the scope of applications of Theorem 2, we first consider the following example.

Example 1. Given $-\infty \leq \alpha < \beta \leq \infty$, let $\mu, \sigma : (\alpha, \beta) \to \mathbb{R}$ be Borel-measurable functions that satisfy the conditions

$$\sigma(u) \neq 0 \quad \text{for all } u \in (\alpha, \beta) \tag{3}$$

$$\text{and} \quad \int_a^b \frac{1 + |\mu(u)|}{\sigma^2(u)} \, du < \infty \quad \text{for all } \alpha < a < b < \beta. \tag{4}$$

It is well-known that the stochastic differential equation

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad X_0 = x \in (\alpha, \beta), \tag{5}$$

where W is a standard Brownian motion, has a weak solution that is unique in law up to a possible explosion time, which is the first exit time of X from (α, β). We denote by ζ the exit time of X from (α, β) and we make the convention that, on the event $\{\zeta < \infty\}$, the process X is absorbed by the endpoint where it exits. Accordingly, the dynamics (5) hold only on the stochastic interval $[0, \zeta)$. Thus defined, the weak solutions to (5) provide a conservative regular continuous strong Markov process. The state space $\mathcal{I}$ of this Markov process is an interval with endpoints α and β that contains α (resp., β) if and only if X reaches α (resp., β) in finite time with positive probability. It is well-known that the scale function p and the restriction $m|_{(\alpha,\beta)}$ of the speed measure in $\big((\alpha, \beta), \mathcal{B}\big((\alpha, \beta)\big)\big)$ are given by the expressions

$$p(u) = \int_{c_1}^u \exp\left(-\int_{c_2}^s \frac{2\mu(r)}{\sigma^2(r)} \, dr\right) ds \quad \text{and} \quad m|_{(\alpha,\beta)}(du) = \frac{2}{\sigma^2(u) p'(u)} \, du, \tag{6}$$

where $c_1, c_2 \in (\alpha, \beta)$ are fixed arbitrary points. These expressions are sufficient to apply Theorem 2 because only the knowledge of p and $m|_{(\alpha,\beta)}$ is required in Theorem 2 (see (I) and (II) after the formulation of Theorem 2). □

In the context of Theorem 1, uniform integrability fails except in the trivial case of a bounded martingale (see Table 1). On the other hand, the situation is different in the context of Theorem 2. Indeed, the following example presents an unbounded $\mathbb{P}_x$-supermartingale that is uniformly integrable for all $x \in (\alpha, \beta)$.

Example 2. Let X be the diffusion in natural scale (i.e., $p(u) = u$) with state space $\mathcal{I} = (0, \infty)$ that identifies with the solutions to the SDE

$$dX_t = X_t^2 \, dW_t, \quad \text{for } X_0 = x > 0,$$

where (W_t) is a standard Brownian motion. It is well-known that $(Y_t) \equiv (X_t)$ can be realised in law as $(\| B_t \|^{-1})$, where (B_t) is a three-dimensional Brownian motion with $\| B_0 \| = x^{-1}$ and $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^3$. A short calculation shows that

$$\sup_{t \geq 0} \mathbb{E}_x \big[Y_t^p \big] < \infty \qquad \text{for all } p \in (1, 2),$$

which implies that (Y_t) is uniformly integrable. Also, (Y_t) is a $\mathbb{P}_x$-supermartingale because it is a bounded from below $\mathbb{P}_x$-local martingale. On the other hand, Theorem 1 implies that (Y_t) is not a $\mathbb{P}_x$-martingale because $\int_{(1,\infty)} p(u) \, m(du) = \int_{(1,\infty)} 2u^{-3} \, du < \infty$, in other words, ∞ is an entrance boundary point. Alternatively, (Y_t) is not a $\mathbb{P}_x$-martingale because, otherwise, it would correspond to case (ii) of Table 1 and its uniform integrability would be contradicted. $\square$

4 Asymptotic behaviour of the expectation

In this section, we consider the context of Example 1 with zero drift $\mu \equiv 0$ and

$$\alpha > -\infty, \quad \beta = \infty.$$

In other words, we consider the weak solution to the SDE

$$dX_t = \eta(X_t) \, dW_t, \quad X_0 = x \in (\alpha, \infty), \tag{7}$$

where $\eta \colon [\alpha, \infty) \to \mathbb{R}$ is a Borel-measurable function satisfying

$$\eta(\alpha) = 0, \tag{8}$$

$$\eta(u) \neq 0 \quad \text{for all } u \in (\alpha, \infty), \tag{9}$$

$$\int_a^b \eta^{-2}(u) \, du < \infty \quad \text{for all } \alpha < a < b < \infty. \tag{10}$$

The role of condition (8) is to make sure that α is absorbing whenever it is accessible. Notice that the scale function of X is the identity function, namely, $p(u) = u$, that is, $(Y_t) \equiv (X_t)$. In what follows we therefore work directly with the process (X_t).

The process (X_t) is a $\mathbb{P}_x$-supermartingale because it is a bounded from below $\mathbb{P}_x$-local martingale. Therefore, the function $t \mapsto \mathbb{E}_x[X_t]$ is constant whenever (X_t) is a martingale, and is decreasing and non-constant otherwise. The following result is concerned with the asymptotic behaviour of this function as $t \to \infty$ whenever (X_t) is a strict supermartingale.

Theorem 5. *The following statements are equivalent:*
(i) (X_t) *is a strict* $\mathbb{P}_x$*-supermartingale.*
(ii) $\int_c^\infty \frac{u}{\eta^2(u)}\, du < \infty$*, where* $c \in (\alpha, \infty)$ *is an arbitrary point.*[2]
(iii) $\lim_{t\to\infty} \mathbb{E}_x[X_t] = \alpha$.

Proof. The equivalence between (i) ad (ii) follows immediately from Theorem 1 (or from Theorem 2). We therefore need to prove that (iii) holds whenever X is a strict supermartingale. Without loss of generality, we assume that $\alpha = 0$ in what follows.

We start by embedding X in a geometric Brownian motion starting from x. To this end, we consider the space $C(\mathbb{R}_+)$ of continuous functions mapping $\mathbb{R}_+$ into $\mathbb{R}$, we denote by Z the coordinate process on this space, which is given by $Z_t(\omega) = \omega(t)$, and we define

$$G_t = xe^{-\frac{1}{2}t+Z_t}, \quad \text{for } t \geq 0.$$

Also, we denote by $(\mathcal{G}_t^0)$ the right-continuous regularisation of the natural filtration of Z, which is defined by $\mathcal{G}_t^0 = \bigcap_{\varepsilon>0} \sigma(Z_s,\, s \in [0, t+\varepsilon])$, we set $\mathcal{G}^0 = \bigvee_{t\geq 0} \mathcal{G}_t^0$, and we denote by $\mathbb{P}$ the probability measure on $\big(C(\mathbb{R}_+), \mathcal{G}^0\big)$ under which Z is a standard $(\mathcal{G}_t^0)$-Brownian motion starting from 0. Furthermore, we define

$$\mathcal{G}_t = \mathcal{G}_t^0 \bigvee \left(\bigcup_{t\geq 0} \mathcal{N}_t^{\mathbb{P}} \right), \quad \text{for } t \geq 0, \quad \text{and} \quad \mathcal{G} = \mathcal{G}^0 \bigvee \left(\bigcup_{t\geq 0} \mathcal{N}_t^{\mathbb{P}} \right), \tag{11}$$

where $\mathcal{N}_t^{\mathbb{P}}$ are the $\mathbb{P}$-null sets in $\mathcal{G}_t^0$. It is worth noting that the resulting filtration $(\mathcal{G}_t)$ is not the usual augmentation of $(\mathcal{G}_t^0)$ because the latter one involves the null sets in $\mathcal{G}^0$ rather than the union of null sets in (11). On the other hand, we may assume that $(\mathcal{G}_t)$ satisfies the "usual conditions" as long as our arguments are reducible to ones involving (increasing) bounded time horizons, which is the case in what follows.

In this context, we consider the $(\mathcal{G}_t)$-adapted continuous strictly increasing process A given by

$$A_t = \int_0^t \frac{G_s^2}{\eta^2(G_s)}\, ds, \quad \text{for } t \geq 0, \tag{12}$$

and we denote by Γ its inverse, which is given by

$$\Gamma_t = \inf\{s \geq 0 \mid \; A_s > t\}, \quad \text{for } t \geq 0,$$

[2] It is worth mentioning that this condition is satisfied if and only if ∞ is an entrance boundary point (see also Footnote 1 in Section 1 for the terminology we use in this paper).

with the usual convention that $\inf \emptyset = \infty$. It is worth noting that Γ is continuous and $\mathbb{P}$-a.s. strictly increasing on the stochastic interval $[0, A_\infty)$ because A has these properties. Moreover, (10) and the occupation times formula imply that $A_t < \infty$, $\mathbb{P}$-a.s., for all $t \in \mathbb{R}_+$, which implies that $\lim_{t\to\infty} \Gamma_t = \infty$, $\mathbb{P}$-a.s. On the other hand, we can have $\mathbb{P}(\Gamma_t = \infty) > 0$ even for finite t.

We now define

$$\bar{\mathcal{G}}_t = \mathcal{G}_{\Gamma_t}, \quad \text{for } t \geq 0, \quad \text{and} \quad \bar{X}_t = \left\{ \begin{matrix} G_{\Gamma_t}, & \text{for } t \in [0, A_\infty) \\ 0, & \text{for } t \in [A_\infty, \infty) \end{matrix} \right\},$$

$$\text{as well as} \quad \bar{W}_t = \int_0^t \eta^{-1}(\bar{X}_s)\, d\bar{X}_s, \quad \text{for } t \in [0, A_\infty), \tag{13}$$

and we note that $\bar{X}$ is continuous because $\lim_{t\to\infty} G_t = 0$. Indeed, on the event $\{A_\infty < \infty\}$, the process $\bar{X}$ hits the point $\alpha \equiv 0$ at time A_∞ and is stopped at this time. Moreover, $\bar{X}$ is a continuous $(\bar{\mathcal{G}}_t)$-local martingale on the stochastic interval $[0, A_\infty)$ (see Revuz and Yor [7, Section V.1]). Let us also note that A_∞ is a predictable $(\bar{\mathcal{G}}_t)$-stopping time. Recalling that Γ is continuous, we can use the time change formula for semimartingales and the fact that $A_{\Gamma_t} = t$ on the event $\{t < A_\infty\}$ to obtain

$$\begin{aligned} \langle \bar{X} \rangle_t = \langle G \rangle_{\Gamma_t} &= \int_0^{\Gamma_t} G_s^2\, ds \\ &= \int_0^{\Gamma_t} \eta^2(G_s)\, dA_s = \int_0^t \eta^2(G_{\Gamma_s})\, ds = \int_0^t \eta^2(\bar{X}_s)\, ds, \quad \mathbb{P}\text{-a.s., on } \{t < A_\infty\}. \end{aligned} \tag{14}$$

In particular, we calculate

$$\langle \bar{W} \rangle_t = \int_0^t \eta^{-2}(\bar{X}_s)\, d\langle \bar{X} \rangle_s = t, \quad \mathbb{P}\text{-a.s., on } \{t < A_\infty\}. \tag{15}$$

Being a continuous $(\bar{\mathcal{G}}_t)$-local martingale on the stochastic interval $[0, A_\infty)$ with angle bracket satisfying (15), the process $\bar{W}$ can be identified with a stopped Brownian motion. Indeed, $\lim_{t\to A_\infty} \bar{W}_t$ exists and is finite on the event $\{A_\infty < \infty\}$, and, if we define $\bar{W}_t = \bar{W}_{A_\infty}$, for $t \in [A_\infty, \infty)$, then $\bar{W}$ is a $(\bar{\mathcal{G}}_t)$-Brownian motion that is stopped at A_∞. Combining this observation with the fact that

$$\bar{X}_t = \int_0^t \eta(\bar{X}_s)\, d\bar{W}_s, \quad \mathbb{P}\text{-a.s., for } t \geq 0,$$

which follows from (13) and the discussion following (13), we can see that the law of $\bar{X}$ under $\mathbb{P}$ is the same as the unique law of the solution to (7) under $\mathbb{P}_x$. By means of the monotone convergence theorem, we therefore obtain

$$\mathbb{E}_x[X_t] = \mathbb{E}_{\mathbb{P}}[\bar{X}_t] = \mathbb{E}_{\mathbb{P}}\big[G_{\Gamma_t} \mathbf{1}_{\{\Gamma_t < \infty\}}\big] = \lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\big[G_{\Gamma_t} \mathbf{1}_{\{\Gamma_t \leq T\}}\big]. \tag{16}$$

To proceed further, we denote by $\mathbb{Q}$ the probability measure on $\big(C(\mathbb{R}_+), \mathcal{G}^0\big)$ under which the process B given by $B_t = Z_t - t$, for $t \geq 0$, is a standard $(\mathcal{G}_t^0)$-Brownian motion starting from 0. The measures $\mathbb{P}$ and $\mathbb{Q}$ are locally equivalent but not equivalent. The local equivalence of $\mathbb{P}$ and $\mathbb{Q}$ implies that the augmentation of $(\mathcal{G}_t^0)$ and $\mathcal{G}$ as in (11) with $\mathbb{Q}$ in place $\mathbb{P}$ results in the same filtration $(\mathcal{G}_t)$ and σ-algebra $\mathcal{G}$ because $\mathcal{N}_t^{\mathbb{P}} = \mathcal{N}_t^{\mathbb{Q}}$ for all $t \geq 0$. Furthermore, the density process of $\mathbb{P}$ and $\mathbb{Q}$ is given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \frac{1}{x} G_T, \quad \text{for } T \geq 0.$$

Combining this observation with the fact that G is a $\mathbb{P}$-martingale, the fact that Γ_t is a $(\mathcal{G}_t)$-stopping time for all $t \geq 0$, and with (16), we can see that

$$\begin{aligned}
\mathbb{E}_x[X_t] &= \lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}[G_T \mid \mathcal{G}_{T\wedge\Gamma_t}]\mathbf{1}_{\{\Gamma_t \leq T\}}\right] \\
&= \lim_{T\to\infty} \mathbb{E}_{\mathbb{P}}\left[G_T \mathbf{1}_{\{\Gamma_t \leq T\}}\right] = \lim_{T\to\infty} x\mathbb{Q}(\Gamma_t \leq T) = x\mathbb{Q}(\Gamma_t < \infty) = x\mathbb{Q}(A_\infty > t).
\end{aligned}$$

In view of this result, we can see that the proof will be complete if we prove that

$$\int_c^\infty \frac{u}{\eta^2(u)}\, du < \infty \quad \Leftrightarrow \quad \mathbb{Q}(A_\infty < \infty) \equiv \mathbb{Q}\left(\int_0^\infty \frac{G_s^2}{\eta^2(G_s)}\, ds < \infty\right) = 1. \tag{17}$$

To show (17), we remark that under $\mathbb{Q}$ the process G satisfies the SDE

$$dG_t = G_t\, dt + G_t\, dB_t, \quad G_0 = x$$

(recall that $(B_t) \equiv (Z_t - t)$ is a $\mathbb{Q}$-Brownian motion), and we apply Proposition 6 with $X = G$ and f defined by

$$f(u) = \frac{u^2}{\eta^2(u)}, \quad \text{for } u > 0.$$

In this case, σ and p of Proposition 6 are given by the formulas

$$\sigma(u) = u \quad \text{and} \quad p(u) = -\frac{1}{u}, \quad \text{for } u > 0.$$

These expressions imply that

$$\int_c^\infty \frac{\big(p(\infty) - p(u)\big) f(u)}{p'(u)\sigma^2(u)}\, du = \int_c^\infty \frac{u}{\eta^2(u)}\, du.$$

Furthermore, the exit time ζ in Proposition 6 is such that

$$\mathbb{Q}(\zeta = \infty) = 1 \quad \text{and} \quad \mathbb{Q}\left(\lim_{t\uparrow\zeta} G_t = \infty\right) \equiv \mathbb{Q}\left(\lim_{t\to\infty} G_t = \infty\right) = 1.$$

In view of these identifications, Proposition 6 implies the required equivalence (17). $\square$

5 Further examples

In this section we use Theorems 2 and 5 to study the following two examples.

Example 3. Let X be the diffusion in natural scale with state space $\mathcal{I} = \mathbb{R}$ that identifies with the solutions to the SDE

$$dX_t = e^{X_t}\, dW_t, \quad X_0 = x \in \mathbb{R}.$$

Theorem 2 and its submartingale counterpart imply that (X_t) is a $\mathbb{P}_x$-supermartingale but not a $\mathbb{P}_x$-submartingale for every initial point x. (It is worth noting that Kotani's result, Theorem 1, establishes only that (X_t) is a strict local martingale and does not indicate whether X is a supermartingale or a submartingale.) In particular, the function $t \mapsto \mathbb{E}_x[X_t]$ is decreasing and non-constant for all $x \in \mathbb{R}$. □

In the next example we observe an interesting effect: the behaviour of the function $t \mapsto \mathbb{E}_x[X_t]$ changes as x changes its sign.

Example 4. Let X be the diffusion in natural scale with state space $\mathcal{I} = \mathbb{R}$ that identifies with the solutions to the SDE

$$dX_t = \cosh(X_t)\, dW_t, \quad X_0 = x \in \mathbb{R}.$$

In this case, (X_t) is neither $\mathbb{P}_x$-supermartingale nor $\mathbb{P}_x$-submartingale, for any initial point x. A straightforward application of Lemma 3 implies that $\mathbb{E}_x\big[|X_t|\big] < \infty$ for all $x \in \mathbb{R}$. Furthermore,

(i) the function $t \mapsto \mathbb{E}_0[X_t]$ is identically zero;

(ii) given any $x > 0$, the function $t \mapsto \mathbb{E}_x[X_t]$ is decreasing and $\lim_{t\to\infty} \mathbb{E}_x[X_t] = 0$;

(iii) given any $x < 0$, the function $t \mapsto \mathbb{E}_x[X_t]$ is increasing and $\lim_{t\to\infty} \mathbb{E}_x[X_t] = 0$.

Statement (i) as well as the implication (ii) $\Rightarrow$ (iii) follow from the even symmetry of the function $x \mapsto \cosh x$. To see (ii), we fix any initial point $x > 0$, and we define the stopping time

$$\tau = \inf\{t \geq 0 \mid \ X_t = 0\}.$$

The stopped process $(X_{t\wedge\tau})$ satisfies (7) with $\eta(x) = \cosh x$, for $x > 0$, and $\eta(0) = 0$. Therefore, Theorem 5 implies that $t \mapsto \mathbb{E}_x[X_{t\wedge\tau}]$ is a decreasing function such that $\lim_{t\to\infty} \mathbb{E}_x[X_{t\wedge\tau}] = 0$. On the other hand, the even symmetry of the function $x \mapsto \cosh x$ implies that the random variable $X_t - X_{t\wedge\tau}$ has a symmetric distribution. Furthermore, this random variable is integrable with respect to $\mathbb{P}_x$ because both X_t and $X_{t\wedge\tau}$ are. Combining these observations with the identity $X_t = X_{t\wedge\tau} + (X_t - X_{t\wedge\tau})$, we obtain statement (ii). □

6 Convergence of integral functionals

This section is concerned with a result that we have used in the proof of Theorem 5. To fix ideas, we consider the setting of Example 1, namely, the weak solution to the SDE (5), where the coefficients μ and σ satisfy (3)–(4). As in Example 1, we denote by ζ the exit time of (X_t) from (α, β), which may be finite with strictly positive probability. For the purposes of this section, we further assume that

$$p(\beta) < \infty, \tag{18}$$

where the scale function p of X is given explicitly by (6). We also recall that (18) is equivalent to β being an attracting boundary point, namely, $\mathbb{P}_x\big(\lim_{t\uparrow\zeta} X_t = \beta\big) > 0$, where ζ can be infinite with probability one. Let $f\colon (\alpha, \beta) \to \mathbb{R}_+$ be a positive Borel-measurable function satisfying

$$\int_a^b \frac{f(u)}{\sigma^2(u)}\, du < \infty \quad \text{for all } \alpha < a < b < \beta, \tag{19}$$

which, by the occupation times formula, is equivalent to

$$\int_0^t f(X_s)\, ds < \infty \quad \text{for all } t < \zeta,\ \mathbb{P}_x\text{-a.s.}$$

The following result has been established in Mijatović and Urusov [6].

Proposition 6. *Assume* (18) *and* (19). *Then the following statements hold true:*
(i) *If the integrability condition*

$$\int_c^\beta \frac{\big(p(\beta) - p(u)\big) f(u)}{p'(u)\sigma^2(u)}\, du < \infty, \tag{20}$$

which is independent of the choice of $c \in (\alpha, \beta)$, is true, then

$$\int_0^\zeta f(X_s)\, ds < \infty, \quad \mathbb{P}_x\text{-a.s. on } \Big\{\lim_{t\uparrow\zeta} X_t = \beta\Big\}.$$

(ii) *If the integrability condition* (20) *fails, then*

$$\int_0^\zeta f(X_s)\, ds = \infty, \quad \mathbb{P}_x\text{-a.s. on } \Big\{\lim_{t\uparrow\zeta} X_t = \beta\Big\}.$$

Acknowledgments

We thank Stefan Ankirchner and Thomas Kruse for a discussion that gave rise to Section 5. The research of Mikhail Urusov is supported by the RNF grant 14-21-00162 from the Russian Scientific Fund.

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