OPTIMAL DIVIDEND AND ISSUANCE OF EQUITY POLICIES IN THE PRESENCE OF PROPORTIONAL COSTS

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Abstract. We consider three optimisation problems faced by a company that can control their liquid reserves by paying dividends and by issuing new equity. The first of these problems involves no issuance of new equity and has been considered by several authors in the literature. The second one aims at maximising the expected discounted dividend payments minus the expected discounted costs of issuing new equity over all strategies associated with positive reserves at all times. The third problem has the same objective as the second one, but with no constraints on the reserves. Assuming proportional issuance of equity costs, we derive closed form solutions and we completely characterise the optimal strategies. We also provide a relationship between the three problems.

1. Introduction

Diffusion models for companies that can control their risk exposure by means of their dividend payments have attracted significant interest in the recent literature. Jeanblanc and Shiryaev [10], Radner and Shepp [12] and Boguslavskaya [2] model the liquid reserves of a company by means of a Brownian motion with drift, while Asmussen, Højgaard and Taksar [1], Choulli, Taksar and Zhou [3] and Højgaard and Taksar [4, 5, 6] consider more general diffusions. A typical application appearing in these references, considers the dividend flow as a controlled process and aims at maximising the expected discounted dividends that are payed up to the company's bankruptcy, which is taken to be the time at which the reserves process hits 0. In a recent paper, Sethi and Taksar [13] consider a model for a company that can control their risk exposure by issuing new equity as well as by paying dividends. These authors consider dynamics for the reserves process that, if uncontrolled, never hit 0 that signals bankruptcy.

The objective of this paper is to study a model for a company that can control their reserves process dynamics as in Sethi and Taksar [13], while still facing the possibility of bankruptcy as in the other references mentioned above. In particular, we model the uncontrolled reserves dynamics by a Brownian motion with drift and we assume that dividend payments and issuance of new equity take the form of "singular" controls. As in Sethi and Taksar [13], the aim is to maximise the expected discounted dividend payments minus the expected discounted costs of issuing new equity. For the resulting optimisation problem to lead to results that have nice economic interpretations, it turns out that we have to define bankruptcy as the time at which the reserves process hits $(-\infty, 0)$ instead

Date: July 14, 2005.

Research supported by EPSRC grant no. GR/S22998/01.

of $(-\infty, 0]$ because, otherwise, there may exist no optimal strategy. It is worth noting that this definition leads to a problem that is non-standard in control theory where, typically, the objective is to control a process up to the exit time from an open domain instead of a closed one.

It turns out that our control problem is associated with qualitatively different optimal strategies, depending on the problem's data, and requires the consideration of two auxiliary control problems. The first one is essentially the same as the one solved by Jeanblanc and Shiryaev [10], which allows for no issuance of new equity. The associated optimal dividend strategy reflects the state process at an appropriate level x^{**} and bankruptcy eventually occurs (see Section 3). The second one aims at maximising the expected discounted dividend payments minus the expected discounted costs of new equity issuance subject to the constraint that bankruptcy never occurs, i.e., subject to the constraint that the associated reserves process remains positive at all times. This problem is interesting on its own. Indeed, banks and insurance companies are restricted by regulations to maintain positive reserves at all times in order to operate. The issuance of new equity provides a strategy that can be used by such institutions in meeting their regulatory requirements, ideally in an optimal way. In this problem, it is optimal to issue minimal new equity so as to reflect the reserves process at 0 and make minimal dividend payments so as to reflect the reserves process at a certain level x^* (see Section 4). With regard to the general problem, it turns out that its value function and its optimal strategy identify with the corresponding ones in either the first auxiliary control problem or the second one, depending on the problem's data (see Section 5). Thus, under the optimal strategy, 0 acts either as an absorbing or as a reflecting boundary point in the reserves process' state space.

The paper is organised as follows. Section 2 is concerned with the formulation of the three stochastic control problems that we solve. In Section 3, we review the solution to the control problem that allows for no issuance of new equity, which was derived by Jeanblanc and Shiryaev [10], and we prove some of its properties that we use later. In Section 4, we solve the control problem that arises when the admissible strategies are constrained to allow for no bankruptcy, while Section 5 is concerned with the solution to the general control problem that involves no constraints on the issuance of new equity or the reserves.

2. Formulation of the control problems

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions and supporting a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a company with an established and stationary cash flow. The company's management can control the reserves by paying out dividends and by raising capital by issuing equity. We denote by L_t the cumulative amount of dividends paid from time zero up to time t, and by G_t the total amount raised by issuing equity from time zero up to time t. We assume that both L and G are increasing, $(\mathcal{F}_t)$-adapted processes and their sample paths are left-continuous with right limits. Given a *dividend process* L and an *issuance of equity process* G, we assume that the *liquid reserves* of the company are modelled by the stochastic differential equation

$$dX_t = \mu\, dt + \sigma\, dW_t - dL_t + dG_t, \quad X_0 = x,$$

where $x \geq 0$ is the reserves at time zero, and μ and σ are parameters describing the growth rate and the volatility of the reserves process, respectively. Note that, contrary to Jeanblanc and Shiryaev [10] and other references discussed in the introduction that assume that $\mu > 0$, we allow for the possibility $\mu \leq 0$.

We make the assumption that the company needs positive reserves in order to operate and the company is considered bankrupt as soon as the reserves become strictly negative. Accordingly, we define the *bankruptcy time* τ by

$$\tau = \inf\{t \geq 0 \,:\, X_t < 0\}.$$

Noting the strict inequality in this definition, we should remark that the random time τ is indeed an $(\mathcal{F}_t)$-stopping time (e.g., see Theorem 1.1.27 Jacod and Shiryaev [9]).

Following Sethi and Taksar [13], our objective is to maximise the performance index J given by

$$J(x; L, G) = \mathbb{E}^x\left[\limsup_{t\to\infty}\left(\int_{[0,t\wedge\tau]} \mathrm{e}^{-rt}\, dL_t - \beta \int_{[0,t\wedge\tau]} \mathrm{e}^{-rt}\, dG_t\right)\right], \tag{2.1}$$

where the discount factor r can be the risk free interest rate and the constant $\beta > 1$ provides a measure for the proportional costs arising from the issuance of equity. Note that $\beta = 1$ corresponds to no costs, while high values of β correspond to high costs associated with raising capital in the market. Also, the purpose of the "lim sup" in (2.1) is to guarantee that the random variable inside the expectation is well defined on the event $\{\tau = \infty\}$.

We consider three distinct optimisation problems, each corresponding to the maximisation of the performance index J over a set of appropriate admissible dividend and issuance of equity strategies.

Definition 2.1. (No restrictions on the issuance of equity or the reserves.) Given an initial condition $x \geq 0$, we denote by $\mathcal{A}(x)$ the set of all dividend and issuance of equity processes (L, G) such that:

(i) L and G are $(\mathcal{F}_t)$-adapted, increasing and càglàd processes,
(ii) $L_0 = G_0 = 0$, and
(iii) $\triangle L_t \leq X_t$, for all $t \geq 0$, P-a.s..

We define the corresponding value function V by

$$V(x) = \sup_{(L,G)\in\mathcal{A}(x)} J(x; L, G), \quad x \geq 0. \tag{2.2}$$

Here, as well as in the following definitions, we impose the condition $\triangle L \leq X$ to rule out the possibility of making dividend payments greater than the company's reserves. Such a condition is essential, because in its absence, the company's management could realise arbitrarily high payoffs by making arbitrarily high dividend payments at time 0, which is plainly unrealistic.

The optimisation problem associated with the following definition corresponds to the problem of maximising the expected discounted dividends paid until default in the absence of an equity issuance possibility, which is a problem addressed by Jeanblanc and

Shiryaev [10], Radner and Shepp [12] and Boguslavskaya [2]. However, note that our version of the problem is slightly different from the one that these authors consider because we have defined the bankruptcy time τ to be the hitting time of $(-\infty, 0)$ instead of $(-\infty, 0]$.

Definition 2.2. (Issuance of equity not permitted.) Given an initial condition $x \geq 0$, denote by $\mathcal{A}_{\rm d}(x) \subseteq \mathcal{A}(x)$ the set of all dividend and issuance of equity processes (L, G) such that:

(i) L is $(\mathcal{F}_t)$-adapted, increasing and càglàd,
(ii) $L_0 = 0$,
(iii) $\triangle L_t \leq X_t$, for all $t \geq 0$, P-a.s., and
(iv) $G_t = 0$, for all $t \geq 0$, P-a.s..

We define the associated value function $V_{\rm d}$ by

$$V_{\rm d}(x) = \sup_{(L,G)\in\mathcal{A}_{\rm d}(x)} J(x; L, G), \quad x \geq 0. \tag{2.3}$$

The next definition is concerned with the optimisation problem that aims at maximising the performance index J over all dividend and issuance of equity strategies that are associated with a positive reserves process. This problem corresponds to maximising the performance index J over all dividend and issuance of equity strategies satisfying the additional condition that the default time τ is infinite.

Definition 2.3. (Restrictions on the reserves.) Given an initial condition $x \geq 0$, denote by $\mathcal{A}_{\rm c}(x) \subseteq \mathcal{A}(x)$ the set of all dividend and issuance of equity processes (L, G) such that:

(i) L and G are $(\mathcal{F}_t)$-adapted, increasing and càglàd,
(ii) $L_0 = G_0 = 0$,
(iii) $\triangle L_t \leq X_t$, for all $t \geq 0$, P-a.s., and
(iv) $X_t \geq 0$, for all $t \geq 0$, P-a.s..

We define the value function $V_{\rm c}$ by

$$V_{\rm c}(x) = \sup_{(L,G)\in\mathcal{A}_{\rm c}(x)} J(x; L, G), \quad x \geq 0. \tag{2.4}$$

Remark 2.1. Since the families of the admissible decision strategies appearing in these definitions satisfy $\mathcal{A}_{\rm d}(x), \mathcal{A}_{\rm c}(x) \subseteq \mathcal{A}(x)$,

$$V(x) \geq \max\big\{V_{\rm d}(x), V_{\rm c}(x)\big\}, \quad \text{for all } x \geq 0. \tag{2.5}$$

This inequality has played an important role in our discovering the optimal strategy of the optimisation problem that involves no constraints (see also Section 5).

Remark 2.2. At this point, we should note that our analysis in the following sections also establishes the fact that the expectation in (2.1) is well defined and takes values in $[-\infty, \infty)$ for any admissible strategy as in Definitions 2.1–2.3. Therefore, the three optimisation problems that we consider are well defined.

3. The solution to the problem that involves no issuance of new equity

The solution to the singular control problem arising when $\mu > 0$ and the admissible strategies are as in Definition 2.2 was obtained in Jeanblanc and Shiryaev [10]. With reference to standard theory of optimal control, the Hamilton-Jacobi-Bellman (HJB) equation corresponding to this problem is given by

$$\max\left\{\frac{1}{2}\sigma^2 w''(x) + \mu w'(x) - rw(x),\ -w'(x) + 1\right\} = 0, \quad x > 0, \tag{3.1}$$

$$w(0) = 0. \tag{3.2}$$

The boundary condition here arises naturally once we observe that the value function is zero at default. With regard to simple economic considerations, we can conjecture that the value function V_{d} identifies with a solution w to this HJB equation satisfying

$$\begin{aligned} \frac{1}{2}\sigma^2 w''(x) + \mu w'(x) - rw(x) = 0, &\quad \text{for } x < x^{**}, \\ -w'(x) + 1 = 0, &\quad \text{for } x \geq x^{**}, \end{aligned}$$

for some constant $x^{**} > 0$. To proceed further, we observe that the general solution to the ordinary differential equation

$$\frac{1}{2}\sigma^2 w''(x) + \mu w'(x) - rw(x) = 0, \tag{3.3}$$

is given by

$$w(x) = c_1 \mathrm{e}^{\alpha_1 x} + c_2 \mathrm{e}^{\alpha_2 x}, \tag{3.4}$$

where, $c_1, c_2 \in \mathbb{R}$ are constants and the real numbers α_1, α_2 are given by

$$\alpha_1 = \frac{1}{\sigma^2}\left(-\mu + \sqrt{\mu^2 + 2\sigma^2 r}\right), \tag{3.5}$$

$$\alpha_2 = \frac{1}{\sigma^2}\left(-\mu - \sqrt{\mu^2 + 2\sigma^2 r}\right). \tag{3.6}$$

Since every solution to (3.4) that satisfies the boundary condition $w(0) = 0$ is associated with the relation $c_1 + c_2 = 0$, it follows that we should consider a solution to the HJB equation (3.1)–(3.2) of the form

$$w(x) = \begin{cases} c_1\left(\mathrm{e}^{\alpha_1 x} - \mathrm{e}^{\alpha_2 x}\right), & \text{for } 0 \leq x < x^{**}, \\ x - x^{**} + w(x^{**}), & \text{for } x \geq x^{**}. \end{cases} \tag{3.7}$$

To specify the parameters c_1 and x^{**}, we appeal to the so-called "smooth-pasting condition" of singular control, which dictates that the value function should be C^2, in particular, at the free boundary point x^{**}. This condition gives rise to the system of equations

$$c_1 \alpha_1 \mathrm{e}^{\alpha_1 x^{**}} - c_1 \alpha_2 \mathrm{e}^{\alpha_2 x^{**}} = 1, \tag{3.8}$$

$$c_1 \alpha_1^2 \mathrm{e}^{\alpha_1 x^{**}} - c_1 \alpha_2^2 \mathrm{e}^{\alpha_2 x^{**}} = 0. \tag{3.9}$$

which is equivalent to

$$x^{**} = -\frac{\ln(\alpha_1^2) - \ln(\alpha_2^2)}{\alpha_1 - \alpha_2}, \tag{3.10}$$

and

$$c_1 = \left[\alpha_1 \left|\frac{\alpha_2}{\alpha_1}\right|^{\frac{2\alpha_1}{\alpha_1-\alpha_2}} - \alpha_2 \left|\frac{\alpha_2}{\alpha_1}\right|^{\frac{2\alpha_2}{\alpha_1-\alpha_2}}\right]^{-1} > 0, \tag{3.11}$$

the inequality following thanks to the fact that $\alpha_2 < 0 < \alpha_1$.

Now, it is a straightforward exercise to show that

$$x^{**} > 0 \quad \Leftrightarrow \quad \mu > 0. \tag{3.12}$$

In view of this observation, the function w given by (3.7) is is well-defined if and only if $\mu > 0$, in which case, it is concave. Indeed, the calculation

$$w'''(x) = c_1\Big(\alpha_1^3 e^{\alpha_1 x} - \alpha_2^3 e^{\alpha_2 x}\Big) > 0, \quad \text{for } x < x^{**},$$

proves that w'' is increasing in $[0, x^{**}]$. This observation and the boundary condition $w''(x^{**}) = 0$ imply that $w''(x) \le 0$, for all $x > 0$, which establishes the claim. However, the concavity of w and the free-boundary condition $w'(x^{**}) = 1$ imply

$$-w'(x) + 1 \le 0, \quad \text{for all } x \in [0, x^{**}],$$

while the fact that w is strictly increasing and the identity $\mu w'(x^{**}) - rw(x^{**}) = 0$, which follows from the construction of w, imply

$$\frac{1}{2}\sigma^2 w''(x) + \mu w'(x) - rw(x) \le 0, \quad \text{for all } x \ge x^{**}.$$

It follows that, when $\mu > 0$, the function w given by (3.7) is well-defined and satisfies the HJB equation (3.1).

In view of (3.12), we conjecture that it is optimal to liquidate the company at time 0 when $\mu \le 0$. With regard to this possibility, it is straightforward to check that the function defined by

$$w(x) = x, \quad \text{for all } x \ge 0, \tag{3.13}$$

satisfies the HJB equation (3.1) if and only if $\mu \le 0$.

For future reference, let us consider the case associated with $\mu > 0$. In this context, we can use (3.10) to see that $e^{(\alpha_1-\alpha_2)x^{**}} = \big(\alpha_2/\alpha_1\big)^2$. In view of this calculation, it follows that

$$\Big(\alpha_1 e^{-\alpha_2 x^{**}} - \alpha_2 e^{-\alpha_1 x^{**}}\Big)\Big(\alpha_1 e^{\alpha_1 x^{**}} - \alpha_2 e^{\alpha_2 x^{**}}\Big) = (\alpha_1 - \alpha_2)^2,$$

which implies

$$\frac{\alpha_1 - \alpha_2}{\alpha_1 e^{\alpha_1 x^{**}} - \alpha_2 e^{\alpha_2 x^{**}}} = \frac{\alpha_1 e^{-\alpha_2 x^{**}} - \alpha_2 e^{-\alpha_1 x^{**}}}{\alpha_1 - \alpha_2}. \tag{3.14}$$

Now, the concavity of w implies that $w'(x) \leq \beta$, for all $x \geq 0$, if and only if $w'(0) = \lim_{x\downarrow 0} w'(0) \leq \beta$. However, combining this observation with (3.14), we can see that, if $\mu > 0$, then

$$w'(x) \leq \beta, \text{ for all } x \geq 0 \quad \Leftrightarrow \quad \alpha_1 \mathrm{e}^{-\alpha_2 x^{**}} - \alpha_2 \mathrm{e}^{-\alpha_1 x^{**}} \leq \beta(\alpha_1 - \alpha_2). \tag{3.15}$$

The following result, the proof of which can be developed following a straightforward modification of the proof of Theorem 5.1, provides the solution to the optimisation problem that we consider in this section.

Theorem 3.1. *Consider the problem of maximising the performance criterion $J(x; L, G)$ over all strategies (L, G) in the set $\mathcal{A}_{\mathrm{d}}(x)$ provided by Definition 2.2. The following cases hold:*

(i) *If $\mu > 0$, then the value function V_{d} identifies with the concave solution w to the HJB equation (3.1)–(3.2) given by (3.7), where the constants c_1, x^{**} are as in (3.10) and (3.11), respectively.*

(ii) *If $\mu \leq 0$, then the value function is given by $V_{\mathrm{d}}(x) = x$, for all $x \geq 0$.*

4. The solution to the problem that does not allow for bankruptcy

We now address the problem that arises in the context of Definition 2.3 and aims at maximising the expected discounted dividend flow minus the expected discounted costs of issuing equity over all dividend and issuance of equity strategies associated with a positive reserves process. With reference to standard theory of singular control, the associated Hamilton-Jacobi-Bellman (HJB) equation takes the form

$$\max\left\{\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - r v(x),\ -v'(x) + 1,\ v'(x) - \beta\right\} = 0, \quad x \geq 0. \tag{4.1}$$

Remark 4.1. In our analysis below, we are going to consider C^2 solutions to this equation with bounded first and second derivatives. Therefore, we are going to consider functions v that satisfy (4.1) for all x, rather than for all x, Lebesgue-a.e., which is often the case in stochastic control. In particular, we are going to consider solutions that satisfy (4.1) for $x = 0$ as well, with $v(0) = \lim_{x\downarrow 0} v(x)$, $v'(0) = \lim_{x\downarrow 0} v'(x)$ and $v''(0) = \lim_{x\downarrow 0} v''(x)$.

Now, considering the time value of money can lead us to the conclusion that it is optimal to postpone the issuance of new equity for as long as possible. We therefore conjecture that it is optimal to issue equity only when the reserves become zero. Such a conjecture indicates that it is optimal for the company's management to take no action as long as the reserves process takes values in $(0, x^*)$, for some $x^* > 0$, take minimal action by issuing new equity so as to prevent the reserves process X from entering $(-\infty, 0)$, and take minimal action to keep X below x^*. With regard to the results discussed in Section 3, this strategy is associated with a solution to the HJB equation (4.1) that is characterised by

$$v'(0) - \beta = 0, \tag{4.2}$$

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - r v(x) = 0, \quad \text{for } 0 < x < x^*, \tag{4.3}$$

$$-v'(x) + 1 = 0, \quad \text{for } x \geq x^*. \tag{4.4}$$

Recalling that the solution to the ordinary differential equation (3.3) is given by (3.4) for some constants $c_1, c_2 \in \mathbb{R}$ and for α_1, α_2 being as in (3.5), (3.6), respectively, every function v satisfying (4.3) and (4.4) is given by

$$v(x) = \begin{cases} c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}, & 0 \leq x < x^*, \\ x - x^* + c_1 e^{\alpha_1 x^*} + c_2 e^{\alpha_2 x^*}, & x \geq x^*. \end{cases} \tag{4.5}$$

To specify the parameters c_1, c_2 and the free boundary point x^*, we use (4.2) and assume that v is C^2 at x^*, which is suggested by the "smooth pasting condition" of singular control. Thus, we obtain the system of equations

$$c_1\alpha_1 + c_2\alpha_2 = \beta, \tag{4.6}$$

$$c_1\alpha_1 e^{\alpha_1 x^*} + c_2\alpha_2 e^{\alpha_2 x^*} = 1, \tag{4.7}$$

$$c_1\alpha_1^2 e^{\alpha_1 x^*} + c_2\alpha_2^2 e^{\alpha_2 x^*} = 0, \tag{4.8}$$

which is equivalent to

$$\alpha_1 e^{-\alpha_2 x^*} - \alpha_2 e^{-\alpha_1 x^*} = \beta(\alpha_1 - \alpha_2), \tag{4.9}$$

$$c_1 = \frac{-\alpha_2}{\alpha_1(\alpha_1 - \alpha_2)} e^{-\alpha_1 x^*} \quad \text{and} \quad c_2 = \frac{\alpha_1}{\alpha_2(\alpha_1 - \alpha_2)} e^{-\alpha_2 x^*}. \tag{4.10}$$

The next result is concerned with showing that the HJB equation (4.1) has a unique solution conforming with the considerations above.

Lemma 4.1. *Equation (4.9) has a unique solution $x^* > 0$. The function v defined by (4.5) with x^* being the unique solution to (4.9) and with c_1, c_2 being given by (4.10) is increasing and concave in $[0, \infty)$, and satisfies the HJB equation (4.1) with boundary condition (4.2). Moreover, $v(0) \geq 0$ if and only if*

$$\mu > 0 \quad \text{and} \quad \alpha_1 e^{-\alpha_2 x^{**}} - \alpha_2 e^{-\alpha_1 x^{**}} \geq \beta\big(\alpha_1 - \alpha_2\big), \tag{4.11}$$

where x^{**} is given by (3.10).

Proof. To prove that (4.9) has a unique solution, we define

$$f(x) = \alpha_1 e^{-\alpha_2 x} - \alpha_2 e^{-\alpha_1 x}. \tag{4.12}$$

Since $\alpha_2 < 0 < \alpha_1$,

$$f''(x) = -\alpha_2\alpha_1^2 e^{-\alpha_1 x} + \alpha_1\alpha_2^2 e^{-\alpha_2 x} > 0.$$

This inequality and the calculation $f'(0) = 0$ imply that $f'(x) > 0$, for all $x > 0$. It follows that f is strictly increasing in $(0, \infty)$, which, combined with the observation

$$f(0) = \alpha_1 - \alpha_2 < \beta(\alpha_1 - \alpha_2),$$

implies that the equation $f(x) = \beta(\alpha_1 - \alpha_2)$ has a unique solution $x^* > 0$. Moreover,

$$\alpha_1 e^{-\alpha_2 x} - \alpha_2 e^{-\alpha_1 x} > \beta(\alpha_1 - \alpha_2) \quad \Leftrightarrow \quad x^* < x. \tag{4.13}$$

With regard to the calculation

$$v(0) = \frac{1}{\alpha_1 - \alpha_2}\left[\frac{\alpha_1}{\alpha_2}\mathrm{e}^{-\alpha_2 x^*} - \frac{\alpha_2}{\alpha_1}\mathrm{e}^{-\alpha_1 x^*}\right],$$

we can see that $v(0) > 0$ if and only if the point $x^* > 0$ satisfies

$$x^* < \frac{\ln(\alpha_2^2) - \ln(\alpha_1^2)}{\alpha_1 - \alpha_2} = x^{**},$$

where x^{**} is as in Section 3. However, combining this inequality with (3.12) and (4.13) above, we can see that (4.11) is true.

Now, we calculate

$$v''(x) = -\frac{\alpha_1\alpha_2}{\alpha_1 - \alpha_2}\left[\mathrm{e}^{\alpha_1(x-x^*)} - \mathrm{e}^{\alpha_2(x-x^*)}\right] < 0, \quad \text{for } x \in (0, x^*), \tag{4.14}$$

with the inequality following because $\alpha_2 < 0 < \alpha_1$. However, this shows that v is concave. Moreover, combining this observation with the boundary condition $v'(0) = \beta > 1$ and $v'(x) = 1$, for $x \geq x^*$, we can see that $v'(x) \geq 1$, for all $x \geq 0$, which proves that v is strictly increasing.

By construction, we will prove that v satisfies the HJB equation (4.1) if we show that

$$v'(x) \leq \beta, \quad \text{for } x \leq x^*, \tag{4.15}$$

$$v'(x) \geq 1, \quad \text{for } x < x^*, \tag{4.16}$$

$$\frac{1}{2}\sigma^2 v''(x) + \mu v'(x) - rv(x) \leq 0, \quad \text{for } x \geq x^*. \tag{4.17}$$

The concavity of v and the boundary conditions $v'(0) = \beta$ and $v'(x^*) = 1$ establish (4.15) and (4.16), while the fact that $v'(x) = 1$, for $x \geq x^*$, and the observation that

$$\frac{1}{2}\sigma^2 v''(x^*) + \mu v'(x^*) - rv(x^*) = \mu - rv(x^*) = 0$$

imply (4.17), and the proof is complete. □

In view of the analysis above, the following result, the proof of which can be developed by a straightforward modification of the arguments used to establish Theorem 5.1 below, provides the solution to the optimisation problem considered in this section

Theorem 4.2. *Consider the problem of maximising the performance index $J(x; L, G)$ over all strategies (L, G) within the class $\mathcal{A}_\mathrm{c}(x)$ provided by Definition 2.3. The value function V_c identifies with the increasing and concave solution v to the HJB equation (4.1) given by (4.5), where the constants x^* and c_1, c_2 are as in (4.9) and (4.10), respectively.*

5. The solution to the general problem

We now address the problem of maximising the expected discounted dividend flow minus the expected discounted costs of issuing equity over all admissible dividend and issuance of equity strategies when there are no restrictions on the issuance of equity or the reserves (see

Definition 2.1). With reference to the theory of singular stochastic control the associated Hamilton-Jacobi-Bellman equation takes the form

$$\max\left\{\frac{1}{2}\sigma^2 h''(x) + \mu h'(x) - rh(x),\ -h'(x)+1,\ h'(x)-\beta\right\} = 0, \quad x \geq 0. \tag{5.1}$$

With regard to the solvability of this equation, we assume in our discussion below that comments similar to the ones made in Remark 4.1 apply.

Now, in view of the Markovian structure of the problem that we study, we can expect that the optimal strategy should either allow for the reserves process to hit $(-\infty, 0)$ by issuing no new equity at any time, which corresponds to the boundary condition $h(0) = 0$ (see Section 3), or should keep the reserves process outside the interval $(-\infty, 0)$, which corresponds to the boundary condition $h'(0) = \beta$ (see Section 4). If this is indeed the case, then we should complement (5.1) with the boundary condition

$$\max\left\{-h(0),\ h'(0)-\beta\right\} = 0. \tag{5.2}$$

Moreover, this observation suggests that the value function V should identify with either $V_{\mathrm{d}} \equiv w$ or $V_{\mathrm{c}} \equiv v$ (see Theorem 3.1 and Theorem 4.2).

With regard to Lemma 4.1, the function v considered in Section 4 satisfies the HJB equation (5.1) with boundary condition (5.2) if and only if (4.11) is true, which in view of the expression for x^{**} provided by (4.9) is equivalent to the inequality

$$\mu > 0 \quad \text{and} \quad \frac{\mu+\delta}{2\delta}\left|\frac{\mu-\delta}{\mu+\delta}\right|^{1-\mu/\delta} - \frac{\mu-\delta}{2\delta}\left|\frac{\mu-\delta}{\mu+\delta}\right|^{-1-\mu/\delta} \geq \beta, \tag{5.3}$$

with $\delta = \sqrt{\mu^2 + 2\sigma^2 r}$, which involves only the problem's original data. Also, when $\mu > 0$, (3.15) implies that the function w defined by (3.7) satisfies the HJB equation (5.1) with the boundary condition (5.2) if and only if

$$\frac{\mu+\delta}{2\delta}\left|\frac{\mu-\delta}{\mu+\delta}\right|^{1-\mu/\delta} - \frac{\mu-\delta}{2\delta}\left|\frac{\mu-\delta}{\mu+\delta}\right|^{-1-\mu/\delta} \leq \beta, \tag{5.4}$$

while the function w defined by (3.13) satisfies the HJB equation (5.1) with the boundary condition (5.2) if and only if $\mu \leq 0$ because $\beta > 1$.

The arguments above suggest that we should identify the value function V with the function v given by (4.5) if (5.3) is true, with w given by (3.7) if $\mu > 0$ and (5.4) is true, and with w given by (3.13) if $\mu \leq 0$. The following result proves that such an identification indeed provides the solution to the optimisation problem considered.

Theorem 5.1. *Fix any initial condition $x \geq 0$, and consider the problem of maximising the performance criterion $J(x; L, G)$ over all dividend and issuance of equity strategies $(L, G) \in \mathcal{A}(x)$. Also, let w be the solution to the HJB equation (3.1) considered in Theorem 3.1, and let v be the solution to the HJB equation (4.1) appearing in Lemma 4.1. The value function V is increasing and concave, and the following cases provide the solution to the control problem:*

(i) *If $\mu \leq 0$, then $v(x) \leq w(x) \equiv x = V(x)$, the optimal dividend and issuance of equity strategy $(L^{\mathrm{o}}, G^{\mathrm{o}})$ consists of paying a dividend of size x at time 0, i.e. L^{o} has a jump of size x at time 0 and $G^{\mathrm{o}}_t = 0$, for all $t > 0$, and bankruptcy occurs at time $\tau^{\mathrm{o}} = 0$.*

(ii) *If $\mu > 0$ and condition (5.4) is true, then $v(x) \leq w(x) = V(x)$. The optimal dividend strategy L^{o} has a jump of size $(x - x^{**})^+$ at time 0 and then reflects the reserves process X at x^{**}, where x^{**} is as in (3.10), while the optimal issuance of equity strategy $G^{\mathrm{o}} \equiv 0$.*

(iii) *If condition (5.3) holds true, then $w(x) \leq v(x) = V(x)$, and the optimal dividend and issuance of equity strategy $(L^{\mathrm{o}}, G^{\mathrm{o}})$ reflects the reserves process X at the end-points of the interval $[0, x^*]$, where x^* is given by (4.9), with L^{o} having a jump of size $(x - x^*)^+$ at time 0. In this case, $\tau = \infty$, P-a.s..*

Proof. Fix any initial condition $x \geq 0$ and any admissible strategy $(L, G) \in \mathcal{A}(x)$, and let τ be the first time that the corresponding reserves process X hits the set $(-\infty, 0)$. Also, let h be the solution to the HJB equation (5.1) with boundary condition (5.2), depending on which of the conditions distinguishing the three cases of the theorem holds. We consider any C^2 continuation of h in $\mathbb{R}$ with bounded first derivative and we use Itô-Tanaka's formula to obtain

$$
\begin{aligned}
\mathrm{e}^{-r(t\wedge\tau)} h(X_{(t\wedge\tau)+}) = v(x) &+ \int_0^{t\wedge\tau} \mathrm{e}^{-rs} \left[\frac{1}{2}\sigma^2 h''(X_s) + \mu h'(X_s) - r h(X_s) \right] ds \\
&+ M_{t\wedge\tau} - \int_0^{t\wedge\tau} \mathrm{e}^{-rs} h'(X_s)\, dL^c_s + \int_0^{t\wedge\tau} \mathrm{e}^{-rs} h'(X_s)\, dG^c_s \\
&+ \sum_{0\leq s\leq t\wedge\tau} \mathrm{e}^{-rs} \big[h(X_s - \triangle L_s) - h(X_s) \big] \\
&+ \sum_{0\leq s\leq t\wedge\tau} \mathrm{e}^{-rs} \big[h(X_s + \triangle G_s) - h(X_s) \big],
\end{aligned}
$$

where $M_t = \sigma \int_0^t \mathrm{e}^{-rs} h'(X_s)\, dW_s$. Here, we denote by L^c, G^c the continuous parts of the processes L, G, respectively, and we have used the fact that $\triangle X_t = -\triangle L_t + \triangle G_t$. It follows that

$$
\begin{aligned}
\int_{[0,t\wedge\tau]} \mathrm{e}^{-rs} \Big(dL_s - \beta\, dG_s \Big) = &- \mathrm{e}^{-r(t\wedge\tau)} h\big(X_{(t\wedge\tau)+} \big) \\
&+ h(x) + \int_0^{t\wedge\tau} \mathrm{e}^{-rs} \left[\frac{1}{2}\sigma^2 h''(X_s) + \mu h'(X_s) - r h(X_s) \right] ds \\
&+ M_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{-rs} \big[1 - h'(X_s) \big]\, dL^c_s \\
&+ \int_0^{t\wedge\tau} e^{-rs} \big[h'(X_s) - \beta \big]\, dG^c_s
\end{aligned}
$$

$$
(5.5) \qquad \begin{aligned} &+ \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-rs} \int_0^{\triangle L_s} \big[1 - h'(X_s+z)\big]\, dz \\ &+ \sum_{0\le s\le t\wedge\tau} \mathrm{e}^{-rs} \int_0^{\triangle G_s} \big[h'(X_s-z) - \beta\big]\, dz. \end{aligned}
$$

Since h satisfies the HJB equation (5.1) and the boundary condition (5.2), this expression implies

$$
(5.6) \qquad \int_{[0,t\wedge\tau]} \mathrm{e}^{-rs}\Big(dL_s - \beta\, dG_s\Big) \le -\mathrm{e}^{-r(t\wedge\tau)} h\big(X_{(t\wedge\tau)+}\big) + h(x) + M_{t\wedge\tau}.
$$

With regard to the restriction on the dividend process L provided by (iii) in Definition 2.1, X enters the set $(-\infty, 0)$ in a "continuous" way. Therefore,

$$
(5.7) \qquad \begin{aligned} \liminf_{t\to\infty} \mathrm{e}^{-r(t\wedge\tau)} h\big(X_{(t\wedge\tau)+}\big) &= \mathrm{e}^{-r\tau} h(0)\mathbf{1}_{\{\tau<\infty\}} + \liminf_{t\to\infty} \mathrm{e}^{-rt} h\big(X_t\big)\mathbf{1}_{\{\tau=\infty\}} \\ &\ge \mathrm{e}^{-r\tau} h(0)\mathbf{1}_{\{\tau<\infty\}}, \end{aligned}
$$

the inequality following because h is bounded from below in $[0,\infty)$. The boundedness of h' in $\mathbb{R}$ implies that

$$
(5.8) \qquad \lim_{t\to\infty} \langle M\rangle_t = \lim_{t\to\infty} \int_0^t \mathrm{e}^{-rs} h'(X_s)\, ds < \infty, \ P\text{-a.s.}.
$$

It follows that M converges P-a.s. and in $\mathcal{L}^1$ to an integrable random variable M_∞, and M is a uniformly integrable martingale on $[0,\infty]$. In view of this observation, inequality (5.7) and the fact that $h(0) \ge 0$, we can take limits in (5.6) and then take expectations, using Doob's optional sampling theorem, to conclude that

$$
(5.9) \qquad J(x; L, G) \le h(x).
$$

Now, consider case (i) of the theorem, and let L^{o}, G^{o} be as in the statement. Since $h(x) \equiv x$ and $\tau^{\mathrm{o}} = 0$, (5.5) implies

$$
J(x; L^{\mathrm{o}}, G^{\mathrm{o}}) = \Delta L_0^{\mathrm{o}} = x,
$$

which, combined with (5.9), implies the optimality of the strategy $(L^{\mathrm{o}}, G^{\mathrm{o}})$.

To proceed further, consider case (ii) of the theorem, and let L^{o}, G^{o} be as in the statement. Apart from a possible jump at time 0, the dividend process L^{o} has continuous sample paths, its pathwise construction can be found, e.g., in Karatzas and Shreve [8, Lemma 3.6.14], and satisfies

$$
(5.10) \qquad L_t^{\mathrm{o}} = (x - x^*)^+ \mathbf{1}_{\{t>0\}} + \int_{(0,t]} \mathbf{1}_{\{X_s = x^{**}\}}\, dL_s^{\mathrm{o}}.
$$

Combining this expression with the fact that the associated reserves process X takes values outside (x^{**}, ∞), we can see that (5.6) holds with equality and

$$
(5.11) \qquad \lim_{t\to\infty} \mathrm{e}^{-r(t\wedge\tau)} h\big(X_{(t\wedge\tau)+}\big) = \mathrm{e}^{-r\tau} h(0)\mathbf{1}_{\{\tau<\infty\}}.
$$

However, these observations and the fact that $h(0) = w(0) = 0$ imply

$$\lim_{t\to\infty} \int_{[0,t\wedge\tau]} \mathrm{e}^{-rs} \Big(dL_s - \beta\, dG_s \Big) = h(x) + M_\tau. \tag{5.12}$$

With regard to case (iii), we can appeal to a simple inductive argument to construct pathwisely the processes L^{o} and G^{o} starting from the solution to Skorohod's equation provided, e.g., by Lemma 3.6.14 in Karatzas and Shreve [8] (note that such a construction can be found, under much more general assumptions on the dynamics of the reserves process X in El Karoui [7]). Apart from a possible jump of L^{o} at time 0, these processes have continuous sample paths and satisfy

$$G^{\mathrm{o}}_t = \int_{[0,t]} \mathbf{1}_{\{X_s=0\}}\, dG^{\mathrm{o}}_s \quad \text{and} \quad L^{\mathrm{o}}_t = (x-x^*)^+ \mathbf{1}_{\{t>0\}} + \int_{(0,t]} \mathbf{1}_{\{X_s=x^*\}}\, dL^{\mathrm{o}}_s. \tag{5.13}$$

In view of these expressions and the fact that the associated reserves process X takes values outside (x^*, ∞) and is reflected at 0, we can see that (5.6) holds with equality, $\tau = \infty$ and

$$\lim_{t\to\infty} \mathrm{e}^{-r(t\wedge\tau)} h\big(X_{(t\wedge\tau)+}\big) = \lim_{t\to\infty} \mathrm{e}^{-rt} h\big(X_t\big) = 0.$$

It follows that

$$\lim_{t\to\infty} \int_{[0,t\wedge\tau]} \mathrm{e}^{-rs} \Big(dL_s - \beta\, dG_s \Big) = h(x) + M_\infty. \tag{5.14}$$

Finally, we can take expectations in (5.12) and (5.14) to conclude that $J(x; L^{\mathrm{o}}, G^{\mathrm{o}}) = h(x)$, which combined with (5.9), establish the optimality of the corresponding strategy $(L^{\mathrm{o}}, G^{\mathrm{o}})$. □

Remark 5.1. With reference to the inequality $h(0) \geq 0$ implied by boundary condition (5.2) and the fact that h is increasing, we can see that (5.5) implies

$$\int_{[0,t\wedge\tau]} \mathrm{e}^{-rs} \Big(dL_s - \beta\, dG_s \Big) \leq h(x) + M_{t\wedge\tau}$$

instead of just (5.6). We adopted a slightly more complicated analysis because we wanted to develop a proof that can trivially be modified to establish Theorem 4.2. In the context of that result, $h(0)$ is not necessarily positive, but every admissible strategy is associated with $\tau = \infty$.

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