Impulse Control of One-dimensional Itô Diffusions with an Expected and a Pathwise Ergodic Criterion*

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Abstract

We consider the problem of controlling a general one-dimensional Itô diffusion by means of an impulse control process. The objective is to minimise a long-term expected criterion as well as a long term pathwise criterion that penalise both deviations of the state process from a given nominal point and the use of impulsive control effort. In particular, each time that the controller deploys an impulse to reposition the system's state, a fixed cost and a cost proportional to the impulse's size are incurred. We solve the resulting optimisation problems and we provide an explicit characterisation of an optimal control strategy under general assumptions. The control of a foreign exchange rate or an inflation rate presents a potential application of the model that we study.

1 Introduction

We consider a stochastic system, the state of which is modelled by the controlled, one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R},$$

where W is a standard, one-dimensional Brownian motion. The controlled process Z is piece-wise constant: the jumps of this process occur at the times when the system's controller

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intervenes to reposition the system's state by an amount equal to the associated jump sizes. The objective of the optimisation problem is to minimise the *long-term average expected* criterion

$$\limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T h(X_t)\,dt + \sum_{t\in[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t>0\}} + \sum_{t\in[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t<0\}}\right],$$

as well as the *long-term average pathwise* criterion

$$\limsup_{T\to\infty} \frac{1}{T} \left[\int_0^T h(X_t)\,dt + \sum_{t\in[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t>0\}} + \sum_{t\in[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t<0\}}\right],$$

over all admissible choices of the controlled process Z. Here, h is a given function that is strictly decreasing in $]-\infty,0[$ and strictly increasing in $]0,\infty[$. Thus, these performance indices penalise deviations of the state process X from the nominal operating point 0. The positive constants c^+ and K^+ (resp., c^- and K^-) provide a fixed and a proportional cost each time that the controller incurs a jump of the system's state in the positive (resp., negative) direction.

One application of the control problems that we solve arises in the context of controlling a foreign exchange (FX) rate or an inflation rate by means of a central bank intervention policy. Indeed, Jeanblanc-Picqué [JP93] considers the problem of controlling in an impulsive way an FX rate, modelled by a Brownian motion with drift, so that it is confined within a given interval $[a,b]$. Mundaca and Øksendal [MØ98] and Cadenillas and Zapatero [CZ99, CZ00] present further contributions in this direction that incorporate an additional central bank intervention policy that takes the form of absolutely continuous control of the drift of the underlying FX rate dynamics. Also, Chiarolla and Haussmann [CH98] study a model for the control of an inflation rate by means of an intervention policy that results in a singular stochastic control problem. In all of these references, *expected discounted* performance criteria are considered.

We can see how the optimisation problem that we consider can be of use to a central bank in its task of controlling an FX rate as follows (for more details, see the references mentioned above). The controlled Itô diffusion X is used to model the stochastic dynamics of the logarithm of the FX rate. The central bank aims at keeping the rate as close as possible to a given nominal rate that translates to 0 in the state space of X. To achieve this aim, the central bank can purchase or sell large amounts of the foreign currency at discrete times, the effect of which actions is incorporated into the model through the jumps of the controlled process Z. To quantify the effects of its decision making, the central bank uses the function h to penalise deviations of the rate from its nominal point, and the fixed and proportional costs provided by the constants $c^+, c^-, K^+, K^- > 0$ to penalise each of its interventions in the FX market. With regard to this application and the existing literature on the subject discussed above, we note that an FX rate is not an asset and the function h does not represent a tangible cost, which implies that the choice of a discounting rate does not have a clear economic interpretation. This observation suggests that addressing this type of applications using a long-term average criterion rather than an expected discounted one conforms better with standard economic theory.

Apart from the range of its potential applications such as the one discussed above, the problem that we solve is also interesting from the perspective of the general theory of stochastic optimal control. Indeed, it provides one of the few non-trivial examples where a control problem admits a solution of an explicit analytic nature. At this point, we should mention that Kushner [Ku78], Karatzas [Ka83], Stettner [S86], Borkar and Ghosh [BG88], Gatarek and Stettner [GS90], Bensoussan and Frehse [BF92], Menaldi, Robin and Taksar [MRT92], Duncan, Maslowski and Pasik-Duncan [DMP98], Kurtz and Stockbridge [KuS98], Borkar [B99], Kruk [Kr00], and Sadowy and Stettner [SS02] provide an incomplete list of notable contributions to the theory of continuous time stochastic control with an ergodic criterion. Also, we should note that ergodic control models with pathwise performance criteria have recently attracted significant interest, e.g., see Rotar [R91], Presman, Rotar and Taksar [PRT93], Dai Pra, Di Masi and Trivellato [DDT01], Dai Pra, Runggaldier and Tolloti [DRT04], and a number of references therein.

2 An impulse stochastic control problem

We consider a stochastic system, the state process X of which is driven by a Brownian motion W and a controlled process Z that affects the system's dynamics in an impulsive way. In particular, we consider the controlled, one-dimensional SDE

$$dX_t = b(X_t)\,dt + dZ_t + \sigma(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R}, \tag{1}$$

where $b, \sigma : \mathbb{R} \to \mathbb{R}$ are given functions, and W is a standard, one-dimensional Brownian motion. Here, the impulse control process Z is a finite variation, piece-wise constant process, the time evolution of which is determined by the system's controller. Such a process can also be described by the collection

$$\mathcal{Z} = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \Delta Z_{\tau_1}, \Delta Z_{\tau_2}, \ldots, \Delta Z_{\tau_n}, \ldots),$$

where $(\tau_n, n \geq 1)$ is the sequence of random times at which the jumps of Z occur, and $(\Delta Z_{\tau_n}, n \geq 1)$ are the sizes of the corresponding jumps. A choice of such a process Z affects the system's state process only by causing a jump of size $\Delta X_{\tau_n} = \Delta Z_{\tau_n}$ at each of the times τ_n. Indeed, the evolution of the state process between any two consecutive times at which Z has a discretionary jump is governed by the uncontrolled SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x \in \mathbb{R}. \tag{2}$$

We impose the following assumption that is required by our analysis of the control problem considered in this paper.

Assumption 1 The functions $b, \sigma : \mathbb{R} \to \mathbb{R}$ are continuous, and there exists a constant $C_1 > 0$ such that

$$0 < \sigma^2(x) \leq C_1(1 + |x|), \quad \text{for all } x \in \mathbb{R}, \tag{3}$$

$\square$

With regard to the standard theory of one-dimensional diffusions, this assumption guarantees that, given any initial condition $x \in \mathbb{R}$, (2) has a weak solution, up to a possible explosion time, that is unique in the sense of probability law. In particular, in the presence of this assumption, conditions (ND)$'$ and (LI)$'$ in Section 5.5 of Karatzas and Shreve [KaS88] are both satisfied. Moreover, the scale function and the speed measure that characterise one-dimensional diffusions, such as the one in (2), given by

$$p_a(a) = 0, \quad p_a'(x) = \exp\left(-2\int_a^x \frac{b(s)}{\sigma^2(s)}\, ds\right), \quad \text{for } x \in \mathbb{R}, \tag{4}$$

and

$$m_a(dx) = \frac{2}{p_a'(x)\sigma^2(x)}\, dx, \tag{5}$$

respectively, for any given choice of $a \in \mathbb{R}$, are well-defined.

We adopt a weak formulation of the control problem that we are going to study.

Definition 1 Given $b, \sigma : \mathbb{R} \to \mathbb{R}$ satisfying Assumption 1 and an initial condition $x \in \mathbb{R}$, an impulse control of a system that obeys the stochastic dynamics (1) is any eight-tuple $\mathbb{C}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, P, W, Z, X, \tau)$, where

$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space satisfying the usual conditions,
W is a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion,
Z is a finite variation, piece-wise constant, càglàd, $(\mathcal{F}_t)$-adapted process, and
X is the unique in law, càglàd, $(\mathcal{F}_t)$-adapted process that satisfies (1) up to its explosion time τ.

We define $\mathcal{C}_x$ to be the family of all such controls $\mathbb{C}_x$. $\square$

With each control $\mathbb{C}_x \in \mathcal{C}_x$, we associate the *long-term average expected* performance criterion defined by

$$\begin{aligned} J^{\mathrm{E}}(\mathbb{C}_x) = \limsup_{T\to\infty} \frac{1}{T} E\biggl[\int_0^T h(X_t)\, dt + \sum_{t\in[0,T]} \left(K^+ \Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} \\ + \sum_{t\in[0,T]} \left(-K^- \Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}}\biggr], \quad \text{if } P(\tau = \infty) = 1, \end{aligned} \tag{6}$$

where $\Delta Z_t = Z_{t+} - Z_t$, and by

$$J^{\mathrm{E}}(\mathbb{C}_x) = \infty, \quad \text{if } P(\tau = \infty) < 1, \tag{7}$$

as well as the *long-term average pathwise* criterion

$$J^{\mathrm{P}}(\mathbb{C}_x) = \infty \mathbf{1}_{\{\tau<\infty\}} + \limsup_{T\to\infty} \frac{1}{T}\Bigg[\int_0^T h(X_t)\, dt + \sum_{t\in[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t>0\}} + \sum_{t\in[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t<0\}}\Bigg]\mathbf{1}_{\{\tau=\infty\}}. \tag{8}$$

Here, $h : \mathbb{R} \to \mathbb{R}$ is a given function that models the running cost resulting from the system's operation and $K^+, c^+, K^-, c^- > 0$, are given constants penalising the use of control effort. With regard to the control's contribution to these performance indices, we observe that, each time the controller deploys an impulse to cause a repositioning of the state process in the positive (resp., negative) direction, a fixed cost c^+ (resp., c^-) and a proportional cost equal to K^+ (resp., K^-) multiplied by the size of the jump are incurred.

The objective is to minimise the performance criteria defined by (6)–(7) and (8) over all controls $\mathbb{C}_x \in \mathcal{C}_x$. The following additional assumption on the problem's data is sufficient for our optimisation problem to be well-posed.

Assumption 2 The following conditions hold:

(a) The function h is continuous, strictly decreasing on $]-\infty, 0[$ and strictly increasing on $]0, \infty[$. Also, $h(0) = 0$, and there exists a constant $C_2 > 0$ such that

$$h(x) \geq C_2(|x| - 1), \quad \text{for all } x \in \mathbb{R}. \tag{9}$$

(b) Given any constants $\gamma, \lambda \in \mathbb{R}$,

$$\lim_{x\to\pm\infty} \frac{1}{\sigma^2(x)} \left[h(x) + b(x)\gamma - \lambda\right] = \infty. \tag{10}$$

(c) There exist $a_- \leq a_+$ such that

$$h(\cdot) + b(\cdot)K^- \begin{cases} \text{is strictly decreasing on }]-\infty, a_-[, \\ \text{is strictly negative inside }]a_-, a_+[, \text{ if } a_- < a_+, \\ \text{is strictly increasing on }]a_+, \infty[. \end{cases} \tag{11}$$

(d) There exist $\alpha_- \leq \alpha_+$ such that

$$h(\cdot) - b(\cdot)K^+ \begin{cases} \text{is strictly decreasing on }]-\infty, \alpha_-[, \\ \text{is strictly negative inside }]\alpha_-, \alpha_+[, \text{ if } \alpha_- < \alpha_+, \\ \text{is strictly increasing on }]\alpha_+, \infty[. \end{cases} \tag{12}$$

(e) $K^+, c^+, K^-, c^- > 0$. □

It is worth noting that the conditions in this assumption involve no convexity assumptions. Also, although they appear to be involved, they are quite general and easy to verify in practice.

Example 1 If we choose

$$b(x) = ax, \quad \sigma(x) = c \quad \text{and} \quad h(x) = \zeta |x|^p,$$

for some constants $a \in \mathbb{R}$, $c \neq 0$, $\zeta > 0$ and $p > 1$, then Assumptions 1 and 2 both hold. □

3 The solution to the impulse control problem

With regard to the general theory of stochastic control, the solution of the control problem formulated in Section 2 can be obtained by finding a sufficiently, for an application of Itô's formula, smooth function w and a constant λ satisfying the Hamilton-Jacobi-Bellman (HJB) equation

$$\min\Biggl\{ \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda, \\ c^+ - w(x) + \inf_{z \geq 0} \bigl[w(x+z) + K^+ z\bigr] , \; c^- - w(x) + \inf_{z \leq 0} \bigl[w(x+z) - K^- z\bigr] \Biggr\} = 0. \quad (13)$$

If such a pair (w, λ) exists, then, subject to suitable technical conditions, we expect the following statements to be true. Given any initial condition $x \in \mathbb{R}$,

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{E}}(\mathbb{C}_x) = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{P}}(\mathbb{C}_x).$$

In particular, the optimal value of the performance criteria is independent of the system's initial condition. The set of all $x \in \mathbb{R}$ such that

$$c^- - w(x) + \inf_{z \leq 0} \bigl[w(x+z) - K^- z\bigr] = 0 \quad (14)$$

defines the part of the state space where the controller should act immediately with an impulse in the negative direction. Similarly, the set of all $x \in \mathbb{R}$ such that

$$c^+ - w(x) + \inf_{z \geq 0} \bigl[w(x+z) + K^+ z\bigr] = 0$$

defines the region of the state space where the controller should act with an impulse in the positive direction. The interior of the set of all $x \in \mathbb{R}$ such that

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0 \quad (15)$$

defines the part of the state space in which the controller should take no action.

We conjecture that the optimal strategy is characterised by four points, $y_2 < y_1 < x_1 < x_2$, and takes a form that can be described as follows (see also Figure 1). If the state space process X assumes any value $x \geq x_2$, then control should be exercised to "push" it instantaneously to the level x_1. Similarly, whenever the state process X assumes a value $x \leq y_2$, control action should be used to reposition it at y_1. As long as the state process is inside the interval $]y_2, x_2[$, the controller should take no action. We therefore look for a solution (w, λ) of the HJB equation (13) such that

$$w(x) = w(x_1) + K^-(x - x_1) + c^-, \quad \text{for } x \geq x_2, \tag{16}$$

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0, \quad \text{for } x \in]y_2, x_2[, \tag{17}$$

$$w(x) = w(y_1) + K^+(y_1 - x) + c^+, \quad \text{for } x \leq y_2. \tag{18}$$

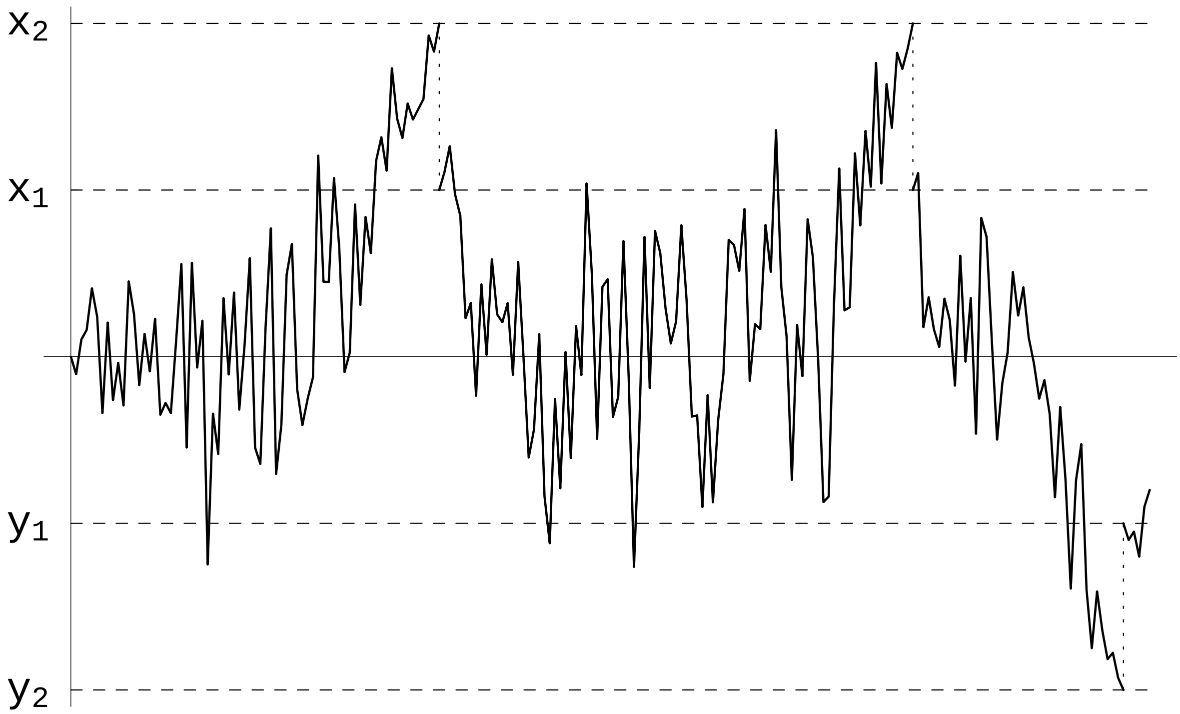

Figure 1: A typical sample path of the optimally controlled state process.

Assuming that this strategy is indeed optimal, we need a system of appropriate equations to determine the free boundary points y_2, y_1, x_1, x_2 and the constant λ. To derive such equations, we argue as follows. By appealing to the so-called "principle of smooth fit", we impose the conditions

$$w'(y_2+) = -K^+ \quad \text{and} \quad w'(x_2-) = K^-. \tag{19}$$

Now, relative to impulses in the negative direction, we consider the inequality

$$c^- - w(x) + \inf_{z \le 0} \left[w(x+z) - K^- z \right] \ge 0.$$

Assuming for a moment that we have somehow calculated w, this inequality implies

$$c^- - w(x_2) + w(x) - K^-(x - x_2) \ge 0, \quad \text{for all } x \le x_2.$$

With regard to (16) and the fact that x_2 is a constant, this observation implies that the function $x \mapsto w(x) - K^- x$ has a local minimum at $x = x_1$, which can be true only if

$$w'(x_1) = K^-. \tag{20}$$

Moreover, with regard to the discussion related to (14), the optimality of a jump from x_2 to x_1 is associated with

$$\int_{x_1}^{x_2} w'(s)\, ds = K^- (x_2 - x_1) + c^-. \tag{21}$$

Similarly, a consideration of impulses in the positive direction leads to

$$w'(y_1) = -K^+ \tag{22}$$

and

$$\int_{y_2}^{y_1} w'(s)\, ds = -K^+ (y_1 - y_2) - c^+. \tag{23}$$

To complete the picture, it turns out that we have to introduce an extra parameter: assuming that w' is continuous, (20), (22) and the strict positivity of K^+ and K^-, imply that w has a local minimum, denoted by a, inside $]y_1, x_1[$.

Summarising the heuristic analysis above, we look for six parameters, namely $y_2 < y_1 < a < x_1 < x_2$ and λ, and a function w such that (16)–(23) and $w'(a) = 0$ are all true. Now, observe that the solution of the ODE (15) with the initial condition $w'(a) = 0$ is given by

$$w'(x) = p_a'(x) \int_a^x [\lambda - h(s)]\, m_a(ds), \quad x \in \mathbb{R}, \tag{24}$$

where p_a and m_a are the scale function and the speed measure of the uncontrolled diffusion (2), defined by (4) and (5), respectively. It follows that, to determine the six parameters $y_2 < y_1 < a < x_1 < x_2$ and λ, we have to solve the system of the following six algebraic, non-linear equations:

$$g(x_2, \lambda, a) = K^-, \tag{25}$$
$$g(x_1, \lambda, a) = K^-, \tag{26}$$
$$g(y_2, \lambda, a) = -K^+, \tag{27}$$
$$g(y_1, \lambda, a) = -K^+, \tag{28}$$
$$\int_{x_1}^{x_2} g(s, \lambda, a)\, ds = K^- (x_2 - x_1) + c^-, \tag{29}$$
$$\int_{y_2}^{y_1} g(s, \lambda, a)\, ds = -K^+ (y_1 - y_2) - c^+, \tag{30}$$

where g is defined by

$$g(x, \lambda, a) = p_a'(x) \int_a^x [\lambda - h(s)]\, m_a(ds), \quad \text{for } x, \lambda, a \in \mathbb{R}. \tag{31}$$

Now, suppose that the system of equations (25)–(30) has a solution of the required form. Although the points $y_2 < y_1 < a < x_1 < x_2$ and λ determine completely the conjectured optimal strategy as well as the associated value of the performance criterion, proving that this strategy is indeed optimal requires a solution (w, λ) of the HJB equation (13). To this end, observe that (24) and (16)–(18) determine w uniquely, modulo an additive constant. The following result, the proof of which is developed in the Appendix, is concerned with these issues.

Lemma 1 *Suppose that Assumptions 1 and 2 hold. The system of equations (25)–(30), where g is defined by (31), has a solution $(y_2, y_1, a, x_1, x_2, \lambda)$ such that $y_2 < y_1 < a < x_1 < x_2$, and, if w is any function satisfying (24) inside the interval $]y_2, x_2[$ and is given by (16) and (18) in the complement of $]y_2, x_2[$, then $w \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{y_2, x_2\})$ and the pair (w, λ) is a solution of the HJB equation (13).*

We can now derive the solution to the control problem that is concerned with the optimisation of the ergodic expected criterion.

Theorem 2 *Consider the stochastic control problem formulated in Section 2 that aims at the minimisation of the long-term expected criterion defined by (6)–(7). Suppose that Assumptions 1 and 2 hold, and let $(y_2, y_1, a, x_1, x_2, \lambda)$ be the solution to (25)–(30), where g is defined by (31), as in Lemma 1. Then, given an initial condition $x \in \mathbb{R}$,*

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{E}}(\mathbb{C}_x), \tag{32}$$

and the points y_2, y_1, x_1, x_2 determine the optimal strategy that has been discussed qualitatively above and can be constructed rigorously as in the proof below.

Proof. Throughout this proof, we fix a solution (w, λ) of the HJB equation (13) that is constructed as in Lemma 1. We also fix any initial condition $x \in \mathbb{R}$.

Consider any admissible control $\mathbb{C}_x \in \mathcal{C}_x$ such that $J^{\mathrm{E}}(\mathbb{C}_x) < \infty$. Using Itô's formula for general semimartingales, we obtain

$$\begin{aligned} w(X_{T+}) &= w(x) + \int_0^T \left[\frac{1}{2}\sigma^2(X_s)w''(X_s) + b(X_s)w'(X_s)\right] ds + \int_{[0,T]} w'(X_s)\, dZ_s \\ &\quad + \int_0^T \sigma(X_s)w'(X_s)\, dW_s + \sum_{s\in[0,T]} \left[w(X_{s+}) - w(X_s) - w'(X_s)\Delta X_s\right] \\ &= w(x) + \int_0^T \left[\frac{1}{2}\sigma^2(X_s)w''(X_s) + b(X_s)w'(X_s)\right] ds \\ &\quad + \int_0^T \sigma(X_s)w'(X_s)\, dW_s + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s)\right], \end{aligned}$$

the second equality following because $\Delta X_s \equiv X_{s+} - X_s = \Delta Z_s$. This implies

$$\begin{aligned} Q_T(\mathbb{C}_x) &:= \int_0^T h(X_s)\, ds + \sum_{s\in[0,T]} \left(K^+\Delta Z_t + c^+\right) \mathbf{1}_{\{\Delta Z_t > 0\}} + \sum_{s\in[0,T]} \left(-K^-\Delta Z_t + c^-\right) \mathbf{1}_{\{\Delta Z_t < 0\}} \\ &= \lambda T + w(x) - w(X_{T+}) + \int_0^T \sigma(X_s)w'(X_s)\, dW_s \\ &\quad + \int_0^T \left[\frac{1}{2}\sigma^2(X_s)w''(X_s) + b(X_s)w'(X_s) + h(X_s) - \lambda\right] ds \\ &\quad + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s) + K^+\Delta Z_s + c^+\right] \mathbf{1}_{\{\Delta Z_s > 0\}} \\ &\quad + \sum_{s\in[0,T]} \left[w(X_s + \Delta Z_s) - w(X_s) - K^-\Delta Z_s + c^-\right] \mathbf{1}_{\{\Delta Z_s < 0\}}. \end{aligned} \tag{33}$$

Since the pair (w, λ) satisfies the HJB equation (13),

$$Q_T(\mathbb{C}_x) \geq \lambda T + w(x) - w(X_{T+}) + \int_0^T \sigma(X_s)w'(X_s)\, dW_s. \tag{34}$$

By construction, w is C^1, $w'(x) = K^-$, for all $x \geq x_2$, and $w'(x) = -K^+$, for all $x \leq y_2$. Therefore, there exists a constant $C_3 > 0$ such that

$$w(x) \leq C_3(1 + |x|) \quad \text{and} \quad |w'(x)| \leq C_3, \quad \text{for all } x \in \mathbb{R}. \tag{35}$$

For such a choice of C_3, (34) yields

$$Q_T(\mathbb{C}_x) \geq \lambda T + w(x) - C_3 - C_3\, |X_{T+}| + \int_0^T \sigma(X_s)w'(X_s)\, dW_s. \tag{36}$$

Now, with respect to Assumption 2.(a),

$$\begin{aligned}\infty &> J^{\mathrm{E}}(\mathbb{C}_x) \\ &\geq \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T h(X_s)\,ds\right] \\ &\geq -C_2 + C_2 \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T |X_s|\,ds\right]. \end{aligned} \tag{37}$$

These inequalities imply

$$E\left[\int_0^T |X_s|\,ds\right] < \infty, \quad \text{for all } T > 0, \tag{38}$$

and

$$\liminf_{T\to\infty} \frac{1}{T} E\left[|X_{T+}|\right] = 0. \tag{39}$$

To see (39), suppose that $\liminf_{T\to\infty} T^{-1} E\left[|X_{T+}|\right] > \varepsilon > 0$. This implies that there exists $T_1 \geq 0$ such that $E\left[|X_{s+}|\right] > \varepsilon s/2$, for all $s \geq T_1$. Since the sample paths of X have countable discontinuities, it follows that

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T |X_s|\,ds\right] &= \limsup_{T\to\infty} \frac{1}{T} E\left[\int_0^T |X_{s+}|\,ds\right] \\ &\geq \limsup_{T\to\infty} \frac{1}{T} \int_{T_1}^T \frac{\varepsilon s}{2}\,ds \\ &= \infty, \end{aligned}$$

which contradicts (37).

With regard to (3) in Assumption 1, the second inequality in (35), and (38), we calculate

$$E\left[\int_0^T \left[\sigma(X_s) w'(X_s)\right]^2 ds\right] \leq C_3^2 C_1 \left[T + E\left[\int_0^T |X_s|\,ds\right]\right] < \infty, \quad \text{for all } T > 0, \tag{40}$$

which proves that the stochastic integral in (36) is a square integrable martingale, and therefore has zero expectation. In view of this observation, we can take expectations in (36) and divide by T to obtain

$$\frac{1}{T} E\left[Q_T(\mathbb{C}_x)\right] \geq \lambda + \frac{w(x)}{T} - \frac{C_3}{T} - \frac{C_3}{T} E\left[|X_{T+}|\right].$$

In view of (39) and the definition of $Q_T(\mathbb{C}_x)$ in (33), we can pass to the limit $T \to \infty$ to obtain $J^{\mathrm{E}}(\mathbb{C}_x) \geq \lambda$.

To prove the reverse inequality, suppose that we can find a control

$$\hat{\mathbb{C}}_x = (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}, \hat{W}, \hat{Z}, \hat{X}, \hat{\tau}) \in \mathcal{C}_x$$

such that

$$\hat{X}_t \in [y_2, x_2], \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{41}$$

$$\Delta \hat{Z}_t \mathbf{1}_{\{\Delta \hat{Z}_t > 0\}} = (y_1 - y_2) \mathbf{1}_{\{\hat{X}_t = y_2\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{42}$$

$$\Delta \hat{Z}_t \mathbf{1}_{\{\Delta \hat{Z}_t < 0\}} = -(x_2 - x_1) \mathbf{1}_{\{\hat{X}_t = x_2\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}. \tag{43}$$

Plainly, (41) implies that $\hat{X}$ is non-explosive, so that $\hat{\tau} = \infty$, $\hat{P}$-a.s.. With regard to the construction of w, we can see that, for such a choice of a control, (33) implies

$$Q_T(\hat{\mathbb{C}}_x) = \lambda T + w(x) - w(\hat{X}_{T+}) + \int_0^T \sigma(\hat{X}_s) w'(\hat{X}_s)\, d\hat{W}_s. \tag{44}$$

Now, (3) in Assumption 1, (35) and (41) imply

$$E\left[\int_0^T \left[\sigma(\hat{X}_s) w'(\hat{X}_s)\right]^2 ds\right] \leq C_3^2 C_1 \left(1 + |y_2| \vee |x_2|\right) T < \infty, \quad \text{for all } T > 0,$$

which proves that the stochastic integral in (44) is a square integrable martingale, and

$$\lim_{T \to \infty} \frac{1}{T} E\left[\left|w(\hat{X}_{T+})\right|\right] \leq \lim_{T \to \infty} \frac{C_3 \left(1 + |y_2| \vee |x_2|\right)}{T} = 0.$$

It follows that

$$\lim_{T \to \infty} \frac{1}{T} E\left[Q_T(\hat{\mathbb{C}}_x)\right] = \lambda,$$

which proves that $J^{\mathrm{E}}(\hat{\mathbb{C}}_x) = \lambda$, and establishes (32).

It remains to construct a control $\hat{\mathbb{C}}_x \in \mathcal{C}_x$ satisfying (41)–(43). Assuming that such a control exists, if we define $\tilde{X}_t = p_a(\hat{X}_t)$, where p_a is the scale function given by (4), then we can use Itô's formula to calculate

$$d\tilde{X}_t = d\tilde{Z}_t + \tilde{\sigma}(\tilde{X}_t)\, d\hat{W}_t, \quad \tilde{X}_0 = p_a(x), \tag{45}$$

where $\tilde{\sigma} = (p_a' \sigma) \circ p_a^{-1}$, and $\tilde{Z}$ is the càglàd, finite variation process defined by

$$\tilde{Z}_t = \sum_{s \in [0,t[} \left[p_a(\hat{X}_{s+}) - p_a(\hat{X}_s)\right].$$

Moreover, we can verify that the processes $\tilde{X}$ and $\tilde{Z}$ satisfy

$$\tilde{X}_t \in [p_a(y_2), p_a(x_2)], \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{46}$$

$$\Delta\tilde{Z}_t \mathbf{1}_{\{\Delta\tilde{Z}_t>0\}} = [p_a(y_1) - p_a(y_2)]\, \mathbf{1}_{\{\tilde{X}_t = p_a(y_2)\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{47}$$

$$\Delta\tilde{Z}_t \mathbf{1}_{\{\Delta\tilde{Z}_t<0\}} = -\, [p_a(x_2) - p_a(x_1)]\, \mathbf{1}_{\{\tilde{X}_t = p_a(x_2)\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}. \tag{48}$$

These calculations show that, if we can find a weak solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}, \hat{W}, \tilde{Z}, \tilde{X})$ to the SDE (45) satisfying (46)–(48) and we define

$$\hat{\hat{Z}} = p_a^{-1}(\tilde{Z}) \quad \text{and} \quad \hat{\hat{X}} = p_a^{-1}(\tilde{X}),$$

then $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}, \hat{W}, \hat{\hat{Z}}, \hat{\hat{X}}, \infty) \in \mathcal{C}_x$ is the required control. Thus, the problem reduces to constructing a control $\mathbb{C}_x \in \mathcal{C}_x$ satisfying (41)–(43) in the case that arises when $b \equiv 0$.

In the context of the simplification developed above, we fix a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \bar{\mathcal{F}}_t, \hat{P})$ satisfying the usual conditions and supporting a standard, one-dimensional Brownian motion $\bar{W}$. By appealing to a simple induction argument, we construct a càglàd, finite variation process $\bar{Z}$ such that, if

$$\bar{X}_t = x + \bar{Z}_t + \bar{W}_t, \tag{49}$$

then

$$\bar{X}_t \in [y_2, x_2], \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{50}$$

$$\Delta\bar{Z}_t \mathbf{1}_{\{\Delta\bar{Z}_t>0\}} = (y_1 - y_2)\mathbf{1}_{\{\bar{X}_t = y_2\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}, \tag{51}$$

$$\Delta\bar{Z}_t \mathbf{1}_{\{\Delta\bar{Z}_t<0\}} = -(x_2 - x_1)\mathbf{1}_{\{\bar{X}_t = x_2\}}, \quad \text{for all } t \geq 0, \ \hat{P}\text{-a.s.}. \tag{52}$$

Now, we consider the continuous, increasing process A defined by

$$A_t = \int_0^t \sigma^{-2}(\bar{X}_s)\, ds,$$

and we observe that $A_t < \infty$, for all $t \geq 0$, and $\lim_{t\to\infty} A_t = \infty$, thanks to the continuity of σ and (3) in Assumption 1, and (50). Also, we denote by C the inverse of A defined by

$$C_t = \inf\{s \geq 0 \mid \ A_s > t\},$$

and we note that $C_t < \infty$, for all $t \geq 0$, and $\lim_{t\to\infty} C_t = \infty$. Since C is continuous, if we define

$$\hat{\mathcal{F}}_t = \bar{\mathcal{F}}_{C_t}, \quad \hat{X}_t = \bar{X}_{C_t}, \quad \hat{Z}_t = \bar{Z}_{C_t} \quad \text{and} \quad L_t = \bar{W}_{C_t}, \tag{53}$$

then $\hat{X}$, $\hat{Z}$ are càglàd, $(\hat{\mathcal{F}}_t)$-adapted processes satisfying (41)–(43), and L is a continuous, $(\hat{\mathcal{F}}_t)$-local martingale. Furthermore, if we define

$$\hat{W}_t = \int_0^t \sigma^{-1}(\hat{X}_s)\, dL_s,$$

then, in view of (49) and (53),

$$d\hat{X}_t = d\hat{Z}_t + \sigma(\hat{X}_t)\, d\hat{W}_t.$$

To show that the collection $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{P}, \hat{W}, \hat{Z}, \hat{X}, \infty)$ thus constructed is the required optimal control, we still have to prove that $\hat{W}$ is an $(\hat{\mathcal{F}}_t)$-Brownian motion. To this end, we observe that

$$\langle L \rangle_t = C_t = \int_0^{C_t} \sigma^2(\bar{X}_s)\, dA_s = \int_0^t \sigma^2(\hat{X}_s)\, ds,$$

the last equality following thanks to the time change formula and the fact that $A_{C_s} = s$. It follows that

$$\langle \hat{W} \rangle_t = \int_0^t \sigma^{-2}(\hat{X}_s)\, d\langle L \rangle_s = t.$$

However, with reference to Lévy's characterisation theorem, this calculation and the fact that $\hat{W}$ is a continuous, $(\hat{\mathcal{F}}_t)$-local martingale imply that $\hat{W}$ is an $(\hat{\mathcal{F}}_t)$-Brownian motion, and the proof is complete. □

The following result is concerned with the solution to the optimisation problem considered with the ergodic pathwise criterion.

Theorem 3 *Consider the stochastic control problem formulated in Section 2 that aims at the minimisation of the long-term pathwise criterion defined by (8). Suppose that Assumptions 1 and 2 hold, and let $(y_2, y_1, a, x_1, x_2, \lambda)$ be the solution of (25)–(30), where g is defined by (31), as in Lemma 1. Then, given an initial condition $x \in \mathbb{R}$,*

$$\lambda = \inf_{\mathbb{C}_x \in \mathcal{C}_x} J^{\mathrm{P}}(\mathbb{C}_x), \tag{54}$$

and the points y_2, y_1, x_1, x_2 determine the optimal strategy that can be constructed as in the proof of Theorem 2.

Proof. Throughout this proof, we fix a solution (w, λ) of the HJB equation (13) that is constructed as in Lemma 1. We also fix any initial condition $x \in \mathbb{R}$.

Consider any admissible control $\mathbb{C}_x \in \mathcal{C}_x$. Using the same arguments as the ones that established (36) in the proof of Theorem 2 above, we can show that

$$\begin{aligned} Q_T(\mathbb{C}_x)\mathbf{1}_{\{T<\tau\}} :=& \bigg(\int_0^T h(X_s)\, ds + \sum_{s\in[0,T]} \left(K^+ \Delta Z_s + c^+ \right) \mathbf{1}_{\{\Delta Z_s > 0\}} \\ &+ \sum_{s\in[0,T]} \left(-K^- \Delta Z_s + c^- \right) \mathbf{1}_{\{\Delta Z_s < 0\}} \bigg) \mathbf{1}_{\{T<\tau\}} \\ &\geq (\lambda T + w(x) - C_3 - C_3 |X_{T+}| + M_T)\, \mathbf{1}_{\{T<\tau\}}, \end{aligned} \tag{55}$$

where

$$M_T = \int_0^T \sigma(X_s) w'(X_s)\, dW_s. \tag{56}$$

With regard to Assumptions 2.(a) and 2.(e), we can see that

$$\begin{aligned}
\infty > J^{\mathrm{P}}(\mathbb{C}_x) \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} &\geq \left(\limsup_{T\to\infty} \frac{1}{T} \int_0^T h(X_s)\, ds \right) \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} \\
&\geq C_2 \left(-1 + \limsup_{T\to\infty} \frac{1}{T} \int_0^T |X_s|\, ds \right) \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}}.
\end{aligned}$$

By appealing to arguments similar those that established (39), we can see that these inequalities imply

$$\liminf_{T\to\infty} \frac{1}{T} |X_{T+}| \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} = 0. \tag{57}$$

Furthermore, they imply that there exists a real-valued random variable Z and a *random* time τ_Z such that

$$\left(\frac{1}{T} \int_0^T |X_s|\, ds \right) \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} \leq Z \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}}, \quad \text{for all } T \geq \tau_Z.$$

In view of (3) in Assumption 1 and the second inequality in (35), it follows that

$$\begin{aligned}
\langle M \rangle_T \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} &\leq C_3^2 C_1 \left(1 + \frac{1}{T} \int_0^T |X_s|\, ds \right) T \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}} \\
&\leq C_3^2 C_1 (1+Z)\, T \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}}, \quad \text{for all } T \geq \tau_Z.
\end{aligned} \tag{58}$$

Now, with regard to the Dambis, Dubins and Schwarz theorem (e.g., see Revuz and Yor [RY94, Theorem V.1.7]), there exists a standard, one-dimensional Brownian motion B defined on a possible extension of $(\Omega, \mathcal{F}, P)$ such that

$$M_T \mathbf{1}_{\{T<\tau\}} = B_{\langle M \rangle_T} \mathbf{1}_{\{T<\tau\}}.$$

In view of this representation, the observation that

$$\{J^{\mathrm{P}}(\mathbb{C}_x) < \infty\} \subseteq \{\tau = \infty\},$$

(58), and the fact that $\lim_{T\to\infty} B_T/T = 0$, we can see that

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} |M_T| \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}\cap\{\langle M \rangle_\infty = \infty\}} &= \lim_{T\to\infty} \frac{1}{T} \left| B_{\langle M \rangle_T} \right| \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}\cap\{\langle M \rangle_\infty = \infty\}} \\
&\leq C_3^2 C_1 (1+Z) \lim_{T\to\infty} \frac{1}{\langle M \rangle_T} \left| B_{\langle M \rangle_T} \right| \mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}\cap\{\langle M \rangle_\infty = \infty\}} \\
&= 0.
\end{aligned} \tag{59}$$

Furthermore, since a continuous local martingale M converges in $\mathbb{R}$, P-a.s., on the event $\{\langle M\rangle_\infty < \infty\}$ (e.g., see Revuz and Yor [RY94, Proposition IV.1.26]),

$$\lim_{T\to\infty} \frac{1}{T}|M_T|\mathbf{1}_{\{J^{\mathrm{P}}(\mathbb{C}_x)<\infty\}\cap\{\langle M\rangle_\infty<\infty\}} = 0. \tag{60}$$

However, combining (59) and (60) with (55) and (57), we can see that

$$J^{\mathrm{P}}(\mathbb{C}_x) \equiv \limsup_{T\to\infty} \frac{1}{T} Q_T(\mathbb{C}_x) \geq \lambda.$$

To prove the reverse inequality, consider the control $\hat{\mathbb{C}}_x \in \mathcal{C}_x$ satisfying (41)–(43), which is associated with $\hat{\tau} = \infty$ and

$$Q_T(\hat{\mathbb{C}}_x) = \lambda T + w(x) - w(\hat{X}_T) + \hat{M}_T,$$

where Q_T and $\hat{M}$ are defined as in (55) and (56), respectively. Since $\hat{X}_t \in [y_2, x_2]$, for all $t \geq 0$, $\hat{P}$-a.s., (3) in Assumption 1 and (35) imply

$$|w(X_T)| \leq C_3\left(1 + |y_2| \vee |x_2|\right) \text{ and } \langle M\rangle_T \leq C_3^2 C_1\left(1 + |y_2| \vee |x_2|\right) T, \text{ for all } T \geq 0.$$

However, in light of these inequalities and an argument such as the one establishing (59) and (60) above, we can see that $J^{\mathrm{P}}(\hat{\mathbb{C}}_x) \equiv \lim_{T\to\infty} \frac{1}{T} Q_T(\hat{\mathbb{C}}_x) = \lambda$, and the proof is complete. □

Appendix

Before addressing the proof of Lemma 1, we establish a series of results concerning the function g defined by (31) and certain aspects of the system of equations (25)–(30) that we want to solve. For future reference, we note that, given any $\lambda, a \in \mathbb{R}$, the partial derivative of g with respect to x is given by

$$g_x(x,\lambda,a) = -\frac{2}{\sigma^2(x)}\left[h(x) + b(x)g(x,\lambda,a) - \lambda\right], \quad \text{for all } x \in \mathbb{R}. \tag{61}$$

The next result is concerned with some first properties of g.

Lemma 4 *The following statements are true:*
(i) *Given any $x, a \in \mathbb{R}$, the function $g(x,\cdot,a)$ is strictly increasing if $x > a$, and strictly decreasing if $x < a$.*
(ii) *Given any $\lambda, a \in \mathbb{R}$, the equation $g(x,\lambda,a) = 0$ has at most two solutions $x \in]a,\infty[$, and at most two solutions $x \in]-\infty, a[$.*
(iii) *Given any $\lambda, a \in \mathbb{R}$,*

$$\lim_{x\to\infty} g(x,\lambda,a),\ \lim_{x\to-\infty} g(x,\lambda,a) \in \{-\infty,\infty\}. \tag{62}$$

Proof. (i) Fix any $x, a \in \mathbb{R}$ such that $x > a$. With regard to (31), we calculate

$$g_\lambda(x, \lambda, a) = p_a'(x) m_a([a, x]) > 0,$$

which proves that $g(x, \cdot, a)$ is strictly increasing.

Given any $x < a$, a similar calculation shows that $g(x, \cdot, a)$ is strictly decreasing.

(ii) Fix any $\lambda, a \in \mathbb{R}$, and consider the solvability of $g(x, \lambda, a) = 0$ for $x \in]a, \infty[$. Assumption 2.(a) implies that there exist at most two points $x > a$ such that $h(x) = \lambda$. Also, (61) implies that

$$g_x(x, \lambda, a) = -\frac{2}{\sigma^2(x)} [h(x) - \lambda], \text{ for all } x > a \text{ such that } g(x, \lambda, a) = 0. \tag{63}$$

Combining these observations with the boundary condition $g(a, \lambda, a) = 0$, we can conclude that the number of solutions of $g(x, \lambda, a) = 0$ inside $]a, \infty[$ is less than or equal to the number of solutions of $h(x) = \lambda$ inside $]a, \infty[$, which is at most two.

Similar arguments show that the number of solutions of $g(x, \lambda, a) = 0$ inside $]-\infty, a[$ is also less than or equal to two.

(iii) Fix any $\lambda, a \in \mathbb{R}$. With reference to part (ii) of this lemma, the conclusion $\lim_{x\to\infty} g(x, \lambda, a) \in \{-\infty, \infty\}$ will follow if we show that either of

$$\liminf_{x\to\infty} g(x, \lambda, a) \in [0, \infty[, \tag{64}$$

$$\limsup_{x\to\infty} g(x, \lambda, a) \in]-\infty, 0], \tag{65}$$

leads to a contradiction. Assuming that (64) is true, we choose a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} g(x_n, \lambda, a) = \liminf_{x\to\infty} g(x, \lambda, a) \quad \text{and} \quad \lim_{n\to\infty} g_x(x_n, \lambda, a) = 0.$$

By passing to a subsequence, if necessary, we assume that either $b(x_n) \geq 0$ for all n, or $b(x_n) < 0$ for all n. If we assume that (65) is true, then we choose a sequence (x_n) in a similar fashion. In either case, we define

$$\gamma = \begin{cases} \inf_{n\geq 1} g(x_n, \lambda, a), & \text{if } b(x_n) \geq 0, \\ \sup_{n\geq 1} g(x_n, \lambda, a), & \text{if } b(x_n) < 0. \end{cases}$$

Now, we observe that $\gamma \in \mathbb{R}$, and we calculate

$$\begin{aligned} 0 &= \lim_{n\to\infty} g_x(x_n, \lambda, a) \\ &= \lim_{n\to\infty} \frac{-2}{\sigma^2(x_n)} [h(x_n) + b(x_n) g(x_n, \lambda, a) - \lambda] \\ &\leq \lim_{n\to\infty} \frac{-2}{\sigma^2(x_n)} [h(x_n) + b(x_n)\gamma - \lambda] \\ &= -\infty, \end{aligned}$$

the last inequality following thanks to Assumption 2.(b). However this calculation provides the required contradiction.

Using similar arguments, we can show that $\lim_{x\to-\infty} g(x,\lambda,a) \in \{-\infty,\infty\}$. $\square$

To proceed further, we need to understand better the asymptotic behaviour of g as $x \to \pm\infty$. To this end, we define

$$\lambda^*(a) = \inf\left\{\lambda \in \mathbb{R} \mid \sup_{x\geq a} g(x,\lambda,a) = \infty\right\}, \quad \text{for } a \in \mathbb{R}, \tag{66}$$

$$^*\lambda(a) = \inf\left\{\lambda \in \mathbb{R} \mid \inf_{x\leq a} g(x,\lambda,a) = -\infty\right\}, \quad \text{for } a \in \mathbb{R}, \tag{67}$$

with the usual convention that $\inf\emptyset = \infty$.

Lemma 5 *Fix any $a \in \mathbb{R}$ and suppose that Assumptions 1 and 2 are true. If $\lambda^*(a)$ and $^*\lambda(a)$ are defined as in (66) and (67), respectively, then $\lambda^*(a), {}^*\lambda(a) \in]0,\infty]$, and*

$$\lim_{x\to\infty} g(x,\lambda,a) = \begin{cases} -\infty, & \text{if } \lambda < \lambda^*(a), \\ \infty, & \text{if } \lambda \in [\lambda^*(a),\infty] \cap \mathbb{R}, \end{cases} \tag{68}$$

$$\lim_{x\to-\infty} g(x,\lambda,a) = \begin{cases} \infty, & \text{if } \lambda < {}^*\lambda(a), \\ -\infty, & \text{if } \lambda \in [{}^*\lambda(a),\infty] \cap \mathbb{R}. \end{cases} \tag{69}$$

Proof. We fix any $a \in \mathbb{R}$, throughout the proof, and we note that we can treat $\lambda^*(a)$ as a given constant. In view of (31) and the positivity of h, we can see that, given any $\lambda \leq 0$,

$$g(x,\lambda,a) \leq 0, \quad \text{for all } x \in \mathbb{R}, \tag{70}$$

which implies that $\lambda^*(a) \in]0,\infty]$. Also, the fact that $g(x,\cdot,a)$ is strictly increasing, for all $x > a$, that we proved in Lemma 4.(i), implies

$$\sup_{x\geq a} g(x,\lambda,a) \begin{cases} < \infty, & \text{for all } \lambda < \lambda^*(a), \\ = \infty, & \text{for all } \lambda \in]\lambda^*(a),\infty] \cap \mathbb{R}. \end{cases}$$

To show that $\sup_{x\geq a} g(x,\lambda^*(a),a) = \infty$ when $\lambda^*(a) < \infty$, and thus, in the light of (62) in Lemma 4.(iii), complete the proof of (68), we argue by contradiction. To this end, we assume that

$$\lim_{x\to\infty} g(x,\lambda^*(a),a) = -\infty.$$

This limit and (9) in Assumption 2.(a) imply that there exists $\hat{x}(a) > a$ such that

$$g(x,\lambda^*(a),a) < 0 \quad \text{and} \quad h(x) - \lambda^*(a) > \eta > 0, \quad \text{for all } x \geq \hat{x}(a), \tag{71}$$

where η is any given constant. Now, we and note that (63) and the second inequality in (71) imply that, given any $\lambda \in [\lambda^*(a), \lambda^*(a) + \eta]$,

$$g_x(x, \lambda, a) < 0, \text{ for all } x \geq \hat{x}(a) \text{ such that } g(x, \lambda, a) = 0.$$

This observation and the fact that $\lim_{x \to \infty} g(x, \lambda, a) = \infty$, for all $\lambda > \lambda^*(a)$, imply that there exists no $x \geq \hat{x}(a)$ such that $g(x, \lambda, a) = 0$ when $\lambda \in \,]\lambda^*(a), \lambda^*(a) + \eta]$, and that

$$g(x, \lambda, a) > 0, \quad \text{for all } x \geq \hat{x}(a) \text{ and } \lambda \in]\lambda^*(a), \lambda^*(a) + \eta].$$

However, this and the first inequality in (71) imply

$$\lim_{\lambda \downarrow \lambda^*(a)} g(x, \lambda, a) \geq 0 > g(x, \lambda^*(a), a), \quad \text{for all } x \geq \hat{x}(a),$$

which contradicts the continuity of g.

Proving the statements relating to ${}^*\lambda(a)$ involves similar arguments. □

The following example shows that, depending on the problem's data, we can have $\lambda^*(a)$, ${}^*\lambda(a) < \infty$.

Example 2 Suppose that

$$b(x) = -x, \quad \sigma(x) \equiv 1 \quad \text{and} \quad h(x) = x^2,$$

and note that Assumptions 1 and 2 are both satisfied. It is straightforward to verify that, for $a = 0$, (31) yields

$$g(x, \lambda, 0) = x + \frac{(2\lambda - 1)\sqrt{\pi}}{2} \operatorname{erf}(x) e^{x^2},$$

where erf is the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt.$$

Recalling that

$$\lim_{x \to \infty} \operatorname{erf}(x) = 1 \quad \text{and} \quad \lim_{x \to -\infty} \operatorname{erf}(x) = -1,$$

we can see that $\lambda^*(0) = {}^*\lambda(0) = 1/2$. □

With regard to the structure of the system of equations (25)–(30), we need to study the functions $g(\cdot, \cdot, \cdot) + K^+$ and $g(\cdot, \cdot, \cdot) - K^-$. To this end we define

$$\lambda_*(a) = \inf \left\{ \lambda > 0 \mid \sup_{x \geq a} g(x, \lambda, a) \geq K^- \right\}, \tag{72}$$

$$_*\lambda(a) = \inf \left\{ \lambda > 0 \mid \inf_{x \leq a} g(x, \lambda, a) \leq -K^+ \right\}. \tag{73}$$

Lemma 6 *Given any* $a \in \mathbb{R}$, $\lambda_*(a) \in]0, \lambda^*(a)[$, *and the equation* $g(x, \lambda, a) = K^-$ *defines uniquely two* C^1 *functions* $x_1(\cdot, a), x_2(\cdot, a) :]\lambda_*(a), \lambda^*(a)[\to]a, \infty[$ *such that*

$$x_1(\lambda, a) < x_2(\lambda, a) \quad \text{and} \quad a_+ < x_2(\lambda, a), \quad \text{for all } \lambda \in]\lambda_*(a), \lambda^*(a)[,$$

where a_+ *is as in Assumption 2.(c). Moreover*

$$x_1(\cdot, a) \text{ is strictly decreasing,} \quad x_2(\cdot, a) \text{ is strictly increasing,} \tag{74}$$

$$\lim_{\lambda \downarrow \lambda_*(a)} x_1(\lambda, a) = \lim_{\lambda \downarrow \lambda_*(a)} x_2(\lambda, a), \tag{75}$$

$$\lim_{\lambda \uparrow \lambda^*(a)} x_2(\lambda, a) = \infty \tag{76}$$

and

$$h(x) + b(x)K^- - \lambda > 0, \quad \text{for all } x > x_2(\lambda, a). \tag{77}$$

Proof. Fix any $a \in \mathbb{R}$. In view of (31) and the assumptions that $h \geq 0$ and $K^- > 0$, we can see that $\lambda_*(a) > 0$ (see also (70) in the proof of Lemma 5). Also, the definitions of $\lambda_*(a)$, $\lambda^*(a)$ and the continuity of g imply trivially that $\lambda_*(a) < \lambda^*(a)$.

Now, observe that a simple inspection of (61) reveals that

$$\text{if } x > a \text{ satisfies } g(x, \lambda, a) = K^-, \text{ then } g_x(x, \lambda, a) = \frac{-2}{\sigma^2(x)} \left[h(x) + b(x)K^- - \lambda \right]. \tag{78}$$

With regard to the definitions of $\lambda_*(a)$ and $\lambda^*(a)$, (68) in Lemma 5, the fact that $g(a, \lambda, a) = 0$, Assumption 2.(c), and the continuity of g, this observation implies the following:

(I) If $\lambda < \lambda_*(a)$, then the equation $g(x, \lambda, a) = K^-$ has no solutions $x \in]a, \infty[$.

(II) If $\lambda \in]\lambda_*(a), \lambda^*(a)[$, then the equation $g(x, \lambda, a) = K^-$ has one solution $x_1(\lambda, a) > a$ such that

$$h(x_1(\lambda, a)) + b(x_1(\lambda, a))K^- - \lambda < 0, \tag{79}$$

and one solution $x_2(\lambda, a) > x_1(\lambda, a)$ such that

$$h(x_2(\lambda, a)) + b(x_2(\lambda, a))K^- - \lambda > 0. \tag{80}$$

(III) If $\lambda \geq \lambda^*(a)$, then the equation $g(x, \lambda, a) = K^-$ has one solution $x_1(\lambda, a) > a$ such that

$$h(x_1(\lambda, a)) + b(x_1(\lambda, a))K^- - \lambda < 0. \tag{81}$$

Since $\lambda_*(a) > 0$, Assumption 2.(c) and (79)–(80) imply that the solution x_2 in (II) above satisfies $x_2(\lambda, a) > a_+$ and that (77) is true. Also, (I) and (II) and the continuity of g imply (75), while (II) and (III) and (74) imply (76).

To prove (74), we differentiate $g(x_j(\lambda, a), \lambda, a) = K^-$ with respect to λ to calculate

$$\begin{aligned}\frac{\partial x_j}{\partial \lambda}(\lambda, a) &= -\frac{g_\lambda(x_j(\lambda, a), \lambda, a)}{g_x(x_j(\lambda, a), \lambda, a)} \\ &= \frac{\sigma^2(x_j(\lambda, a)) g_\lambda(x_j(\lambda, a), \lambda, a)}{2\left[h(x_j(\lambda, a)) + b(x_j(\lambda, a))K^- - \lambda\right]},\end{aligned}$$

for all $\lambda \in]\lambda_*(a), \lambda^*(a)[$, $j = 1, 2$. However, this calculation, the result of Lemma 4.(i), and (79)–(80) imply that the function $x_1(\cdot, a)$ (resp., $x_2(\cdot, a)$) is strictly decreasing (resp., increasing), and the proof is complete. □

With regard to the problem's data symmetry, we can trivially modify the arguments of the preceding proof to establish the following result.

Lemma 7 *Given any $a \in \mathbb{R}$, ${}_*\lambda(a) \in]0, {}^*\lambda(a)[$, and the equation $g(x, \lambda, a) = -K^+$ defines uniquely two C^1 functions $y_1(\cdot, a), y_2(\cdot, a) :]{}_*\lambda(a), {}^*\lambda(a)[\to]-\infty, a[$ such that*

$$y_2(\lambda, a) < y_1(\lambda, a) \quad \text{and} \quad y_2(\lambda, a) < \alpha_-, \quad \text{for all } \lambda \in]{}_*\lambda(a), {}^*\lambda(a)[\tag{82}$$

where α_- is as in Assumption 2.(d). Furthermore,

$$y_2(\cdot, a) \text{ is strictly decreasing}, \quad y_1(\cdot, a) \text{ is strictly increasing}, \tag{83}$$

$$\lim_{\lambda \downarrow {}_*\lambda(a)} y_1(\lambda, a) = \lim_{\lambda \downarrow {}_*\lambda(a)} y_2(\lambda, a), \tag{84}$$

$$\lim_{\lambda \uparrow {}^*\lambda(a)} y_2(\lambda, a) = -\infty, \tag{85}$$

and

$$h(x) - b(x)K^+ - \lambda > 0, \quad \text{for all } x < y_2(\lambda, a) \tag{86}$$

Proof of Lemma 1. We fix any $a \in \mathbb{R}$ and, with regard to (29)–(30), we define

$$q^*(\lambda, a) = \int_{x_1(\lambda,a)}^{x_2(\lambda,a)} \left[g(s, \lambda, a) - K^-\right] ds - c^-, \quad \text{for } \lambda \in]\lambda_*(a), \lambda^*(a)[, \tag{87}$$

$${}^*q(\lambda, a) = \int_{y_2(\lambda,a)}^{y_1(\lambda,a)} \left[g(s, \lambda, a) + K^+\right] ds + c^+, \quad \text{for } \lambda \in]{}_*\lambda(a), {}^*\lambda(a)[, \tag{88}$$

where x_1, x_2 are as in Lemma 6, and y_1, y_2 are as in Lemma 7. Given these definitions, we will establish the claim regarding the solvability of the system of equations (25)–(30) if we prove that

$$\begin{aligned}&\text{there exist } \tilde{a}, \tilde{\lambda} \in \mathbb{R} \text{ such that } \lambda_*(\tilde{a}) \vee {}_*\lambda(\tilde{a}) < \lambda^*(\tilde{a}) \wedge {}^*\lambda(\tilde{a}),\\ &\tilde{\lambda} \in]\lambda_*(\tilde{a}) \vee {}_*\lambda(\tilde{a}), \lambda^*(\tilde{a}) \wedge {}^*\lambda(\tilde{a})[\text{ and } q^*(\tilde{\lambda}, \tilde{a}) = {}^*q(\tilde{\lambda}, \tilde{a}) = 0.\end{aligned} \tag{89}$$

Differentiating (87) with respect to λ, and using the fact that both $g(x_1(\lambda, a), \lambda, a)$ and $g(x_2(\lambda, a), \lambda, a)$ are equal to the constant K^-, we calculate

$$q^*_\lambda(\lambda, a) = \int_{x_1(\lambda,a)}^{x_2(\lambda,a)} g_\lambda(s, \lambda, a)\, ds > 0, \quad \text{for } \lambda \in]\lambda_*(a), \lambda^*(a)[. \tag{90}$$

the inequality following thanks to Lemma 4.(i). Also, with regard to the continuity of g, (68) in Lemma 5, and (74)–(76) in Lemma 6, we can see that

$$\lim_{\lambda \uparrow \lambda^*(a)} q^*(\lambda, a) = \infty \quad \text{and} \quad \lim_{\lambda \downarrow \lambda_*(a)} q^*(\lambda, a) = -c^- < 0. \tag{91}$$

However, (90) and (91) imply that, given any $a \in \mathbb{R}$, there exists a unique point $\Lambda^*(a) \in]\lambda_*(a), \lambda^*(a)[$ such that $q^*(\Lambda^*(a), a) = 0$. Similarly, we can show that, given any $a \in \mathbb{R}$, there exists a unique point ${}^*\Lambda(a) \in]_*\lambda(a), {}^*\lambda(a)[$ such that ${}^*q({}^*\Lambda(a), a) = 0$. With regard to these calculations, (89) will follow if we prove that

$$\text{there exists } \tilde{a} \in \mathbb{R} \text{ such that } \Lambda^*(\tilde{a}) = {}^*\Lambda(\tilde{a}). \tag{92}$$

To establish (92), we first differentiate $q^*(\Lambda^*(a), a) = 0$ with respect to a to obtain

$$\frac{d}{da}\Lambda^*(a) = -\frac{q^*_a(\Lambda^*(a), a)}{q^*_\lambda(\Lambda^*(a), a)}.$$

In view of the calculation

$$q^*_a(\lambda, a) = 2\frac{h(a) - \lambda}{\sigma^2(a)} \int_{x_1(\lambda,a)}^{x_2(\lambda,a)} p'_a(s)\, ds.$$

and (90), it follows that

$$\frac{d}{da}\Lambda^*(a) > 0, \quad \text{for all } a \in \mathbb{R} \text{ such that } h(a) < \Lambda^*(a). \tag{93}$$

Using similar arguments, we can also show that

$$\frac{d}{da}{}^*\Lambda(a) < 0, \quad \text{for all } a \in \mathbb{R} \text{ such that } h(a) < {}^*\Lambda(a). \tag{94}$$

Now, if we assume that $h(a) < \Lambda^*(a)$, for all $a \in \mathbb{R}$, then (93) implies

$$h(a) < \Lambda^*(a) < \Lambda^*(0), \quad \text{for all } a < 0,$$

which contradicts Assumption 2.(a). With respect to the usual convention $\sup \emptyset = -\infty$, this shows that $\sup \{a \in \mathbb{R} \mid \ \Lambda^*(a) \leq h(a)\} > -\infty$. Moreover, the definition (31) of g and Assumption 2.(a) imply that

$$g(x, \lambda, a) \leq 0, \quad \text{for all } x \geq a \geq 0 \text{ and } \lambda \leq h(a).$$

Combining this observation with the definition (72) of $\lambda_*(a)$, we can see that $h(a) < \lambda_*(a)$, for all $a \geq 0$, which, together with the inequality $\lambda_*(a) < \Lambda^*(a)$, implies that $h(a) \leq \Lambda^*(a)$, for all $a \geq 0$. It follows that

$$A_- := \sup \{a \in \mathbb{R} \mid \ \Lambda^*(a) \leq h(a)\} \in]-\infty, 0[. \tag{95}$$

Using a similar reasoning, we can also show that

$$A_+ := \inf \{a \in \mathbb{R} \mid \ {}^*\Lambda(a) \leq h(a)\} \in]0, \infty[. \tag{96}$$

With regard to (93)–(96), we can see that

$$\text{the function } \Lambda^*(\cdot) - {}^*\Lambda(\cdot) \text{ is strictly increasing on the interval }]A_-, A_+[. \tag{97}$$

To proceed further, let us assume that ${}^*\Lambda(A_+) \geq \Lambda^*(A_+)$, so that $h(A_+) \geq {}^*\Lambda(A_+) \geq \Lambda^*(A_+)$. In this context, we can combine the definition (31) of g and Assumption 2.(a) with the fact that $A_+ > 0$ to obtain

$$g(x, \Lambda^*(A_+), A_+) < 0, \quad \text{for all } x > A_+.$$

Since $\Lambda^*(A_+) > \lambda_*(A_+)$, this inequality and Lemma 4.(i) imply that

$$g(x, \lambda_*(A_+), A_+) < 0, \quad \text{for all } x > A_+,$$

which contradicts the definition (72) of λ_*. However, this proves that

$$\Lambda^*(A_+) - {}^*\Lambda(A_+) > 0. \tag{98}$$

Similarly, we can show that $\Lambda^*(A_-) - {}^*\Lambda(A_-) < 0$, which, combined with (97) and (98), implies (92), and, therefore, (89).

With regard to its construction, we will prove that a function w defined as in the statement of the lemma satisfies the HJB equation (13) if we show that

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda \geq 0, \quad \text{for all } x > x_2, \tag{99}$$

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda \geq 0, \quad \text{for all } x < y_2, \tag{100}$$

$$w(x+z) - w(x) - K^- z + c^- \geq 0, \quad \text{for } z < 0, \ x \in \mathbb{R}, \tag{101}$$

$$w(x+z) - w(x) + K^+ z + c^+ \geq 0, \quad \text{for } z > 0, \ x \in \mathbb{R}. \tag{102}$$

In view of (24) and (31), we note that, if $\tilde{\alpha}$, $\tilde{\lambda}$ are as in (89), then $w'(x) = g(x, \tilde{\lambda}, \tilde{a})$, for all $x \in [y_2, x_2] \equiv [y_2(\tilde{\lambda}, \tilde{a}), x_2(\tilde{\lambda}, \tilde{a})]$. Given this observation, inequalities (99) and (100) follow by a straightforward calculation that shows that they are implied by (77) and (86) respectively. Inequality (101) is equivalent to

$$-\int_{x+z}^{x} \left[w'(s) - K^-\right] ds + c^- \geq 0, \quad \text{for } z < 0, \ x \in \mathbb{R}. \tag{103}$$

With regard to the inequalities

$$w'(x) \begin{cases} < K^-, & \text{for } x < x_1, \\ > K^-, & \text{for } x \in]x_1, x_2[, \\ = K^-, & \text{for } x > x_2, \end{cases}$$

and equation (87) it is a tedious, but totally straightforward exercise to show that (103) is true. Finally, the proof of (102) is similar. □

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