LONG-TERM OPTIMAL INVESTMENT STRATEGIES IN THE PRESENCE OF ADJUSTMENT COSTS

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ABSTRACT. We consider the problem of determining in a dynamical way the optimal capacity level of an investment project that operates within a random economic environment. In particular, we consider an investment project that yields payoff at a rate that depends on its installed capacity level and on a random economic indicator such as the price of the project's output commodity. We model this economic indicator by means of a general one-dimensional ergodic diffusion. At any time, the project's capacity level can be increased or decreased at given proportional costs. The aim is to maximise an ergodic performance criterion that reflects the long-term average payoff resulting from the project's management. We solve this genuinely two-dimensional stochastic control problem by constructing an explicit solution to an appropriate Hamilton-Jacobi-Bellman equation and by fully characterising an optimal investment strategy.

Dedicated to Professor Ioannis Karatzas on the occasion of his 60th birthday.

1. Introduction

We consider an investment project in a random environment that yields a payoff rate that depends on its installed capacity and on a stochastic economic indicator such as the price of or the demand for one unit of the project's output. We model this economic indicator by the one-dimensional diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{1}$$

where W is a standard one-dimensional Brownian motion. We assume that the functions b and σ satisfy general assumptions such that (1) has a unique solution X in $\mathbb{R}_+$ that is ergodic as well as recurrent. We denote by Y_t the project's installed capacity at time t, and we assume that this can be increased or decreased dynamically over time. Also, we assume that there is no capital depreciation, so

$$Y_t = y + Y_t^+ - Y_t^- \geq 0, \tag{2}$$

where $y \geq 0$ is the project's initial capital invested at time 0, while Y_t^+ (resp., Y_t^-) is the total additional capital that has been invested (resp., disinvested) by time t. The objective of the optimisation problem that we study is to maximise the long-term average payoff resulting from the management of the project, which is given by

$$J_{x,y}(Y) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T h(X_t, Y_t)\,dt - K^+Y_T^+ + K^-Y_T^-\right], \tag{3}$$

Research supported by EPSRC grant no. GR/S22998/01.

where h models the running payoff resulting from the project's operation, $K^+ > 0$ is the cost of increasing the capacity by one unit and $K^- \in [0, K^+)$ is the return of capital resulting from reducing the capacity by one unit.

Irreversible capacity expansion models have attracted considerable interest in the literature and can be traced beck to Manne [38]; see Van Mieghem [47] for a survey. Relevant models that have been studied in the mathematics literature include Davis, Dempster, Sethi and Vermes [15], Davis [14], Kobila [32], Øksendal [41], Wang [48], Chiarolla and Haussmann [13], Bank [6], Alvarez [2, 3], and references therein. The first reversible capacity expansion model was studied by Abel and Eberly [1] who considered the discounted version of (3) that arises when X is a geometric Brownian motion and $h(x, y) = x^\mu y^\nu$, for some constants $\mu, \nu > 0$. Later, Merhi and Zervos [40] solved the problem that arises when X is a geometric Brownian motion and (3) takes the form of a discounted criterion with a general running payoff function h. In the context of such models, Guo and Tomecek [22, 23] and Guo, Kaminsky, Tomecek and Yuen [20] established interesting connections between singular control and sequential switching. Another related model of partially reversible investment was studied by Guo and Pham [21].

All of the above references consider expected discounted performance indices. By their nature, such indices attach higher values to payoffs realised in the shorter term horizon, which may be associated with unfairness if one considers the payoffs received by successive generations. The use of exponential discounting can make this criticism appear as a non-issue because it typically results in stationary optimal strategies. However, this approach to modelling assumes that all generations agree on the same discounting rate. Moreover, recent economics theory has questioned the appropriateness of exponential discounting in the timeframe of a single generation (e.g., see Haldane [25]). Long-term average performance criteria could be considered as an alternative that bypasses such issues regarding discounting choices. In particular, they could be considered as better suited to decision making in the context of sustainable development because they assign the same values to payoffs enjoyed by present and future generations.

Recently, Løkka and Zervos [37] studied an ergodic irreversible capacity expansion model. In this paper, we solve the first reversible capacity expansion problem with an ergodic performance criterion that has appeared in the literature. In particular, we consider state dynamics that are modelled by a general one-dimensional diffusion rather than a geometric Brownian motion, which is more appropriate for several practical applications. Indeed, it has been well-documented in the economics and finance literature that a range of economic indicators are more realistically modelled by means of mean-reverting processes rather than a geometric Brownian motion (e.g., see Geman [18] and references therein). Also, it is worth noting that we could replace the assumption $K^- \in [0, K^+)$ with the more general $K^+ - K^- > 0$ in all of our analysis with the exception of just one point, which would require restricting the set of admissible investment strategies to ergodic ones (see Remark 3).

Models involving the ergodic control of one-dimensional diffusions have been studied by Bensoussan and Borkar [7, 8], who consider absolutely continuous control of the drift, Karatzas [30], who considers singular stochastic control, Jack and Zervos [28, 29], who consider impulse and absolutely continuous control, Bronstein and Zervos [12], who consider a sequential entry and exit decision model, and Irle and Sass [27], who consider the problem of maximising the asymptotic growth rate of a portfolio in the presence of fixed and proportional costs. For further general theory of stochastic optimal ergodic control, the reader is referred to Kushner [36], Gatarek and Stettner [17], Borkar and Gosh [11], Bensoussan and Frehse [9], Menaldi, Robin and Taksar [39], Ghosh, Arapostathis and Marcus [19], Duncan, Maslowski and Pasik-Duncan [16], Kurtz and Stockbridge [34, 35], Borkar [10], Kruk [33], Sadowy and Stettner [46], Arapostathis and Borkar [4], the recent monograph by Arapostathis, Borkar and Ghosh [5], and several references therein.

From a control theoretic perspective, the singular stochastic control problem that we solve presents an addition to a rather small list of explicitly solvable genuinely two-dimensional stochastic control problems. This list includes the problems arising in the context of the capacity expansion models discussed above as well as certain optimal stopping problems involving the running maximum of the state process (e.g., see Peskir [43], Pedersen [42], Hobson [26], Guo and Zervos [24], and references therein). To the best of our knowledge, we derive here the first explicit solution to a genuinely two-dimensional non-trivial problem involving the ergodic control of a diffusion process. Furthermore, we allow for general state process dynamics and we do not make any Lipschitz assumptions.

By their nature, ergodic performance criteria result in non-unique optimal strategies. Indeed, any two decision strategies that differ on an arbitrary long, but finite, time period are associated with the same value of the performance index. From an applications' perspective, this observation presents a real issue. However, this can be addressed in practice by means of appropriate levels of regulation and transparency, which are fundamental in the context of sustainable development. From a theoretical point of view, this observation renders Bellman's principle of optimality and the use of dynamic programming techniques inapplicable. As a result, different methodologies have been devised to solve ergodic control problems. The more recent one is based on reformulating the control problem at hand as an infinite dimensional linear program and then devising numerical schemes by means of appropriate finite-dimensional relaxations (see Kurtz and Stockbridge [35] for the analysis of ergodic singular stochastic control problems following this approach). The so-called "vanishing discounting" approach analyses ergodic control problems by considering them as limiting cases of appropriate infinite time horizon discounted problems as the discounting rate tends to 0 (see Menaldi, Robin and Taksar [39] for the analysis of ergodic singular stochastic control problems by means of this approach, as well as Arapostathis, Borkar and Ghosh [5, Chapter 3] for other general theory). The approach that we follow here is better suited to problems that admit explicit solutions and is closely related to part of the analysis in Karatzas [30] as well as in Menaldi, Robin and Taksar [39]: we construct an explicit solution to the Hamilton-Jacobi-Bellman (HJB) equation suggested by the "vanishing discounting" approach and we use this to identify a strategy that we prove to be optimal by means of a "verification theorem".

The paper is organised as follows. In Section 2, we formulate the problem that we study and we list all of the assumptions that we make. We construct an appropriate solution to an associated HJB equation in Section 3 and we derive the solution of the control problem in Section 4. Finally, we consider a number of special cases in Section 5. To focus on the main results of the paper, we include most proofs in Appendices I–III.

2. Problem formulation

We start with the following assumption on the functions $b, \sigma : \mathbb{R}_+^* \to \mathbb{R}$ that define the one-dimensional diffusion given by (1)[1]. The inequalities (6)–(7) may appear involved at first glance. However, they are quite general and easy to verify in practice (see also Remark 1 at the end of this section as well as the examples in Section 5).

Assumption 1. The functions $b, \sigma : \mathbb{R}_+^* \to \mathbb{R}$ are continuous and

$$\sigma^2(x) > 0 \quad \text{for all } x \in \mathbb{R}_+^*. \tag{4}$$

Also, there exist constants

$$\varepsilon_0 \in (0,1), \quad C_0 > 0 \quad \text{and} \quad 0 < \underline{\chi} < 1 < \overline{\chi} \tag{5}$$

such that[2]

$$\frac{2b(x)}{\sigma^2(x)} \geq \frac{1+\varepsilon_0}{x} - C_0 \quad \text{and} \quad \sigma^2(x) \geq \varepsilon_0 x^2 \quad \text{for all } x \leq \underline{\chi}, \tag{6}$$

$$-\frac{2b(x)}{\sigma^2(x)} \geq 2\varepsilon_0 \frac{\ln x}{x} \quad \text{and} \quad \sigma^2(x) \geq \frac{\varepsilon_0}{x^{C_0}} \quad \text{for all } x \geq \overline{\chi}. \tag{7}$$

□

The continuity of b, σ and (4) are sufficient conditions for (1) to have a solution that is unique in the sense of probability law (e.g., see Karatzas and Shreve [31, Section 5.5]). Given an initial condition $x > 0$, we assume such a weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ fixed throughout the paper. Also, the continuity of b, σ and (4) ensure that the scale function p and the speed measure m, given by

$$p(1) = 0 \quad \text{and} \quad p'(x) = \exp\left(-\int_1^x \frac{2b(s)}{\sigma^2(s)}\, ds\right), \quad \text{for } x \in (0,\infty), \tag{8}$$

and

$$m(dx) = \frac{2}{\sigma^2(x) p'(x)}\, dx, \tag{9}$$

respectively, are well-defined.

We can check that the estimates (119) and (120) in Appendix I imply that

$$\lim_{x \downarrow 0} p(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} p(x) = \infty,$$

which are sufficient for the solution of (1) to be non-explosive as well as recurrent (see Karatzas and Shreve [31, Proposition 5.5.22]). Also, (118) in Appendix I with $n = 0$ yields

[1]Throughout the paper, we use the notation $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{R}_+^* = (0,\infty)$.

[2]In practice, we can verify each of the inequalities in (6)–(7) independently of each other: if we then take the smallest (resp., largest) value of ε_0 (resp., C_0), then they all hold for the same ε_0 and C_0.

$m(\mathbb{R}_+^*) < \infty$, which implies that the process X is ergodic. This observation, (118) in Appendix I and Exercise X.3.18 in Revuz and Yor [45] imply that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T X_t^n\,dt\right] = \frac{1}{m(\mathbb{R}_+^*)}\int_0^\infty x^n\,m(dx) < \infty \tag{10}$$

for every constant $n \geq 0$.

We will use the following result, which we consider in slightly greater generality than we actually need and we prove in Appendix II.

Lemma 1. *Consider the one-dimensional diffusion given by (1) and suppose that the functions $b, \sigma : \mathbb{R}_+^* \to \mathbb{R}$ are continuous, $\sigma^2(x) > 0$ for all $x \in \mathbb{R}_+^*$ and*

$$-\frac{2b(x)}{\sigma^2(x)} \geq 2\varepsilon_0 \frac{\ln x}{x} \quad \textit{for all } x \geq \overline{\chi}, \tag{11}$$

for some constants $\varepsilon_0 \in (0,1)$ and $\overline{\chi} > 1$. Then

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\sup_{0\leq t\leq T} X_t^\eta\right] = 0 \tag{12}$$

for every constant $\eta \geq 1$.

We now introduce the family of all admissible investment strategies.

Definition 1. Given an initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, an investment strategy is any $(\mathcal{F}_t)$-adapted càglàd finite-variation process Y such that $Y_0 = y$ and $Y_T \geq 0$ for all $T \geq 0$, $\mathbb{P}_x$-a.s.. An investment strategy is admissible if

$$\mathbb{E}^x\left[Y_T^+ + Y_T^-\right] < \infty \quad \text{for all } T \geq 0, \tag{13}$$

where, if we denote by $\check{Y}$ is the total variation process of Y, then Y^+, Y^- are the unique $(\mathcal{F}_t)$-adapted càglàd increasing processes satisfying

$$Y = y + Y^+ - Y^- \quad \text{and} \quad \check{Y} = Y^+ + Y^-. \tag{14}$$

We denote by $\mathcal{Y}_{x,y}$ the set of all such admissible investment strategies. □

The aim of our optimisation problem is to determine the strategy that maximises the performance index defined by (3). To this end, we define the problem's value function V by

$$V(x, y) = \sup_{Y\in\mathcal{Y}_{x,y}} J_{x,y}(Y), \tag{15}$$

which turns out to be identically equal to a constant (see Theorem 6, our main result).

For the optimisation problem defined by (1)–(3) to be well-posed and admit a solution that conforms with economic intuition, we need to make additional assumptions. If we define

$$H(x, y) = \frac{\partial h}{\partial y}(x, y), \tag{16}$$

then, for $\Delta y > 0$ small, $H(x, y)\,\Delta y$ represents the additional running payoff rate that the project yields if the underlying economic indicator takes the value x and the project's capacity level is $y + \Delta y$ instead of y. If the economic indicator process X is interpreted as "price" or "demand", then it is reasonable to assume that this additional payoff rate is increasing as a function of x and becomes strictly positive, if not tend to ∞, as x tends to ∞. On the other hand, this additional payoff rate should be decreasing as a function of y and should become strictly negative as y tends to ∞. These considerations are reflected by (17), (18) and (20) below, provided we assume that, whatever the value x of the underlying economic indicator is, it is always profitable to marginally increase the project's capacity from 0 to a strictly positive value. In fact, we can dispense of this last requirement by introducing a point x_0^* and replacing the function $y^* : \mathbb{R}_+^* \to \mathbb{R}_+^*$ appearing in (19)–(20) by a function $\tilde{y}^* : (x_0^*, \infty) \to \mathbb{R}_+^*$ having the same properties otherwise; the analysis that we develop can easily be modified to account for such a relaxation, but this would involve extra notational complexity and would make the paper significantly longer. We also need to make the additional technical assumptions (21)–(22) (see also Remark 2 below).

Assumption 2. The inequalities $K^+ > K^- \geq 0$ hold true, the function h is $C^{2,2}$,

$$h(x, \cdot) \text{ is strictly concave for all } x > 0, \tag{17}$$

$$\lim_{x\to\infty} H(x, y) > 0 \quad \text{for all } y > 0, \tag{18}$$

and there exists a continuous strictly increasing function $y^* : \mathbb{R}_+^* \to \mathbb{R}_+^*$ of polynomial growth, i.e.,

$$0 < y^*(x) \leq \eta\big(1 + x^\eta\big) \quad \text{for all } x > 0, \tag{19}$$

for some constant $\eta \geq 1$, such that

$$H(x, y) \begin{cases} > 0, & \text{if } y < y^*(x), \\ = 0, & \text{if } y = y^*(x), \\ < 0, & \text{if } y > y^*(x). \end{cases} \tag{20}$$

Also, there exist constants $C_1, \mu, \zeta, \varepsilon_1 > 0$ and $\nu \in (0, 1)$ such that

$$-C_1(1 + y) \leq h(x, y) \leq C_1\left(1 + x^\mu y^\nu + x^\zeta\right) - \varepsilon_1 y \quad \text{for all } (x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+, \tag{21}$$

$$-C_1(1 + y) \leq H(x, y) \leq C_1\left(1 + y^{-C_1}\right)\left(1 + x^{C_1}\right) \quad \text{for all } (x, y) \in {\mathbb{R}_+^*}^2. \tag{22}$$

□

For future reference, we note that, if we define

$$y_0^* = \lim_{x\downarrow 0} y^*(x) \quad \text{and} \quad y_\infty^* = \lim_{x\uparrow\infty} y^*(x), \tag{23}$$

and we let $x^* : (y_0^*, y_\infty^*) \to \mathbb{R}_+^*$ be the inverse function of y^*, then x^* is strictly increasing,

$$\lim_{y\downarrow y_0^*} x^*(y) = 0 \quad \text{and} \quad \lim_{y\uparrow y_\infty^*} x^*(y) = \infty. \tag{24}$$

Combining the assumption that H is a continuous function satisfying the integrability condition (117) in Lemma 7 in Appendix I, which our assumptions thus far imply, with (17) and the monotone and the dominated convergence theorems, we can see that the function $y \mapsto \int_0^\infty H(u,y)\, m(du)$ is continuous and strictly decreasing in $(0,\infty)$. To simplify our analysis and keep the paper within a reasonable size, we make the following additional assumption, which rules out the possibility for never decreasing or never increasing the project's capacity to be optimal (such an assumption can easily be relaxed).

Assumption 3. The function H defined by (16) satisfies

$$\lim_{y\to\infty} \int_0^\infty H(x,y)\, m(dx) < 0 < \lim_{y\downarrow 0} \int_0^\infty H(x,y)\, m(dx). \tag{25}$$

□

For future reference, we note that this assumption and its preceding discussion imply that there exists a unique point $y^\dagger \in (y_0^*, y_\infty^*)$ such that

$$\int_0^\infty H(u,y)\, m(du) \begin{cases} > 0, & \text{for } y < y^\dagger, \\ = 0, & \text{for } y = y^\dagger, \\ < 0, & \text{for } y > y^\dagger. \end{cases} \tag{26}$$

The following result, which we prove in Appendix II, is concerned with the well-posedness of the control problem that we study as well as with certain estimates that we will need.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. Given any initial condition $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the following statements are true:*
(I) $-\infty < V(x,y) < \infty$*;*
(II) *every investment strategy Y satisfying $\mathbb{E}^x\left[\int_0^T Y_t\, dt\right] < \infty$ for all $T \geq 0$ is such that*

$$\mathbb{E}^x\left[\int_0^T |h(X_t, Y_t)|\, dt\right] < \infty; \tag{27}$$

(III) *every admissible investment strategy $Y \in \mathcal{Y}_{x,y}$ such that $J_{x,y}(Y) > -\infty$ satisfies*

$$\liminf_{T\to\infty} \frac{\mathbb{E}^x\left[Y_T\right]}{T} = 0. \tag{28}$$

Remark 1. The diffusion $Z = p(X)$ satisfies the SDE

$$dZ_t = \sigma\big(p^{-1}(Z_t)\big) p'\big(p^{-1}(Z_t)\big)\, dW_t, \tag{29}$$

where p^{-1} is the inverse of the scale function p. Since the performance index $J_{x,y}(Y)$ depends on X only through the running payoff function h, we can make the state space transformation associated with (29) to derive an equivalent model with the underlying diffusion being in natural scale. Starting from such an equivalent setting would have simplified the assumptions that we have made by making redundant the first inequalities in (6) and (7), which are needed to derive the estimates (119) and (120) in Appendix I. However, we would still need assumptions on the volatility function, which would require considerable analysis to verify for standard diffusions such as the ones we consider in Section 5, which have been extensively studied in the context of several applications, because this would involve their scale functions (see (29)). On the other hand, appropriate state space transformations can in principle be used to study examples with state spaces other than $\mathbb{R}_+^*$ by means of the analysis that we develop, though, one has to be careful because state space transformations other than the one we have discussed above do not necessarily preserve ergodicity. $\square$

Remark 2. Among the technical assumptions (21)–(22), the upper bound of h is the strongest one. The first term of this upper bound is quite general and consistent with (17). On the other hand, we need its second term to exclude the possibility for the value function V to be identically equal to ∞. Indeed, if we allow for $\varepsilon_1 = 0$ and we choose $h(x,y) = x^\mu y^\nu$, for some $\mu > 0$ and $\nu \in (0,1)$, then the strategy that increases the project's capacity by $\ell > 0$ at time 0 and then makes no further adjustments has payoff

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x \left[\int_0^T X_t^\mu (y+\ell)^\nu \, dt - K^+\ell\right] = \frac{(y+\ell)^\nu}{m\big(\mathbb{R}_+^*\big)} \int_0^\infty x^\mu \, m(dx),$$

the equality being true thanks to (10). It follows that $V(x,y) = \infty$ for every initial condition $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+$. $\square$

3. The Hamilton-Jacobi-Bellman (HJB) equation

We will solve the stochastic control problem formulated in the previous section by first constructing an appropriate solution to the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2 w_{xx} + b w_x + h,\ w_y - K^+,\ -w_y + K^-\right\} = 0 \tag{30}$$

(see Proposition 5 below, which is the main result of this section). This HJB equation is suggested by the so-called "vanishing discounting" approach, which we have discussed at the end of the penultimate paragraph of the introduction: it arises by setting the discounting rate equal to 0 in the HJB equation of the corresponding problem with expected discounting criterion.

Motivated by Merhi and Zervos [40], we look for a solution to this equation that is characterised by two strictly increasing continuous functions $F : (\underline{y}_F, \overline{y}_F) \to \mathbb{R}_+$ and $G : (\underline{y}_G, \overline{y}_G) \to \mathbb{R}_+$ that divide $\mathbb{R}_+^* \times \mathbb{R}_+$ into three connected subsets (see Figure 1). In the presence of the assumptions that we have made, it turns out that the points defining the domains of F and G satisfy

$$y_0^* = \underline{y}_G \leq \underline{y}_F \leq y^\dagger \leq \overline{y}_G \leq \overline{y}_F = y_\infty^*,$$

where y_0^*, y_∞^* and $y^\dagger$ are as in Assumptions 2 and 3, while,

$$\big\{G(y) :\ y \in (\underline{y}_G, \overline{y}_G)\big\} = \big\{F(y) :\ y \in (\underline{y}_F, \overline{y}_F)\big\} = \mathbb{R}_+^*.$$

In this context, the solution of (30) satisfies

$$-w_y(x,y) + K^- = 0, \quad \text{for } (x,y) \in \mathcal{D}, \tag{31}$$

$$\frac{1}{2}\sigma^2(x)w_{xx}(x,y)+b(x)w_x(x,y)+h(x,y)=0, \quad \text{for } (x,y)\in\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3, \tag{32}$$

$$w_y(x,y)-K^+=0, \quad \text{for } (x,y)\in\mathcal{I}, \tag{33}$$

where the disinvestment region $\mathcal{D}$, the continuation region $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2\cup\mathcal{C}_3$ and the investment region $\mathcal{I}$ are given by

$$\mathcal{D}=\big\{(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+ :\ y\geq F^{-1}(x)\big\} \tag{34}$$

$$\mathcal{C}_1=\begin{cases}\emptyset, & \text{if } \underline{y}_G=\underline{y}_F,\\ \big\{(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+ :\ y\in(\underline{y}_G,\underline{y}_F] \text{ and } x<G(y)\big\}, & \text{if } \underline{y}_G<\underline{y}_F,\end{cases} \tag{35}$$

$$\mathcal{C}_2=\begin{cases}\emptyset, & \text{if } \underline{y}_F=\overline{y}_G,\\ \big\{(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+ :\ y\in(\underline{y}_F,\overline{y}_G) \text{ and } x\in(F(y),G(y))\big\}, & \text{if } \underline{y}_F<\overline{y}_G,\end{cases} \tag{36}$$

$$\mathcal{C}_3=\begin{cases}\emptyset, & \text{if } \overline{y}_G=\overline{y}_F,\\ \big\{(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+ :\ y\in[\overline{y}_G,\overline{y}_F) \text{ and } x>F(y)\big\}, & \text{if } \overline{y}_G<\overline{y}_F,\end{cases} \tag{37}$$

$$\mathcal{I}=\big\{(x,y)\in\mathbb{R}_+^*\times\mathbb{R}_+ :\ y\leq G^{-1}(x)\big\}. \tag{38}$$

In view of the calculation

$$p''(x)=-\frac{2b(x)}{\sigma^2(x)}p'(x), \tag{39}$$

which follows from the definition (8) of the scale function p, we can verify that every solution to the ordinary differential equation

$$\frac{1}{2}\sigma^2(x)v_{xx}(x,y)+b(x)v_x(x,y)+h(x,y)=0 \tag{40}$$

that w should satisfy inside the waiting region $\mathcal{C}$ is given by

$$v(x,y)=B(y)+\int_1^x p'(s)\left[A(y)-\int_1^s h(u,y)\,m(du)\right]ds, \tag{41}$$

for some functions A and B. Combining this observation with (31) and (33), we can see that a solution w to the HJB equation (30) having the form that we have discussed above should be given by

$$w(x,y)=\begin{cases}v(x,G^{-1}(x))-K^+\big[G^{-1}(x)-y\big], & \text{if } (x,y)\in\mathcal{I},\\ v(x,y), & \text{if } (x,y)\in\mathcal{C},\\ v(x,F^{-1}(x))+K^-\big[y-F^{-1}(x)\big], & \text{if } (x,y)\in\mathcal{D}.\end{cases} \tag{42}$$

To determine the functions A, B and the free-boundary functions F, G, we require that w is $C^{2,1}$. In particular, we require that

$$\lim_{x\downarrow F(y)} v_y(x,y)\equiv\lim_{x\downarrow F(y)} w_y(x,y)=\lim_{x\uparrow F(y)} w_y(x,y)=K^-, \quad \text{if } y\in(\underline{y}_F,\overline{y}_F), \tag{43}$$

$$\lim_{x\uparrow G(y)} v_y(x,y)=\lim_{x\uparrow G(y)} w_y(x,y)=\lim_{x\downarrow G(y)} w_y(x,y)=K^+, \quad \text{if } y\in(\underline{y}_G,\overline{y}_G), \tag{44}$$

$$\lim_{x \downarrow F(y)} v_{xy}(x,y) \equiv \lim_{x \downarrow F(y)} w_{xy}(x,y) = \lim_{x \uparrow F(y)} w_{xy}(x,y) = 0, \quad \text{if } y \in (\underline{y}_F, \overline{y}_F), \tag{45}$$

$$\lim_{x \uparrow G(y)} v_{xy}(x,y) \equiv \lim_{x \uparrow G(y)} w_{xy}(x,y) = \lim_{x \downarrow G(y)} w_{xy}(x,y) = 0, \quad \text{if } y \in (\underline{y}_G, \overline{y}_G). \tag{46}$$

If the problem data is such that $\mathcal{C}_1 \neq \emptyset$, then it turns out that, given any $y \in (\underline{y}_G, \underline{y}_F]$,

$$A(y) = -\int_0^1 h(u,y)\, m(du)$$

is the appropriate choice for A (see also (116) in Appendix I), which results in the expression

$$v(x,y) = B(y) - \int_1^x p'(s) \int_0^s h(u,y)\, m(du)\, ds, \quad \text{for } (x,y) \in \mathcal{C}_1. \tag{47}$$

This expression and (46) imply that G should satisfy

$$\int_0^{G(y)} H(u,y)\, m(du) = 0 \quad \text{for all } y \in (\underline{y}_G, \underline{y}_F]. \tag{48}$$

Also, (44) and (47) imply that

$$B'(y) = K^+ + \int_1^{G(y)} p'(s) \int_0^s H(u,y)\, m(du)\, ds,$$

which, combined with (47), yields

$$v_y(x,y) = K^+ + \int_x^{G(y)} p'(s) \int_0^s H(u,y)\, m(du)\, ds, \quad \text{for } (x,y) \in \mathcal{C}_1. \tag{49}$$

If the problem data is such that $\mathcal{C}_3 \neq \emptyset$, then we can argue in the same way to see that F should satisfy

$$\int_{F(y)}^{\infty} H(u,y)\, m(du) = 0 \quad \text{for all } y \in [\overline{y}_G, \overline{y}_F), \tag{50}$$

and derive the expression

$$v_y(x,y) = K^- + \int_{F(y)}^x p'(s) \int_s^{\infty} H(u,y)\, m(du)\, ds, \quad \text{for } (x,y) \in \mathcal{C}_3. \tag{51}$$

The following result, which we prove in Appendix III, is concerned with the solvability of (48) and (50).

Lemma 3. *There exist strictly increasing C^1 functions $\beta : (y_0^*, y^\dagger) \to \mathbb{R}_+^*$ and $\alpha : (y^\dagger, y_\infty^*) \to \mathbb{R}_+^*$ such that*

$$\int_0^x H(u,y)\, m(du) \begin{cases} < 0, & \text{if } x < \beta(y), \\ = 0, & \text{if } x = \beta(y), \\ > 0, & \text{if } x > \beta(y), \end{cases} \quad \text{for all } y \in (y_0^*, y^\dagger). \tag{52}$$

and

$$\int_x^\infty H(u,y)\,m(du)\begin{cases}<0, & \text{if } x<\alpha(y),\\ =0, & \text{if } x=\alpha(y),\\ >0, & \text{if } x>\alpha(y),\end{cases} \quad \text{for all } y\in(y^\dagger, y^*_\infty), \tag{53}$$

Also, these functions satisfy

$$x^*(y)<\beta(y) \quad \text{for all } y\in(y^*_0, y^\dagger), \tag{54}$$

$$\alpha(y)<x^*(y) \quad \text{for all } y\in(y^\dagger, y^*_\infty), \tag{55}$$

$$\lim_{y\downarrow y^*_0}\beta(y)=\lim_{y\downarrow y^\dagger}\alpha(y)=0 \quad \text{and} \quad \lim_{y\uparrow y^\dagger}\beta(y)=\lim_{y\uparrow y^*_\infty}\alpha(y)=\infty, \tag{56}$$

where x^* is the strictly increasing inverse of the function y^* appearing in Assumption 2.

If the problem data is such that $\mathcal{C}_2\neq\emptyset$, then, given any $y\in(\underline{y}_F, \overline{y}_G)$, we can see that (45)–(46) are equivalent to

$$A'(y)=\int_1^{F(y)} H(u,y)\,m(du) \quad \text{and} \quad A'(y)=\int_1^{G(y)} H(u,y)\,m(du), \tag{57}$$

respectively, which imply that F and G should satisfy

$$\int_{F(y)}^{G(y)} H(u,y)\,m(du)=0 \quad \text{for all } y\in(\underline{y}_F, \overline{y}_G). \tag{58}$$

Using the first identity in (57) to substitute for A' in the expression for v_y resulting from (41), we can see that (43), (44) imply that

$$B'(y)+\int_1^{F(y)} p'(s)\int_s^{F(y)} H(u,y)\,m(du)\,ds=K^-, \tag{59}$$

$$B'(y)+\int_1^{G(y)} p'(s)\int_s^{F(y)} H(u,y)\,m(du)\,ds=K^+, \tag{60}$$

respectively. Combining these identities with (58) and (122) in Appendix I, we obtain

$$\begin{aligned}\int_{F(y)}^{G(y)} p(u)H(u,y)\,m(du) &= -\int_{F(y)}^{G(y)} p'(s)\int_{F(y)}^{s} H(u,y)\,m(du)\,ds\\ &= K^+-K^- \qquad \text{for all } y\in(\underline{y}_F, \overline{y}_G).\end{aligned} \tag{61}$$

Furthermore, we note that this equation, the expression (41) for v and the expressions (57) and (59)–(60) for the functions A' and B' imply that

$$\begin{aligned}v_y(x,y) &= K^+ + \int_x^{G(y)} p'(s)\int_{F(y)}^{s} H(u,y)\,m(du)\,ds\\ &= K^- - \int_{F(y)}^{x} p'(s)\int_{F(y)}^{s} H(u,y)\,m(du)\,ds, \quad \text{for } (x,y)\in\mathcal{C}_2.\end{aligned} \tag{62}$$

To determine the solvability of the system of equations (58) and (61), we need to consider the functions $Q_\beta : (y_0^*, y^\dagger) \to (-\infty, \infty]$ and $Q_\alpha : (y^\dagger, y_\infty^*) \to (-\infty, \infty]$ defined by

$$Q_\beta(y) = \int_0^{\beta(y)} p(u) H(u, y)\, m(du) - \bigl(K^+ - K^-\bigr)$$

$$= -\int_0^{\beta(y)} p'(s) \int_0^s H(u, y)\, m(du)\, ds - \bigl(K^+ - K^-\bigr), \quad \text{for } y \in (y_0^*, y^\dagger), \tag{63}$$

$$Q_\alpha(y) = \int_{\alpha(y)}^\infty p(u) H(u, y)\, m(du) - \bigl(K^+ - K^-\bigr)$$

$$= \int_{\alpha(y)}^\infty p'(s) \int_s^\infty H(u, y)\, m(du)\, ds - \bigl(K^+ - K^-\bigr), \quad \text{for } y \in (y^\dagger, y_\infty^*), \tag{64}$$

where the identities follow from Lemma 3 above and Lemma 8 in Appendix I. It is worth noting that the integrals appearing in this definition can indeed be equal to ∞ because $\lim_{x \downarrow 0} p(x) = -\infty$, $\lim_{x \to \infty} p(x) = \infty$ and H satisfies (20) in Assumption 2 (see also special cases considered in Section 5).

We prove the following result in Appendix III.

Lemma 4. *The function Q_β defined by (63) is increasing, while, the function Q_α defined by (64) is decreasing. In particular, if the strict inequality*

$$\lim_{y \uparrow y^\dagger} Q_\beta(y) = \lim_{y \downarrow y^\dagger} Q_\alpha(y) = \int_0^\infty p(u) H(u, y^\dagger)\, m(du) - (K^+ - K^-) > 0 \tag{65}$$

is true, then there exist unique $\underline{y}_F \in [y_0^*, y^\dagger)$ and $\overline{y}_G \in (y^\dagger, y_\infty^*]$ such that

$$Q_\beta(y) \begin{cases} < 0, & \text{if } \underline{y}_F > y_0^* \text{ and } y \in (y_0^*, \underline{y}_F), \\ = 0, & \text{if } \underline{y}_F > y_0^* \text{ and } y = \underline{y}_F, \\ > 0, & \text{if } y \in (\underline{y}_F, y^\dagger), \end{cases} \tag{66}$$

$$Q_\alpha(y) \begin{cases} > 0, & \text{if } y \in (y^\dagger, \overline{y}_G), \\ = 0, & \text{if } \overline{y}_G < y_\infty^* \text{ and } y = \overline{y}_G, \\ < 0, & \text{if } \overline{y}_G < y_\infty^* \text{ and } y \in (\overline{y}_G, y_\infty^*). \end{cases} \tag{67}$$

The system of equations (58) and (61) has a unique solution $F(y) < G(y)$ if and only if (65) is true and $y \in (\underline{y}_F, \overline{y}_G)$, in which case

$$0 < F(y) < x^*(y) < G(y) < \beta(y) \quad \text{for all } y \in (\underline{y}_F, y^\dagger), \tag{68}$$

$$\alpha(y) < F(y) < x^*(y) < G(y) < \infty \quad \text{for all } y \in (y^\dagger, \overline{y}_G), \tag{69}$$

where β, α are as in Lemma 3. Furthermore, if (65) is true, then the resulting functions $F, G : (\underline{y}_F, \overline{y}_G) \to \mathbb{R}_+^*$ are C^1 and strictly increasing,

$$\int_{F(y)}^x H(u, y)\, m(du) < 0 \quad \text{for all } x \in \bigl(F(y), G(y)\bigr), \tag{70}$$

$$\lim_{y \downarrow \underline{y}_F} F(y) = 0, \quad \lim_{y \uparrow \overline{y}_G} G(y) = \infty, \tag{71}$$

$$\lim_{y \uparrow \overline{y}_G} F(y) = \infty, \quad \textit{if } \overline{y}_G = y^*_\infty, \qquad \lim_{y \downarrow \underline{y}_F} G(y) = 0, \quad \textit{if } \underline{y}_F = y^*_0, \tag{72}$$

$$\lim_{y \downarrow \underline{y}_F} G(y) = \beta(\underline{y}_F) \quad \textit{and} \quad \lim_{y \downarrow \underline{y}_F} G'(y) = \beta'(\underline{y}_F), \quad \textit{if } \underline{y}_F > y^*_0, \tag{73}$$

$$\lim_{y \uparrow \overline{y}_G} F(y) = \alpha(\overline{y}_G) \quad \textit{and} \quad \lim_{y \uparrow \overline{y}_G} F'(y) = \alpha'(\overline{y}_G), \quad \textit{if } \overline{y}_G < y^*_\infty. \tag{74}$$

If (65) is true, then we define $\underline{y}_G = y^*_0$ and $\overline{y}_F = y^*_\infty$, we extend the functions $F : (\underline{y}_F, \overline{y}_G) \to \mathbb{R}^*_+$ and $G : (\underline{y}_F, \overline{y}_G) \to \mathbb{R}^*_+$ given by the previous lemma to $(\underline{y}_F, \overline{y}_F)$ and $(\underline{y}_G, \overline{y}_G)$ by defining

$$F(y) = \alpha(y) \quad \text{for all } y \in [\overline{y}_G, \overline{y}_F), \text{ if } \overline{y}_G < \overline{y}_F, \tag{75}$$

and

$$G(y) = \beta(y) \quad \text{for all } y \in (\underline{y}_G, \underline{y}_F], \text{ if } \underline{y}_G < \underline{y}_F, \tag{76}$$

respectively, and we note that these extensions are strictly increasing C^1 functions thanks to Lemmas 3 and 4. On the other hand, if (65) is false, then we define $\underline{y}_G = y^*_0$, $\overline{y}_G = \underline{y}_F = y^\dagger$ and $\overline{y}_F = y^*_\infty$, and we set

$$F(y) = \alpha(y) \text{ for all } y \in (\underline{y}_F, \overline{y}_F) \quad \text{and} \quad G(y) = \beta(y) \text{ for all } y \in (\underline{y}_G, \overline{y}_G), \tag{77}$$

The following result, which we prove in Appendix III, is concerned with the solution to the HJB equation (30).

Proposition 5. *Consider the control problem formulated in Section 2, let*

$$y^*_0 = \underline{y}_G \le \underline{y}_F \le y^\dagger \le \overline{y}_G \le \overline{y}_F = y^*_\infty \tag{78}$$

be as in Lemma 4 and the discussion above, and let $G : (\underline{y}_G, \overline{y}_G) \to \mathbb{R}$ *and* $F : (\underline{y}_F, \overline{y}_F) \to \mathbb{R}$ *be the strictly increasing* C^1 *functions that are given by Lemma 4 and (75)–(77). The functions* F *and* G *satisfy*

$$\lim_{y \downarrow \underline{y}_F} F(y) = \lim_{y \downarrow \underline{y}_G} G(y) = 0 \quad \textit{and} \quad \lim_{y \uparrow \overline{y}_F} F(y) = \lim_{y \uparrow \overline{y}_G} G(y) = \infty,$$

and they partition $\mathbb{R}^*_+ \times \mathbb{R}_+$ *into the regions* $\mathcal{D}$, $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, $\mathcal{I}$ *defined by (34)–(38). Also, the function* $v_y : \mathbb{R}^*_+ \times (y^*_0, y^*_\infty) \to \mathbb{R}$ *defined by (49) if* $\underline{y}_G < \underline{y}_F$ *and* $y \in (\underline{y}_G, \underline{y}_F]$, *by (51) if* $\overline{y}_G < \overline{y}_F$ *and* $y \in [\overline{y}_G, \overline{y}_F)$, *and by (62) if* $\underline{y}_F < \overline{y}_G$ *and* $y \in (\underline{y}_F, \overline{y}_G)$, *is* $C^{2,1}$. *Furthermore, if we define*

$$v(x, y) = \int_{y^\dagger}^{y} v_y(x, r)\, dr, \quad \textit{for } (x, y) \in \mathbb{R}^*_+ \times (y^*_0, y^*_\infty),$$

then the function $w : \mathbb{R}^*_+ \times \mathbb{R}_+ \to \mathbb{R}$ *given by (42) is a* $C^{2,1}$ *solution to the HJB equation (30).*

4. The solution to the control problem

The solution to the HJB equation (30) that we constructed in Proposition 5 suggests the investment strategy introduced by the following definition, which we will prove to be optimal.

Definition 2. Consider the functions F, G and the associated domains $\mathcal{D}$, $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$, $\mathcal{I}$ appearing in Proposition 5. Given an initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, we denote by Y° the investment strategy that instantaneously increases or decreases the project's capacity to the closest boundary point of $\mathcal{C}$ if $(x, y) \notin \mathcal{C}$, and then takes minimal action so as to reflect the process (X, Y°) in the boundary G of $\mathcal{C}$ along the positive y-direction and in the boundary F of $\mathcal{C}$ along the negative y-direction. In particular, the process Y° has a positive jump of size $G^{-1}(x) - y$, if $G^{-1}(x) > y$, and a negative jump of size $y - F^{-1}(x)$, if $y > F^{-1}(x)$, at time 0, and satisfies

$$dY_t^\circ = \left[\mathbf{1}_{\{Y_t^\circ = G^{-1}(X_t)\}} - \mathbf{1}_{\{Y_t^\circ = F^{-1}(X_t)\}}\right] dY_t^\circ \quad \text{for all } t > 0. \tag{79}$$

□

Motivated by the solution to Skorokhod's equation (e.g., see Karatzas and Shreve [31, Lemma 3.6.C.14], we can construct such a process Y° iteratively in a pathwise sense as follows. First, we fix a sample path $X(\omega)$ of any continuous positive stochastic process X such that $X(\omega)$ is "recurrent", namely, $\inf_{u \geq t} X_u(\omega) = 0$ and $\sup_{u \geq t} X_u(\omega) = \infty$ for all $t \geq 0$, and we drop the argument "ω" for notational simplicity. We then define the times

$$\tau_0^+ = \inf\bigl\{t \geq 0 : \ (X_t, y) \in \operatorname{int} \mathcal{I}\bigr\} \quad \text{and} \quad \tau_0^- = \inf\bigl\{t \geq 0 : \ (X_t, y) \in \operatorname{int} \mathcal{D}\bigr\},$$

and we assume that $\tau_0^+ < \tau_0^-$ in what follows; if $\tau_0^- < \tau_0^+$, then only straightforward revisions of the arguments are required. We define

$$Y_t^{(1)+} = \left[G^{-1}\left(\sup_{u \leq t} X_u\right) - y\right]^+ \mathbf{1}_{\{0<t\}}, \quad Y_t^{(1)-} = 0, \quad Y_t^{(1)} = y + Y_t^{(1)+} - Y_t^{(1)-},$$

$$\tau_1 = \inf\left\{t \geq 0 : \ \bigl(X_t, Y_t^{(1)}\bigr) \in \operatorname{int} \mathcal{D}\right\} > \tau_0^+,$$

and we note that $(X, Y^{(1)})$ is reflecting in G in the positive y-direction,

$$(X_t, Y_t^{(1)}) \in \operatorname{cl} \mathcal{C} \text{ for all } t \leq \tau_1 \quad \text{and} \quad Y_{\tau_1}^{(1)} = F^{-1}(X_{\tau_1}),$$

where $\operatorname{cl} \mathcal{C}$ is the closure of $\mathcal{C}$ in $\mathbb{R}_+^* \times \mathbb{R}_+$.

If $\underline{y}_F = y^\dagger = \overline{y}_G$, then the free-boundaries F, G identify with the functions α, β depicted by Figure 2, $\tau_1 = \infty$ and the construction is complete. On the other hand, if $\underline{y}_F < y^\dagger < \overline{y}_G$, then $\tau_1 < \infty$ and we continue the construction as follows (see also Figure 1 depicting this generic case).

We define

$$Y_t^{(2)+} = Y_{t \wedge \tau_1}^{(1)+}, \quad Y_t^{(2)-} = \left[Y_{\tau_1}^{(1)} - F^{-1}\left(\inf_{\tau_1 \leq u \leq t} X_u\right)\right] \mathbf{1}_{\{\tau_1 \leq t\}},$$

$$Y_t^{(2)} = y + Y_t^{(2)+} - Y_t^{(2)-} \quad \text{and} \quad \tau_2 = \inf\left\{t \geq 0 : \ \bigl(X_t, Y_t^{(2)}\bigr) \in \operatorname{int} \mathcal{I}\right\} > \tau_1.$$

An inspection of these definitions reveals that $(X, Y^{(2)})$ is reflecting in G in the positive y-direction and in F in the negative y-direction up to time τ_2,

$$Y_t^{(2)} = Y_t^{(1)} \text{ for all } t \le \tau_1,$$
$$(X_t, Y_t^{(2)}) \in \operatorname{cl}\mathcal{C} \text{ for all } t \le \tau_2 \quad \text{and} \quad Y_{\tau_2}^{(2)} = G^{-1}(X_{\tau_2}).$$

We then iterate these constructions by defining

$$Y_t^{(2n+1)+} = Y_{t\wedge\tau_{2n}}^{(2n)+} + \left[G^{-1}\left(\sup_{\tau_{2n}\le u\le t} X_u\right) - Y_{\tau_{2n}}^{(2n)+}\right] \mathbf{1}_{\{\tau_{2n}\le t\}}, \quad Y_t^{(2n+1)-} = Y_{t\wedge\tau_{2n}}^{(2n)-},$$
$$Y_t^{(2n+1)} = y + Y_t^{(2n+1)+} - Y_t^{(2n+1)-}, \quad \tau_{2n+1} = \inf\left\{t \ge 0: \ \left(X_t, Y_t^{(2n+1)}\right) \in \operatorname{int}\mathcal{D}\right\} > \tau_{2n},$$
$$Y_t^{(2n)+} = Y_{t\wedge\tau_{2n-1}}^{(2n-1)+}, \quad Y_t^{(2n)-} = Y_{t\wedge\tau_{2n-1}}^{(2n-1)-} + \left[Y_{\tau_{2n-1}}^{(2n-1)+} - F^{-1}\left(\inf_{\tau_{2n-1}\le u\le t} X_u\right)\right] \mathbf{1}_{\{\tau_{2n-1}\le t\}},$$
$$Y_t^{(2n)} = y + Y_t^{(2n)+} - Y_t^{(2n)-} \quad \text{and} \quad \tau_{2n} = \inf\left\{t \ge 0: \ \left(X_t, Y_t^{(2n)}\right) \in \operatorname{int}\mathcal{I}\right\} > \tau_{2n-1},$$

for $n \ge 1$, and we note that, given any $m, \ell \ge 1$,

$$Y_t^{(m+\ell)} = Y_t^{(m)} = y + Y_t^{(m)+} - Y_t^{(m)-} \quad \text{and} \quad (X_t, Y_t^{(m)}) \in \operatorname{cl}\mathcal{C} \quad \text{for all } t \le \tau_m.$$

The recurrence of the sample path X that we have considered implies that $\lim_{n\to\infty} \tau_n = \infty$. Therefore, we can define $(Y^\circ)^+$, $(Y^\circ)^-$ and Y° by

$$(Y^\circ)_t^+ = Y_t^{(m)+}, \quad (Y^\circ)_t^- = Y_t^{(m)-} \quad \text{and} \quad Y_t^\circ = Y_t^{(m)}$$

for any $m \ge 1$ such that $t < \tau_m$. The finite variation function Y° thus constructed satisfies (79) because this is true for all of the functions $Y^{(n)}$. Indeed, an inspection of the iterative algorithm that we have developed reveals that $Y^{(n)}$ increases (resp., decreases) on the set

$$\left\{t \ge 0: \ Y_t^{(n)} = G^{-1}(X_t) \text{ and } X_t = \sup_{0\le u\le t} X_u\right\}$$
$$\left(\text{resp., } \left\{t \ge 0: \ Y_t^{(n)} = F^{-1}(X_t) \text{ and } X_t = \inf_{0\le u\le t} X_u\right\}\right).$$

This construction defines operators $\mathbb{F}^+(\cdot; y)$, $\mathbb{F}^-(\cdot; y)$ and $\mathbb{F}(\cdot; y)$ mapping the set $C_+^{\mathrm{r}}(\mathbb{R}_+)$ of all continuous functions $g: \mathbb{R}_+ \to \mathbb{R}_+^*$ that are recurrent into the set of all càglàd finite variation functions that are continuous in $\mathbb{R}_+^*$. In particular, given an initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$ and the solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ of (1) that we have associated with it, a process Y° that is as in Definition 2 is given by

$$Y_t^\circ = \mathbb{F}_t(X; y) = y + \mathbb{F}_t^+(X; y) - \mathbb{F}_t^-(X; y) \quad \text{for all } t \ge 0, \tag{80}$$

where, e.g., $\mathbb{F}_t(g; y)$ is the evaluation of the function $\mathbb{F}(g; y)$ at t, for $g \in C_+^{\mathrm{r}}(\mathbb{R}_+)$.

We can now prove the main result of the paper.

Theorem 6. *Consider the stochastic control problem formulated in Section 2. Given any initial condition $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}_+$, the associated investment strategy $Y^\circ \in \mathcal{Y}_{x,y}$ given by*

Definition 2 is optimal, namely, $V(x,y) = J_{x,y}(Y^\circ)$. *Furthermore, the value function* V *is constant, i.e.,* V *does not depend on the initial condition* (x,y).

Proof. Consider any initial condition $(x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+$ fixed. In view of the fact that $x^*(y) < G(y)$ for all $y \in (\underline{y}_G, \overline{y}_G)$, we can see that

$$G^{-1}(x) \leq y^*(x) \leq \eta\big(1 + x^\eta\big), \tag{81}$$

where $\eta \geq 1$ is as in (19) in Assumption 2. Combining this estimate with the fact that Y° increases through reflection of the process (X, Y°) in the boundary function G, we can see that

$$\begin{aligned} Y_T^\circ = \mathbb{F}_T(X;y) &\leq \sup_{0 \leq t \leq T} \mathbb{F}_t(X;y) \\ &\leq y + \left[G^{-1}\left(\sup_{0 \leq t \leq T} X_t\right) - y\right]^+ \leq y + \eta + \eta \sup_{0 \leq t \leq T} X_t^\eta. \end{aligned} \tag{82}$$

These inequalities and Lemma 1 imply that

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}^x \left[\sup_{0 \leq t \leq T} Y_t^\circ\right] = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^x \left[\sup_{0 \leq t \leq T} \mathbb{F}_t(X;y)\right] = 0. \tag{83}$$

Given any admissible investment strategy $Y \in \mathcal{Y}_{x,y}$, an application of Itô's formula yields

$$\begin{aligned} w(X_T, Y_T) = w(x,y) &+ \int_0^T \left[\frac{1}{2}\sigma^2(X_t) w_{xx}(X_t, Y_t) + b(X_t) w_x(X_t, Y_t)\right] dt \\ &+ \int_0^T w_y(X_t, Y_t)\, d(Y^+)_t^{\mathrm{c}} - \int_0^T w_y(X_t, Y_t)\, d(Y^-)_t^{\mathrm{c}} \\ &+ \sum_{0 \leq t < T} \big[w\big(X_t, Y_t + \triangle(Y^+)_t\big) - w(X_t, Y_t)\big] \\ &+ \sum_{0 \leq t < T} \big[w\big(X_t, Y_t - \triangle(Y^-)_t\big) - w(X_t, Y_t)\big] + \int_0^T \sigma(X_t) w_x(X_t, Y_t)\, dW_t, \end{aligned}$$

where the process $(Y^+)^{\mathrm{c}}$ (resp., $(Y^-)^{\mathrm{c}}$) is the continuous part of Y^+ (resp., Y^-) and the process $\triangle(Y^+)$ (resp., $\triangle(Y^-)$) is the jump part of Y^+ (resp., Y^-). This identity implies that

$$\begin{aligned} &\int_0^T h(X_t, Y_t)\, dt - K^+ Y_T^+ + K^- Y_T^- \\ &\qquad = w(x,y) - w(X_T, Y_T) \\ &\qquad\quad + \int_0^T \left[\frac{1}{2}\sigma^2(X_t) w_{xx}(X_t, Y_t) + b(X_t) w_x(X_t, Y_t) + h(X_t, Y_t)\right] dt \\ &\qquad\quad + \int_0^T \big[w_y(X_t, Y_t) - K^+\big]\, d(Y_t^+)^{\mathrm{c}} - \int_0^T \big[w_y(X_t, Y_t) - K^-\big]\, d(Y_t^-)^{\mathrm{c}} \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0 \le t < T} \int_0^{\triangle(Y^+)_t} \big[w_y(X_t, Y_t + u) - K^+ \big] \, du \\
&- \sum_{0 \le t < T} \int_{-\triangle(Y^-)_t}^0 \big[w_y(X_t, Y_t + u) - K^- \big] \, du + \int_0^T \sigma(X_t) w_x(X_t, Y_t) \, dW_t \\
\le\ & w(x, y) - w(X_T, Y_T) + \int_0^T \sigma(X_t) w_x(X_t, Y_t) \, dW_t,
\end{aligned}
$$

the inequality following because w satisfies the HJB equation (30). In light of the construction of the strategy Y°, we can see that the same calculations yield

$$
\begin{aligned}
&\int_0^T h(X_t, Y_t^\circ) \, dt - K^+ (Y^\circ)_T^+ + K^- (Y^\circ)_T^- \\
&\qquad = w(x, y) - w(X_T, Y_T^\circ) + \int_0^T \sigma(X_t) w_x(X_t, Y_t^\circ) \, dW_t.
\end{aligned}
$$

From these inequalities, it follows that

$$
\begin{aligned}
&\int_0^T h(X_t, Y_t) \, dt - K^+ Y_T^+ + K^- Y_T^- \\
&\qquad \le \int_0^T h(X_t, Y_t^\circ) \, dt - K^+ (Y^\circ)_T^+ + K^- (Y^\circ)_T^- \\
&\qquad\quad + w(X_T, Y_T^\circ) - w(X_T, Y_T) + \int_0^T \sigma(X_t) \big[w_x(X_t, Y_t) - w_x(X_t, Y_t^\circ) \big] \, dW_t.
\end{aligned} \tag{84}
$$

Since w satisfies the HJB equation (30) and $K^+ > K^- \ge 0$, we can see that

$$
\big| w(X_T, Y_T^\circ) - w(X_T, Y_T) \big| = \left| \int_{Y_T}^{Y_T^\circ} w_y(X_T, y) \, dy \right| \le K^+ \big| Y_T^\circ - Y_T \big| \le K^+ \left(Y_T^\circ + Y_T \right). \tag{85}
$$

This estimate implies that

$$
\mathbb{E}^x \left[\sup_{t \in [0,T]} \big| w(X_t, Y_t^\circ) - w(X_t, Y_t) \big| \right] \le K^+ \mathbb{E}^x \left[\sup_{0 \le t \le T} Y_t^\circ + Y_T^+ + Y_T^- \right] < \infty, \tag{86}
$$

the second inequality following thanks to the admissibility condition (13) that the process Y satisfies and (83). Similarly, we can see that

$$
\begin{aligned}
w(X_T, Y_T^\circ) - w(X_T, Y_T) &= \mathbf{1}_{\{Y_T < Y_T^\circ\}} \int_{Y_T}^{Y_T^\circ} w_y(X_T, y) \, dy - \mathbf{1}_{\{Y_T \ge Y_T^\circ\}} \int_{Y_T^\circ}^{Y_T} w_y(X_T, y) \, dy \\
&\le K^+ \big(Y_T^\circ - Y_T \big) \mathbf{1}_{\{Y_T < Y_T^\circ\}} - K^- \big(Y_T - Y_T^\circ \big) \mathbf{1}_{\{Y_T \ge Y_T^\circ\}} \\
&\le K^+ Y_T^\circ.
\end{aligned}
$$

Combining these inequalities with (83), we can see that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\big[w(X_T, Y^\circ_T) - w(X_T, Y_T)\big] \le K^+ \lim_{T\to\infty} \frac{\mathbb{E}^x\big[Y^\circ_T\big]}{T} = 0. \tag{87}$$

In view of the inequality

$$\begin{aligned}&\int_0^T \sigma(X_t)\big[w_x(X_t, Y_t) - w_x(X_t, Y^\circ_t)\big]\, dW_t \\ &\ge \int_0^T h(X_t, Y_t)\, dt - \int_0^T h(X_t, Y^\circ_t)\, dt - K^+ Y^+_T + K^- Y^-_T - w(X_T, Y^\circ_T) + w(X_T, Y_T),\end{aligned}$$

which follows from (84) and the fact that $K^- < K^+$, we can see that (13), which the admissible process Y satisfies, part (II) of Lemma 2, (83) and (86) imply that the process

$$t \mapsto \int_0^t \sigma(X_r)\big[w_x(X_r, Y_r) - w_x(X_r, Y^\circ_r)\big]\, dW_r, \quad t \in [0, T],$$

is bounded from below by an integrable random variable. Therefore, if (τ_n) is a localising sequence for this stochastic integral, then Fatou's lemma implies that

$$\begin{aligned}&\mathbb{E}^x\left[\int_0^T \sigma(X_t)\big[w_x(X_t, Y_t) - w_x(X_t, Y^\circ_t)\big]\, dW_t\right] \\ &\le \liminf_{n\to\infty} \mathbb{E}^x\left[\int_0^{T\wedge\tau_n} \sigma(X_t)\big[w_x(X_t, Y_t) - w_x(X_t, Y^\circ_t)\big]\, dW_t\right] = 0.\end{aligned} \tag{88}$$

Also, the inequality

$$\begin{aligned}(K^+ - K^-)(Y^\circ)^+_T \le& \int_0^T h(X_t, Y^\circ_t)\, dt - \int_0^T h(X_t, Y_t)\, dt + K^+ Y^+_T - K^- Y^\circ_T \\ &+ w(X_T, Y^\circ_T) - w(X_T, Y_T) + \int_0^T \sigma(X_t)\big[w_x(X_t, Y_t) - w_x(X_t, Y^\circ_t)\big]\, dW_t,\end{aligned}$$

which follows from (84), (13), part (II) of Lemma 2, (86), (83) and (88) imply that $\mathbb{E}^x\big[(Y^\circ)^+_T\big] < \infty$. It follows that Y° is admissible because it satisfies (13). Furthermore, we can take expectations in (84) and use (87) to obtain

$$\begin{aligned}J_{x,y}(Y) =& \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T h(X_t, Y_t)\, dt - K^+ Y^+_T + K^- Y^-_T\right] \\ \le& \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T h(X_t, Y^\circ_t)\, dt - K^+ (Y^\circ)^+_T + K^- (Y^\circ)^-_T\right] \\ &+ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\big[w(X_T, Y^\circ_T) - w(X_T, Y_T)\big] \\ =& \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T h(X_t, Y^\circ_t)\, dt - K^+ (Y^\circ)^+_T + K^- (Y^\circ)^-_T\right] \\ =& J_{x,y}(Y^\circ),\end{aligned}$$

which establishes the optimality of Y°.

To prove that V is constant, we consider any initial conditions $(x, y), (\bar{x}, \bar{y}) \in \mathbb{R}_+^* \times \mathbb{R}_+$, we denote by $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$, $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \mathbb{P}_{\bar{x}}, \bar{W}, \bar{X})$ the weak solutions to (1) that we have associated with the initial conditions x, $\bar{x}$, and we let $Y^\circ = \mathbb{F}(X; y) \in \mathcal{Y}_{x,y}$, $\bar{Y}^\circ = \mathbb{F}(\bar{X}; \bar{y}) \in \mathcal{Y}_{\bar{x},\bar{y}}$ be the associated optimal investment strategies, where the operators $\mathbb{F}(\cdot; y)$, $\mathbb{F}(\cdot; \bar{y})$ are as in (80). The recurrence of the diffusion X implies that the $(\mathcal{F}_t)$-stopping time $\tau_{\bar{x}}$ defined by

$$\tau_{\bar{x}} = \inf\{t \geq 0 : \; X_t = \bar{x}\} > 0, \tag{89}$$

is finite $\mathbb{P}_x$-a.s. and $\mathbb{E}^x[\tau_{\bar{x}}] < \infty$. On the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$, we consider the investment strategy $Y^\ddagger \in \mathcal{Y}_{x,y}$ that involves no capacity adjustments up to time $\tau_{\bar{x}}$, adjusts the capacity level from y to $\bar{y}$ at time $\tau_{\bar{x}}$, and then replicates $\bar{Y}^\circ$. In particular, we recall (80) and we define

$$\begin{aligned} Y_t^\ddagger &= y \mathbf{1}_{\{t \leq \tau_{\bar{x}}\}} + \mathbb{F}_{t-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \mathbf{1}_{\{\tau_{\bar{x}} < t\}} \\ &= y \mathbf{1}_{\{t \leq \tau_{\bar{x}}\}} + \Big(\bar{y} + \mathbb{F}^+_{t-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^-_{t-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \Big) \mathbf{1}_{\{\tau_{\bar{x}} < t\}}, \end{aligned}$$

where $\theta_s : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is the shift operator defined by $(\theta_s g)_t = g(s+t)$ for $g \in C(\mathbb{R}_+)$ and $s, t \geq 0$. Next, we calculate

$$\begin{aligned} V(x, y) &\geq J_{x,y}(Y^\ddagger) \\ &= \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}^x \biggl[\int_0^{\tau_{\bar{x}} \wedge T} h(X_t, y)\, dt + \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \int_{\tau_{\bar{x}}}^T h\big(X_t, \mathbb{F}_{t-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y})\big)\, dt \\ &\qquad - K^+ \mathbb{F}^+_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \mathbf{1}_{\{\tau_{\bar{x}} < T\}} + K^- \mathbb{F}^-_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \biggr] \\ &= \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}^x \biggl[\int_0^{\tau_{\bar{x}} \wedge T} h(X_t, y)\, dt - \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \int_{T-\tau_{\bar{x}}}^T h\big((\theta_{\tau_{\bar{x}}} X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}} X; \bar{y})\big)\, dt \\ &\qquad + \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \biggl(\int_0^T h\big((\theta_{\tau_{\bar{x}}} X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}} X; \bar{y})\big)\, dt - K^+ \mathbb{F}^+_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) + K^- \mathbb{F}^-_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) \biggr) \\ &\qquad + K^+ \big[\mathbb{F}^+_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^+_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \\ &\qquad - K^- \big[\mathbb{F}^-_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^-_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \biggr]. \end{aligned} \tag{90}$$

In view of (21) in Assumption 2, we can see that

$$\int_0^{\tau_{\bar{x}} \wedge T} h(X_t, y)\, dt \geq -C_1 (1 + y)(\tau_{\bar{x}} \wedge T). \tag{91}$$

Also, we can use the inequality $K^+ > K^- \geq 0$, (80) and (82) to obtain

$$\begin{aligned} &K^+ \big[\mathbb{F}^+_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^+_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] \mathbf{1}_{\{\tau_{\bar{x}} < T\}} - K^- \big[\mathbb{F}^-_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^-_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \\ &\quad \geq K^+ \Big[\big[\mathbb{F}^+_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^+_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] - \big[\mathbb{F}^-_T(\theta_{\tau_{\bar{x}}} X; \bar{y}) - \mathbb{F}^-_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}} X; \bar{y}) \big] \Big] \mathbf{1}_{\{\tau_{\bar{x}} < T\}} \end{aligned}$$

$$
\begin{aligned}
&= K^{+}\big[\mathbb{F}_T(\theta_{\tau_{\bar{x}}}X;\bar{y}) - \mathbb{F}_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}}X;\bar{y})\big]\mathbf{1}_{\{\tau_{\bar{x}}<T\}} \\
&\geq -K^{+}\mathbb{F}_{T-\tau_{\bar{x}}}(\theta_{\tau_{\bar{x}}}X;\bar{y})\mathbf{1}_{\{\tau_{\bar{x}}<T\}} \\
&\geq -K^{+}\left(\bar{y} + \eta + \eta \sup_{0\leq t\leq T}(\theta_{\tau_{\bar{x}}}X)^{\eta}_{t}\right), \qquad (92)
\end{aligned}
$$

as well as

$$
\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T} \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\,dt \leq \tau_{\bar{x}}\left(\bar{y} + \eta + \eta \sup_{0\leq t\leq T}(\theta_{\tau_{\bar{x}}}X)^{\eta}_{t}\right). \tag{93}
$$

In light of the estimates (91)–(92), we can see that (90) implies that

$$
\begin{aligned}
V(x,y) \geq \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^{x}\Bigg[&-C_1(1+y)(\tau_{\bar{x}}\wedge T) \\
&+ \mathbf{1}_{\{\tau_{\bar{x}}<T\}}\left(\int_0^T h\big((\theta_{\tau_{\bar{x}}}X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt - K^{+}\mathbb{F}^{+}_{T}(\theta_{\tau_{\bar{x}}}X;\bar{y}) + K^{-}\mathbb{F}^{-}_{T}(\theta_{\tau_{\bar{x}}}X;\bar{y})\right) \\
&- \mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T} h\big((\theta_{\tau_{\bar{x}}}X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt - K^{+}\left(\bar{y} + \eta + \eta \sup_{0\leq t\leq T}(\theta_{\tau_{\bar{x}}}X)^{\eta}_{t}\right)\Bigg]. \qquad (94)
\end{aligned}
$$

To proceed further, we note that the definition (89) of $\tau_{\bar{x}}$ implies that

$$
\begin{aligned}
(\theta_{\tau_{\bar{x}}}X)_T \equiv X_{\tau_{\bar{x}}+T} &= \bar{x} + \int_{\tau_{\bar{x}}}^{\tau_{\bar{x}}+T} b(X_t)\,dt + \int_{\tau_{\bar{x}}}^{\tau_{\bar{x}}+T}\sigma(X_t)\,dW_t \\
&= \bar{x} + \int_0^T b\big((\theta_{\tau_{\bar{x}}}X)_t\big)\,dt + \int_0^T \sigma\big((\theta_{\tau_{\bar{x}}}X)_t\big)\,dW_{\tau_{\bar{x}}+t},
\end{aligned}
$$

where the second identity follows from the time change formulae in Revuz and Yor [45, Propositions V.1.4, V.1.5]. Recalling that the process $(W_{\tau_{\bar{x}}+t} - W_{\tau_{\bar{x}}},\ t\geq 0)$ is a standard Brownian motion that is independent of $\mathcal{F}_{\tau_{\bar{x}}}$ (e.g., see Revuz and Yor [45, Exercise IV.3.21]), we can see that this observation implies that the process $\theta_{\tau_{\bar{x}}}X$ is independent of $\mathcal{F}_{\tau_{\bar{x}}}$ and has the same distribution under $\mathbb{P}_x$ as the process $\bar{X}$ under $\mathbb{P}_{\bar{x}}$, thanks to the uniqueness in distribution of the solution to the SDE (1). It follows that

$$
\begin{aligned}
&\mathbb{E}^{x}\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\left(\int_0^T h\big((\theta_{\tau_{\bar{x}}}X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt - K^{+}\mathbb{F}^{+}_{T}(\theta_{\tau_{\bar{x}}}X;\bar{y}) + K^{-}\mathbb{F}^{-}_{T}(\theta_{\tau_{\bar{x}}}X;\bar{y})\right)\right] \\
&\qquad = \mathbb{P}_x\big(\tau_{\bar{x}}<T\big)\mathbb{E}^{\bar{x}}\left[\int_0^T h\big(\bar{X}_t, \mathbb{F}_t(\bar{X};\bar{y})\big)\,dt - K^{+}\mathbb{F}^{+}_{T}(\bar{X};\bar{y}) + K^{-}\mathbb{F}^{-}_{T}(\bar{X};\bar{y})\right] \\
&\qquad = \mathbb{P}_x\big(\tau_{\bar{x}}<T\big)\mathbb{E}^{\bar{x}}\left[\int_0^T h(\bar{X}_t, \bar{Y}^{\circ}_t)\,dt - K^{+}(\bar{Y}^{\circ})^{+}_{T} + K^{-}(\bar{Y}^{\circ})^{-}_{T}\right]. \qquad (95)
\end{aligned}
$$

Also, since $\mathbb{E}^{x}[\tau_{\bar{x}}]<\infty$, (93) and Lemma 1 imply that

$$
0 \leq \lim_{T\to\infty}\frac{1}{T}\mathbb{E}^{x}\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\,dt\right]
$$

$$
\begin{aligned}
&\le \lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\big[\tau_{\bar{x}}\big]\left(\bar{y}+\eta+\eta\mathbb{E}^x\left[\sup_{0\le t\le T}(\theta_{\tau_{\bar{x}}}X)_t^{\eta}\right]\right)\\
&= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\big[\tau_{\bar{x}}\big]\left(\bar{y}+\eta+\eta\mathbb{E}^{\bar{x}}\left[\sup_{0\le t\le T}\bar{X}_t^{\eta}\right]\right)\\
&= 0. \qquad (96)
\end{aligned}
$$

and

$$
\lim_{T\to\infty}\frac{1}{T}\mathbb{E}^x\left[-C_1(1+y)(\tau_{\bar{x}}\wedge T)-K^+\left(\bar{y}+\eta+\eta\sup_{0\le t\le T}(\theta_{\tau_{\bar{x}}}X)_t^{\eta}\right)\right]=0. \qquad (97)
$$

Using the upper bound in (21) of Assumption 1, (135), (96) and Hölder's inequality, we can calculate

$$
\begin{aligned}
&\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T} h\big((\theta_{\tau_{\bar{x}}}X)_t,\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt\right]\\
&\quad\le C_1\mathbb{E}^x\big[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\tau_{\bar{x}}\big]+C_1\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{\zeta}\,dt\right]\\
&\qquad+C_1\mathbb{E}^x\left[\int_0^T\big(\mathbf{1}_{\{0<T-\tau_{\bar{x}}\le t\}}(\theta_{\tau_{\bar{x}}}X)_t\big)^{\mu}\big(\mathbf{1}_{\{0<T-\tau_{\bar{x}}\le t\}}\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)^{\nu}\,dt\right]\\
&\qquad-\varepsilon_1\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt\right]\\
&\quad\le C_1\mathbb{E}^x[\tau_{\bar{x}}]+C_1\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{\zeta}\,dt\right]-\varepsilon_1\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\,dt\right]\\
&\qquad+C_1\left(\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{\mu/(1-\nu)}\,dt\right]\right)^{1-\nu}\left(\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}\mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\,dt\right]\right)^{\nu}\\
&\quad\le C_1\mathbb{E}^x[\tau_{\bar{x}}]+C_1\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{\zeta}\,dt\right]\\
&\qquad+\frac{\varepsilon_1(1-\nu)}{\nu}\left(\frac{\nu C_1}{\varepsilon_1}\right)^{1/(1-\nu)}\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{\mu/(1-\nu)}\,dt\right]. \qquad (98)
\end{aligned}
$$

Also, given any constants $\delta\in(0,1)$ and $n\ge 0$, we can use Hölder's inequality to obtain

$$
\begin{aligned}
\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^{T}(\theta_{\tau_{\bar{x}}}X)_t^{n}\,dt\right]&=\mathbb{E}^x\left[\int_0^T\mathbf{1}_{\{0<T-\tau_{\bar{x}}\le t\}}(\theta_{\tau_{\bar{x}}}X)_t^{n}\,dt\right]\\
&\le\left(\mathbb{E}^x\left[\int_0^T\mathbf{1}_{\{0<T-\tau_{\bar{x}}\le t\}}^{1/\delta}\,dt\right]\right)^{\delta}\left(\mathbb{E}^x\left[\int_0^T(\theta_{\tau_{\bar{x}}}X)_t^{n/(1-\delta)}\,dt\right]\right)^{1-\delta}
\end{aligned}
$$

$$\leq \big(\mathbb{E}^x[\tau_{\bar{x}}]\big)^\delta \left(\mathbb{E}^{\bar{x}}\left[\int_0^T \bar{X}_t^{n/(1-\delta)}\, dt\right]\right)^{1-\delta}.$$

Since $\mathbb{E}^x[\tau_{\bar{x}}] < \infty$, these inequalities, (98) and (10) imply that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^T h\big((\theta_{\tau_{\bar{x}}}X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\, dt\right] \leq 0. \tag{99}$$

On the other hand, we can see that the lower bound in (21) of Assumption 1 and (96) imply that

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^T h\big((\theta_{\tau_{\bar{x}}}X)_t, \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\big)\, dt\right] \\
&\qquad \geq -C_1 \lim_{T\to\infty}\frac{1}{T}\left(\mathbb{E}^x[\tau_{\bar{x}}] + \mathbb{E}^x\left[\mathbf{1}_{\{\tau_{\bar{x}}<T\}}\int_{T-\tau_{\bar{x}}}^T \mathbb{F}_t(\theta_{\tau_{\bar{x}}}X;\bar{y})\, dt\right]\right) = 0.
\end{aligned} \tag{100}$$

Combining (94) with (95), (97) and (99)–(100), we obtain

$$\begin{aligned}
V(x,y) &\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{P}_x\big(\tau_{\bar{x}} < T\big)\mathbb{E}^{\bar{x}}\left[\int_0^T h(\bar{X}_t, \bar{Y}_t^\circ)\, dt - K^+(\bar{Y}^\circ)_T^+ + K^-(\bar{Y}^\circ)_T^-\right] \\
&= V(\bar{x},\bar{y}),
\end{aligned}$$

which establishes the independence of V from the initial condition (x,y) because the initial conditions $(x,y), (\bar{x},\bar{y}) \in \mathbb{R}_+^* \times \mathbb{R}_+$ have been arbitrary. □

Remark 3. The validity of (87) is the only reason we have assumed that $K^+ > K^- \geq 0$ rather than the more general inequalities $K^+, K^+ - K^- > 0$. To relax the assumption $K^+ > K^- \geq 0$ to $K^+, K^+ - K^- > 0$ in a straightforward way, we have to restrict the set of admissible strategies to ergodic ones, namely, to strategies $Y \in \mathcal{Y}_{x,y}$ such that

$$\text{the limit } \lim_{T\to\infty}\frac{\mathbb{E}_x[Y_T]}{T} \text{ exists.} \tag{101}$$

Indeed, we can combine (83) and (85) with part (III) of Lemma 2 and (101) to see that, given any $Y \in \mathcal{Y}_{x,y}$ such that $J_{x,y}(Y) > -\infty$,

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^x\big[\big|w(X_T, Y_T^\circ) - w(X_T, Y_T)\big|\big] \leq K^+ \lim_{T\to\infty}\frac{\mathbb{E}^x\big[Y_T^\circ\big]}{T} + K^+ \lim_{T\to\infty}\frac{\mathbb{E}^x\big[Y_T\big]}{T} = 0,$$

which can substitute for (87) in the proof of Theorem 6, our main result. □

5. Special cases

It is straightforward to check that any of the diffusions that we now consider satisfies all of the conditions in Assumption 1. The solution to the SDE

$$dX_t = k(\vartheta - X_t)\, dt + \sigma\sqrt{X_t}\, dW_t, \tag{102}$$

where $k, \vartheta, \sigma > 0$ are constants such that $k\vartheta - \frac{1}{2}\sigma^2 > 0$, is the mean-reverting square-root process that identifies with the short interest rate process in the Cox-Ingersoll-Ross model. In this case, the derivative of the scale function p is given by

$$p'(x) = x^{-2k\vartheta/\sigma^2} e^{2k(x-1)/\sigma^2}.$$

We can check that the derivative of the scale function of the constant-elasticity-of-variance (CEV) process, which is the solution to the SDE

$$dX_t = k(\vartheta - X_t)\, dt + \sigma X_t^\ell \, dW_t,$$

for some constants $k, \vartheta, \sigma > 0$ and $\ell \in (\frac{1}{2}, 1)$, is given by

$$p'(x) = \exp\left(\frac{2k}{\sigma^2}\left\{\frac{x^{2(1-\ell)}}{2(1-\ell)} + \frac{\vartheta x^{-(2\ell-1)}}{2\ell-1} - \frac{\vartheta}{2\ell-1} - \frac{1}{2(1-\ell)}\right\}\right).$$

Also, the stochastic logistic equation

$$dX_t = k\big(\vartheta - X_t\big)X_t\, dt + \sigma X_t \, dW_t, \tag{103}$$

where $k, \vartheta, \sigma > 0$ are constants such that $k\vartheta - \frac{1}{2}\sigma^2 > 0$, defines a diffusion that has the same scale function as the square-root mean-reverting process given by (102).

The running payoff function given by

$$h(x, y) = C_1 x^\mu y^\nu - \varepsilon_1 y, \tag{104}$$

for some constants $C_1, \mu, \varepsilon_1 > 0$ and $\nu \in (0, 1)$, is a modification of the so-called Cobb-Douglas production function $(x, y) \mapsto x^\mu y^\nu$ that is appropriate for the problem that we study here. We can check that such a choice satisfies all of the requirements in Assumption 2. In particular, the functions y^* and x^* are given by

$$y^*(x) = \left(\frac{C_1\nu}{\varepsilon_1}\right)^{1/(1-\nu)} x^{\mu/(1-\nu)} \quad \text{and} \quad x^*(y) = \left(\frac{\varepsilon_1}{C_1\nu}\right)^{1/\mu} y^{(1-\nu)/\mu},$$

and therefore, $y_0^* = 0$ and $y_\infty^* = \infty$.

In the special cases that we consider below, we show that the inequalities in (78) that the points defining the domains of the free-boundaries F and G satisfy may be strict. Indeed, we show that the points $\underline{y}_F < \overline{y}_F$ and $\underline{y}_G < \overline{y}_G$ may satisfy (106), (110), (112)–(113) or (115), which reveal that our general assumptions give rise to a rich family of optimal investment strategies. In particular, we show that (65) in Lemma 4 may be false, in which case, the free-boundaries F, G identify with the functions α, β, respectively.

5.1. **The case of the mean-reverting square-root process given by (102) and the running payoff function given by (104).** In this case, the point $y^\dagger$ associated with (26) in Assumption 3 is given by

$$y^\dagger = \left[\frac{\nu C_1}{\varepsilon_1}\left(\frac{\sigma^2}{2k}\right)^\mu \frac{\Gamma\left(\frac{2k\vartheta}{\sigma^2} + \mu\right)}{\Gamma\left(\frac{2k\vartheta}{\sigma^2}\right)}\right]^{1/(1-\nu)},$$

where Γ is the Gamma function defined by $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$. We used matlab to compute $y^\dagger = 4.214$ if

$$\begin{aligned} &k = 1, \quad \vartheta = 5, \quad \sigma = 0.8, \quad C_1 = 1, \quad \mu = 1, \\ &\nu = 0.8, \quad \varepsilon_1 = 3 \quad \text{and} \quad K^+ - K^- = 1. \end{aligned} \tag{105}$$

Also, we plot the results of an unsophisticated simulation that calculated the free-boundaries F and G for this data in Figure 3.

For any appropriate values of the parameters in (102) and (104), the points defining the domains of the free-boundaries F and G satisfy

$$0 = y_0^* = \underline{y}_G < \underline{y}_F < y^\dagger < \overline{y}_G = \overline{y}_F = \infty. \tag{106}$$

To see this claim, we first consider any $\lambda \geq 0$ and we calculate

$$\begin{aligned} \int_0^x p'(s) \int_0^s u^\lambda\, m(du)\, ds &= \frac{2}{\sigma^2} \int_0^x s^{-\frac{2k\vartheta}{\sigma^2}} e^{\frac{2k}{\sigma^2} s} \int_0^s u^{\lambda + \frac{2k\vartheta}{\sigma^2} - 1} e^{-\frac{2k}{\sigma^2} u}\, du\, ds \\ &\leq \frac{2}{\sigma^2 (\lambda + 1)(\lambda + \frac{2k\vartheta}{\sigma^2})} x^{\lambda+1} e^{\frac{2k}{\sigma^2} x}. \end{aligned} \tag{107}$$

Using this inequality with $\lambda = 0$ and $\lambda = \mu$, respectively, we can see that the function Q_β defined by (63) is real-valued,

$$\begin{aligned} \lim_{y \downarrow 0} Q_\beta(y) &\leq \frac{\varepsilon_1}{k\vartheta} \lim_{y \downarrow 0} \beta(y) e^{\frac{2k}{\sigma^2} \beta(y)} - (K^+ - K^-) \\ &= -(K^+ - K^-), \end{aligned}$$

and

$$\begin{aligned} \lim_{y \downarrow 0} Q_\beta(y) &\geq -\frac{2\nu C_1}{\sigma^2 (\mu + 1)(\mu + \frac{2k\vartheta}{\sigma^2})} \lim_{y \downarrow 0} y^{\nu - 1} \beta^{\mu+1}(y) e^{\frac{2k}{\sigma^2} \beta(y)} - (K^+ - K^-) \\ &\geq -\frac{2\nu C_1}{\sigma^2 (\mu + 1)(\mu + \frac{2k\vartheta}{\sigma^2})} \lim_{y \downarrow 0} y^{\nu - 1} \big[x^*(y)\big]^{\mu+1} e^{\frac{2k}{\sigma^2} x^*(y)} - (K^+ - K^-) \\ &= -(K^+ - K^-). \end{aligned}$$

It follows that

$$Q_\beta(y) \in \mathbb{R} \text{ for all } y \in (0, y^\dagger) \quad \text{and} \quad \lim_{y \downarrow 0} Q_\beta(y) = -(K^+ - K^-). \tag{108}$$

On the other hand, an application of L'Hôpital's rule yields

$$\begin{aligned} \lim_{s \to \infty} \frac{\int_s^\infty u^{\lambda + \frac{2k\vartheta}{\sigma^2} - 1} e^{-\frac{2k}{\sigma^2} u} du}{s^{\lambda + \frac{2k\vartheta}{\sigma^2} - 1} e^{-\frac{2k}{\sigma^2} s}} &= \lim_{s \to \infty} \frac{-s^{\lambda + \frac{2k\vartheta}{\sigma^2} - 1} e^{-\frac{2k}{\sigma^2} s}}{(\lambda + \frac{2k\vartheta}{\sigma^2} - 1) s^{\lambda + \frac{2k\vartheta}{\sigma^2} - 2} e^{-\frac{2k}{\sigma^2} s} - \frac{2k}{\sigma^2} s^{\lambda + \frac{2k\vartheta}{\sigma^2} - 1} e^{-\frac{2k}{\sigma^2} s}} \\ &= \frac{\sigma^2}{2k}, \end{aligned} \tag{109}$$

which, for $\lambda = 0$, implies that

$$\int_x^\infty p'(s) \int_s^\infty m(du)\, ds = \frac{2}{\sigma^2} \int_x^\infty s^{-1} \frac{\int_s^\infty u^{\frac{2k\vartheta}{\sigma^2}-1} e^{-\frac{2k}{\sigma^2}u}\, du}{s^{\frac{2k\vartheta}{\sigma^2}-1} e^{-\frac{2k}{\sigma^2}s}}\, ds = \infty.$$

Therefore, the function Q_α defined by (64) is identically equal to ∞ thanks to the fact that $\lim_{x\to\infty} H(x,y) = \infty$ for all $y \in \mathbb{R}_+^* \supseteq (y^\dagger, \infty)$. The inequalities in (106) now follow from this observation, (108), Lemma 4 and the discussion after that result.

5.2. **Further special cases.** By means of calculations similar to the ones in (107)–(109), we can show that the case arising when X solves the stochastic logistic equation (103) and h is given by (104), give rise to free-boundaries F, G such that

$$0 = y_0^* = \underline{y}_G = \underline{y}_F < y^\dagger < \overline{y}_G < \overline{y}_F = \infty. \tag{110}$$

Instead of pursuing further this case, we consider the diffusion with data

$$b(x) = \begin{cases} k(\vartheta - x), & \text{if } x \le 1, \\ k(\vartheta - x)x, & \text{if } x \ge 1, \end{cases} \quad \text{and} \quad \sigma(x) = \begin{cases} \sigma\sqrt{x}, & \text{if } x \le 1, \\ \sigma x, & \text{if } x \ge 1. \end{cases} \tag{111}$$

for some constants $k, \vartheta, \sigma > 0$ such that $k\vartheta - \frac{1}{2}\sigma^2 > 0$, which is a hybrid of the square-root mean-reverting process given by (102) and the stochastic logistic equation given by (103). To reduce the number of possible forms that the optimal strategy can take, we consider the running payoff function given by (104) for $\mu \in (0,1)$.

An interesting feature of this special case is that the optimal strategy takes qualitatively different forms, depending on parameter values. To see this claim, we consider (107), (123) and the fact that $p(1) = 0$ (see (8)), and we calculate

$$0 \le -\int_0^1 p(u)|H(u,y)|\, m(du) = \int_0^1 p'(s) \int_0^s |H(u,y)|\, m(du)\, ds < \infty.$$

Also, we use (109) with $\lambda = \mu - 1 \in (-1, 0)$ and $\lambda = -1$ together with (122) to obtain

$$\begin{aligned} 0 &\le \int_1^\infty p(u)|H(u,y)|\, m(du) = \int_1^\infty p'(s) \int_s^\infty |H(u,y)|\, m(du)\, ds \\ &\le \frac{2\nu C_1 y^{\nu-1}}{\sigma^2} \int_1^\infty s^{\mu-2} \frac{\int_s^\infty u^{\mu+\frac{2k\vartheta}{\sigma^2}-2} e^{-\frac{2k}{\sigma^2}u}\, du}{s^{\mu+\frac{2k\vartheta}{\sigma^2}-2} e^{-\frac{2k}{\sigma^2}s}}\, ds \\ &\quad + \frac{2\varepsilon_1}{\sigma^2} \int_1^\infty s^{-2} \frac{\int_s^\infty u^{\frac{2k\vartheta}{\sigma^2}-2} e^{-\frac{2k}{\sigma^2}u}\, du}{s^{\frac{2k\vartheta}{\sigma^2}-2} e^{-\frac{2k}{\sigma^2}s}}\, ds < \infty. \end{aligned}$$

These calculations imply that

$$\int_0^\infty |p(u)H(u,y)|\, m(du) < \infty \quad \text{for all } y > 0.$$

It follows that (65) may be false, in which case,

$$0 = \underline{y}_G < \overline{y}_G = y^\dagger = \underline{y}_F < \overline{y}_F = \infty \tag{112}$$

and the free-boundaries F, G identify with the functions α, β, respectively, or it may be true, in which case,

$$0 = \underline{y}_G < \underline{y}_F < y^{\dagger} < \overline{y}_G < \overline{y}_F = \infty. \tag{113}$$

In the latter case, we can show that the inequalities $\underline{y}_G < \underline{y}_F$ and $\overline{y}_G < \overline{y}_F$ are indeed true using arguments similar to the ones that we considered in the previous subsection.

Finally, we note that the diffusion with data

$$b(x) = \begin{cases} k(\vartheta - x)x, & \text{if } x \leq 1, \\ k(\vartheta - x), & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad \sigma(x) = \begin{cases} \sigma x, & \text{if } x \leq 1, \\ \sigma\sqrt{x}, & \text{if } x \geq 1. \end{cases} \tag{114}$$

for some constants $k, \vartheta, \sigma > 0$ such that $k\vartheta - \frac{1}{2}\sigma^2 > 0$, and the running payoff function given by (104) is associated with free-boundaries F, G such that

$$0 = y_0^* = \underline{y}_G = \underline{y}_F < y^{\dagger} < \overline{y}_G = \overline{y}_F = \infty. \tag{115}$$

Acknowledgement

We are grateful to two anonymous referees for several valuable comments and suggestions that improved our original manuscript.

Appendix I: analytic estimates and identities

Lemma 7. *In the presence of Assumptions 1 and 2,*

$$\int_0^{\infty} |h(s,y)| \, m(ds) < \infty \quad \textit{for all } y \geq 0, \tag{116}$$

$$\int_0^{\infty} |H(s,y)| \, m(ds) < \infty \quad \textit{for all } y > 0, \tag{117}$$

and

$$\int_0^{\infty} s^n \, m(ds) < \infty, \tag{118}$$

for all constants $n \geq 0$.

Proof. We first use the inequalities in (6)–(7) to calculate

$$\begin{aligned} \frac{p'(x_1)}{p'(x_2)} &\geq \exp\left(\int_{x_1}^{x_2} \left[\frac{1+\varepsilon_0}{s} - C_0 \right] ds \right) \\ &= x_1^{-(1+\varepsilon_0)} x_2^{1+\varepsilon_0} e^{-C_0(x_2 - x_1)} \quad \text{for all } 0 < x_1 < x_2 < \underline{\chi}, \end{aligned} \tag{119}$$

and

$$\begin{aligned} \frac{p'(x_2)}{p'(x_1)} &\geq \exp\left(2\varepsilon_0 \int_{x_1}^{x_2} \frac{\ln s}{s} \, ds \right) \\ &= x_1^{-\varepsilon_0 \ln x_1} x_2^{\varepsilon_0 \ln x_2} \quad \text{for all } \overline{\chi} < x_1 < x_2. \end{aligned} \tag{120}$$

We now fix any $y > 0$; (116) for $y = 0$ involves no additional arguments. The inequalities (21) and (22) in Assumption 1 imply that there exist constants $K_1 = K_1(n, y) > 0$ and $\ell = \ell(n) > 0$ such that

$$x^n + |h(x, y)| + |H(x, y)| \le K_1 \quad \text{for all } x \le \underline{\chi},$$
$$x^n + |h(x, y)| + |H(x, y)| \le K_1 x^\ell \quad \text{for all } x \ge \overline{\chi},$$

where $0 < \underline{\chi} < 1 < \overline{\chi}$ are as in Assumption 2. The first of these estimates, the bound of σ^2 in (6) and (119) imply that

$$\begin{aligned}\int_0^{\underline{\chi}} \big[s^n + |h(s, y)| + |H(s, y)|\big]\, m(ds) &\le \frac{2K_1}{\varepsilon_0} \frac{1}{p'(\underline{\chi})} \int_0^{\underline{\chi}} s^{-2} \frac{p'(\underline{\chi})}{p'(s)}\, ds \\ &\le \frac{2K_1 e^{C_0 \underline{\chi}}}{\varepsilon_0} \frac{\underline{\chi}^{-(1+\varepsilon_0)}}{p'(\underline{\chi})} \int_0^{\underline{\chi}} s^{-2} e^{-C_0 s} s^{1+\varepsilon_0}\, ds \\ &< \infty,\end{aligned}$$

while, the second one, the bound of σ^2 in (7) and (120) imply that

$$\begin{aligned}&\int_{\overline{\chi}}^\infty \big[s^n + |h(s, y)| + |H(s, y)|\big]\, m(ds) \\ &\quad\le \frac{2K_1}{\varepsilon_0} \frac{1}{p'(\overline{\chi})} \int_{\overline{\chi}}^\infty s^{C_0+\ell} \frac{p'(\overline{\chi})}{p'(s)}\, ds \\ &\quad\le \frac{2K_1}{\varepsilon_0} \frac{\overline{\chi}^{\varepsilon_0 \ln \overline{\chi}}}{p'(\overline{\chi})} \int_{\overline{\chi}}^\infty s^{-\varepsilon_0 \ln s + C_0 + \ell}\, ds \\ &\quad< \infty.\end{aligned}$$

Combining these results with the inequality

$$\int_{\underline{\chi}}^{\overline{\chi}} \big[s^n + |h(s, y)| + |H(s, y)|\big]\, m(ds) < \infty,$$

which follows from the fact that m is a locally finite measure and the continuity of the functions $|h|$ and $|H|$, we obtain (116), (117) as well as (118). □

Lemma 8. *In the presence of Assumption 1, if $g : \mathbb{R}_+^* \to \mathbb{R}$ is any function such that*

$$\int_0^\infty |g(s)|\, ds < \infty \quad \text{and} \quad g(x) \begin{cases} \le 0, & \text{if } x < x^\ddagger, \\ \ge 0, & \text{if } x > x^\ddagger, \end{cases} \tag{121}$$

for some $x^\ddagger > 0$, then

$$\int_x^z p'(s) \int_x^s g(u)\, m(du)\, ds = p(z) \int_x^z g(u)\, m(du) - \int_x^z p(u) g(u)\, m(du) \in [-\infty, \infty) \tag{122}$$

for all $0 \le x < z < \infty$ and $y > 0$, and

$$\int_x^z p'(s) \int_s^z g(u)\, m(du)\, ds$$

$$= -p(x)\int_x^z g(u)\,m(du) + \int_x^z p(u)g(u)\,m(du) \in (-\infty,\infty] \tag{123}$$

for all $0 < x < z \le \infty$ *and* $y > 0$*, in which identities all integrals are well-defined.*

Proof. Using Fubini's theorem, we can see that

$$\int_x^z \int_x^z p'(s)g(u)\mathbf{1}_{\{u\le s\}}\,m(du)\,ds = \int_x^z \left(\int_x^z p'(s)\mathbf{1}_{\{u\le s\}}\,ds\right) g(u)\,m(du),$$

which implies the identity in (122) for every $0 < x < z < \infty$. The monotone convergence theorem and the last set of inequalities in (121) imply that the last integral in (122) is well-defined and takes values in $[-\infty,\infty)$ if $x = 0$. It follows that the identity is true if $x = 0$ because the second integral remains finite as x tends to 0, thanks to the integrability condition in (121).

Similarly, we use Fubini's theorem to establish the identity in (123) when $0 < x < z < \infty$. The monotone convergence theorem and (121) then imply that the third integral in this identity is well-defined and takes values in $(-\infty,\infty]$ for $z = \infty$. As a consequence, the identity is true for $z = \infty$ because the second integral remains finite as z tends to ∞, thanks to (121). □

Appendix II: proofs of results in Section 2

We will need the following result for the proof of Lemma 1.

Lemma 9. *Consider the function* $\Phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ *given by*

$$\Phi(x) = \int_1^x \tilde{m}\big([1,s]\big)\tilde{p}'(s)\,ds, \tag{124}$$

where $\tilde{p}$, $\tilde{m}$ *are defined as in (8), (9) for some continuous functions* $\tilde{b}, \tilde{\sigma} : \mathbb{R}_+^* \to \mathbb{R}$ *such that* $\tilde{\sigma}^2(z) > 0$ *for all* $z > 0$ *and*

$$-\frac{2\tilde{b}(z)}{\tilde{\sigma}^2(z)} \ge 2\tilde{\varepsilon}\frac{\ln z}{z} \quad \text{for all } z \ge \tilde{\chi}, \tag{125}$$

for some constants $\tilde{\varepsilon} \in (0,1)$ *and* $\tilde{\chi} > 1$*. Then,*

$$\sup_{z>1}\left\{\frac{\Phi(z)}{z}\int_z^\infty \frac{ds}{\Phi(s)}\right\} < \infty, \tag{126}$$

and

$$\lim_{u\to\infty}\frac{\Phi^{-1}(u)}{u} = 0, \tag{127}$$

where Φ^{-1} *is the inverse of* Φ*, so that* $\Phi\circ\Phi^{-1}(u) = u$ *and* $\Phi^{-1}\circ\Phi(z) = z$*.*

Proof. The estimate (120) in Appendix I follows from the lower bound in the first set of inequalities in (7) in Assumption 1, which is the same as (125). Therefore, we can use (120) with the appropriate changes in notation and L'Hôpital's rule to calculate

$$\lim_{z\to\infty}\frac{\tilde{p}(z)}{z\tilde{p}'(z)}=\lim_{z\to\infty}\frac{\tilde{p}(\tilde{\chi})}{z\tilde{p}'(z)}+\lim_{z\to\infty}\frac{1}{z}\int_{\tilde{\chi}}^{z}\frac{\tilde{p}'(s)}{\tilde{p}'(z)}\,ds\le\lim_{z\to\infty}\frac{\int_{\tilde{\chi}}^{z}s^{\varepsilon_0\ln s}\,ds}{z^{1+\varepsilon_0\ln z}}=0.$$

In light of this result, we can use the definition of Φ to obtain

$$\lim_{z\to\infty}\frac{z\Phi'(z)}{\Phi(z)}\ge\lim_{z\to\infty}\frac{z\tilde{p}'(z)}{\int_1^z\tilde{p}'(s)\,ds}=\lim_{z\to\infty}\frac{z\tilde{p}'(z)}{\tilde{p}(z)}=\infty. \tag{128}$$

Therefore, given any $n>0$, there exists a constant $\hat{\chi}=\hat{\chi}(n)>0$ such that $\Phi'(z)/\Phi(z)\ge(n+1)/z$ for all $z\ge\hat{\chi}$, which implies that

$$\ln\Phi(z)-\ln\Phi(\hat{\chi})\ge\ln z^{n+1}-\ln\hat{\chi}^{n+1}.$$

It follows that

$$\lim_{z\to\infty}\frac{\Phi(z)}{z^n}\ge\lim_{z\to\infty}\frac{\Phi(\hat{\chi})}{\hat{\chi}^{n+1}}z=\infty. \tag{129}$$

For $n=2$, this limit implies that

$$0\le\int_z^\infty\frac{ds}{\Phi(s)}\le\left(\min_{s\ge z}\frac{\Phi(s)}{s^2}\right)^{-1}\int_z^\infty s^{-2}\,ds<\infty\quad\text{for all } z>1. \tag{130}$$

Using the definition (9) of the speed measure $\tilde{m}$, we can see that

$$\Phi(x)\le\sup_{s,u\in[1,2]}\frac{\tilde{p}'(s)}{\tilde{\sigma}^2(u)\tilde{p}'(u)}(x-1)^2\quad\text{for all } x\in[1,2],$$

which implies that

$$\lim_{z\downarrow1}\int_z^\infty\frac{ds}{\Phi(s)}\ge\left(\sup_{s,u\in[1,2]}\frac{\tilde{p}'(s)}{\tilde{\sigma}^2(u)\tilde{p}'(u)}\right)^{-1}\int_1^2\frac{ds}{(s-1)^2}=\infty.$$

Therefore, we use L'Hôpital's rule and the identity

$$\Phi''(z)=\tilde{p}''(z)\tilde{m}\big([1,z]\big)+\frac{2}{\tilde{\sigma}^2(z)}$$

to calculate

$$\lim_{z\downarrow1}\left(\frac{\Phi(z)}{z}\int_z^\infty\frac{ds}{\Phi(s)}\right)=\lim_{z\downarrow1}\frac{\int_z^\infty\frac{ds}{\Phi(s)}}{\frac{z}{\Phi(z)}}=\lim_{z\downarrow1}\frac{\Phi(z)}{z\Phi'(z)-\Phi(z)}=\lim_{z\downarrow1}\frac{\Phi'(z)}{z\Phi''(z)}=0, \tag{131}$$

Furthermore, we can use (128) and L'Hôpital's rule to obtain

$$\lim_{z\to\infty}\left(\frac{\Phi(z)}{z}\int_z^\infty\frac{ds}{\Phi(s)}\right)=\lim_{z\to\infty}\frac{\int_z^\infty\frac{ds}{\Phi(s)}}{\frac{z}{\Phi(z)}}=\lim_{z\to\infty}\frac{1}{\frac{z\Phi'(z)}{\Phi(z)}-1}=0.$$

Combining this limit with (130) and (131), we derive (126).

Finally, (129) with $n = 1$ implies that

$$\lim_{u\to\infty} \frac{u}{\Phi^{-1}(u)} = \lim_{u\to\infty} \frac{\Phi(\Phi^{-1}(u))}{\Phi^{-1}(u)} = \infty, \tag{132}$$

and (127) has been established. □

Proof of Lemma 1. Given any constant $\eta \geq 1$ fixed, we consider the process $Z = X^\eta$ and we use Itô's formula to calculate

$$dZ_t = \tilde{b}(Z_t)\,dt + \tilde{\sigma}(Z_t)\,dW_t,$$

where

$$\tilde{b}(z) = \eta z^{(\eta-1)/\eta} b(z^{1/\eta}) + \frac{\eta(\eta-1)}{2} z^{(\eta-2)/\eta}\sigma^2(z^{1/\eta}) \quad \text{and} \quad \tilde{\sigma}(z) = \eta z^{(\eta-1)/\eta}\sigma(z^{1/\eta}).$$

Using (11), we can see that

$$-\frac{2\tilde{b}(z)}{\tilde{\sigma}^2(z)} = -\frac{z^{(1-\eta)/\eta}}{\eta}\frac{2b(z^{1/\eta})}{\sigma^2(z^{1/\eta})} - \frac{\eta-1}{\eta}z^{-1} \geq \left(\frac{2\varepsilon_0}{\eta^2}\ln z - \frac{\eta-1}{\eta}\right)\frac{1}{z}.$$

It follows that $\tilde{b}$, $\tilde{\sigma}$ satisfy (125) in Lemma 9 if we choose any

$$\tilde{\varepsilon} \in (0, \varepsilon_0/\eta^2) \quad \text{and} \quad \tilde{\chi} > \exp\left(\frac{\eta(\eta-1)}{2(\varepsilon_0 - \tilde{\varepsilon}\eta^2)}\right).$$

The result now follows from Lemma 9 and Theorem 2.5 in Peskir [44]. □

Proof of Lemma 2. The lower bound in (21) of Assumption 2 imply that $J_{x,y}(Y^0) \geq -C_1(1+y)$, where Y^0 is the strategy that involves no adjustments, namely, the strategy defined by $Y^+ = Y^- = 0$, and the inequality $-\infty < V(x,y)$ follows.

The inequality $\mathbb{E}^x\left[\int_0^T Y_t\,dt\right] < \infty$, which an admissible investment strategy Y must satisfy, Hölder's inequality and (10) imply that

$$\mathbb{E}^x\left[\int_0^T X_t^\mu Y_t^\nu\,dt\right] \leq \left(\mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\,dt\right]\right)^{1-\nu}\left(\mathbb{E}^x\left[\int_0^T Y_t\,dt\right]\right)^{\nu} < \infty, \tag{133}$$

where $\mu > 0$ and $\nu \in (0,1)$ are the constants appearing in Assumption 2. In view of these observations and (21) in Assumption 2, we can see that

$$\mathbb{E}^x\left[\int_0^T |h(X_t, Y_t)|\,dt\right] \leq C_1\mathbb{E}^x\left[\int_0^T \left[1 + X_t^\mu Y_t^\nu + X_t^\zeta + Y_t\right]dt\right] < \infty,$$

which establishes part (II) of the lemma.

In view of (21) in Assumption 1, we can see that

$$\mathbb{E}^x\left[\int_0^T h(X_t, Y_t)\,dt\right] \leq C_1 T + \mathbb{E}^x\left[\int_0^T \left[C_1 X_t^\mu Y_t^\nu + C_1 X_t^\zeta - \varepsilon_1 Y_t\right]dt\right]. \tag{134}$$

Also, given any constants $Q, \varepsilon > 0$, we can verify that

$$Qz^\nu - \varepsilon z \le \frac{\varepsilon(1-\nu)}{\nu} \left(\frac{\nu}{\varepsilon}\right)^{1/(1-\nu)} Q^{1/(1-\nu)} \quad \text{for all } z \ge 0. \tag{135}$$

Combining these calculations with (133), we obtain

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^T h(X_t, Y_t)\, dt\right] \\
&\quad \le C_1 T + C_1 \mathbb{E}^x\left[\int_0^T X_t^\zeta\, dt\right] \\
&\qquad + C_1\left(\mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\, dt\right]\right)^{1-\nu}\left(\mathbb{E}^x\left[\int_0^T Y_t\, dt\right]\right)^\nu - \varepsilon_1 \mathbb{E}^x\left[\int_0^T Y_t\, dt\right] \\
&\quad \le C_1 T + C_1 \mathbb{E}^x\left[\int_0^T X_t^\zeta\, dt\right] + \frac{\varepsilon_1(1-\nu)}{\nu}\left(\frac{\nu C_1}{\varepsilon_1}\right)^{1/(1-\nu)} \mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\, dt\right]. \qquad (136)
\end{aligned}$$

This estimate, the inequalities

$$-K^+ Y_T^+ + K^- Y_T^- \le -K^+\left(Y_T^+ - Y_T^-\right) = -K^+ Y_T + K^+ y,$$

which follow from (14) and the inequality $K^+ > K^- \ge 0$, and (10) imply that

$$\begin{aligned}
J_{x,y}(Y) &\le \limsup_{T\to\infty} \frac{1}{T}\left\{K^+ y + \mathbb{E}^x\left[\int_0^T h(X_t, Y_t)\, dt - K^+ Y_T\right]\right\} \\
&\le C_1 + \frac{1}{m(\mathbb{R}_+^*)}\int_0^\infty \left[C_1 x^\zeta + \frac{\varepsilon_1(1-\nu)}{\nu}\left(\frac{\nu C_1}{\varepsilon_1}\right)^{1/(1-\nu)} x^{\mu/(1-\nu)}\right] m(dx).
\end{aligned}$$

These inequalities and (118) in Lemma 7 prove that $V(x,y) < \infty$, and part (I) of the lemma follows.

To establish part (III) of the lemma, we first consider any strategy $Y \in \mathcal{Y}_{x,y}$ that is associated with

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t\, dt\right] = \infty. \tag{137}$$

Since $\mu > 0$ and $\nu < 1$, (10) implies that, given any constant $\xi > 0$,

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t \mathbf{1}_{\left\{Y_t < \xi X_t^{\mu/(1-\nu)}\right\}}\, dt\right] \le \xi \lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\, dt\right] < \infty.$$

Therefore, (137) is true if and only if

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t \mathbf{1}_{\left\{Y_t \ge \xi X_t^{\mu/(1-\nu)}\right\}}\, dt\right] \\
&\quad \ge \lim_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t\, dt\right] - \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t \mathbf{1}_{\left\{Y_t < \xi X_t^{\mu/(1-\nu)}\right\}}\, dt\right] \\
&\quad = \infty. \qquad (138)
\end{aligned}$$

Now, we choose any $\xi > 0$ such that $\varepsilon_1 - C_1\xi^{-(1-\nu)} > 0$, and we note that

$$\begin{aligned}
&\mathbb{E}^x\left[\int_0^T [C_1 X_t^\mu Y_t^\nu - \varepsilon_1 Y_t]\,dt\right] \\
&\quad = C_1\mathbb{E}^x\left[\int_0^T X_t^\mu Y_t^\nu \mathbf{1}_{\left\{Y_t<\xi X_t^{\mu/(1-\nu)}\right\}}\,dt\right] + C_1\mathbb{E}^x\left[\int_0^T X_t^\mu Y_t^\nu \mathbf{1}_{\left\{Y_t\geq\xi X_t^{\mu/(1-\nu)}\right\}}\,dt\right] \\
&\qquad - \varepsilon_1\mathbb{E}^x\left[\int_0^T Y_t\,dt\right] \\
&\quad \leq C_1\xi^\nu\mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\mathbf{1}_{\left\{Y_t<\xi X_t^{\mu/(1-\nu)}\right\}}\,dt\right] \\
&\qquad - \left(\varepsilon_1 - C_1\xi^{-(1-\nu)}\right)\mathbb{E}^x\left[\int_0^T Y_t\mathbf{1}_{\left\{Y_t\geq\xi X_t^{\mu/(1-\nu)}\right\}}\,dt\right] \\
&\quad \leq C_1\xi^\nu\mathbb{E}^x\left[\int_0^T X_t^{\mu/(1-\nu)}\,dt\right] - \left(\varepsilon_1 - C_1\xi^{-(1-\nu)}\right)\mathbb{E}^x\left[\int_0^T Y_t\mathbf{1}_{\left\{Y_t\geq\xi X_t^{\mu/(1-\nu)}\right\}}\,dt\right].
\end{aligned}$$

Combining this inequality with (10), (134) and (138), we can see that

$$\begin{aligned}
J_{x,y}(Y) &\leq C_1 + \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}^x\left[\int_0^T \left[C_1X_t^\mu Y_t^\nu + C_1X_t^\zeta - \varepsilon_1 Y_t\right]dt\right] \\
&= -\infty.
\end{aligned}$$

Therefore, a strategy $Y \in \mathcal{Y}_{x,y}$ satisfies $J_{x,y}(Y) > -\infty$ only if

$$\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t\,dt\right] < \infty, \tag{139}$$

which implies (28). To see the latter claim, we argue by contradiction and we assume that $\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}^x\big[Y_T\big] \geq \varepsilon$, for some $\varepsilon > 0$, which implies that there exists a constant $C > 0$ such that $\mathbb{E}^x\big[Y_t\big] \geq \varepsilon t$ for all $t \geq C$. In this context, we can calculate

$$\liminf_{T\to\infty}\frac{1}{T}\mathbb{E}^x\left[\int_0^T Y_t\,dt\right] \geq \liminf_{T\to\infty}\frac{1}{T}\int_C^T \mathbb{E}^x\big[Y_t\big]\,dt \geq \lim_{T\to\infty}\frac{\varepsilon}{T}\int_C^T t\,dt = \infty,$$

which contradicts (139). □

Appendix III: proofs of results in Section 3

Proof of Lemma 3. The existence of functions α and β satisfying (53) and (52) as well as (55)–(54) follows immediately from an inspection of the inequalities (20) and (26). By continuity, (20) and (26) also imply that $\lim_{y\downarrow y^\dagger}\alpha(y) = 0$ and $\lim_{y\uparrow y^\dagger}\beta(y) = \infty$. In view of the first of the limits in (24) and the lower bound in (22), we can see that

$$\lim_{y\downarrow y_0^*}\int_0^{x^*(y)} H(u,y)\,m(du) \geq -C_1\lim_{y\downarrow y_0^*}(1+y)\int_0^{x^*(y)} m(du) = 0.$$

This observation, (52) and (54) imply that

$$0 = \lim_{y \downarrow y_0^*} \int_0^{\beta(y)} H(u,y)\, m(du) \geq \lim_{y \downarrow y_0^*} \int_0^{x^*(y)} H(u,y)\, m(du) \geq 0.$$

It follows that $\lim_{y \downarrow y_0^*} \beta(y) = 0$ because $\lim_{y \downarrow y_0^*} H(u,y) > 0$ for all $u > \lim_{y \downarrow y_0^*} x^*(y) = 0$. Similarly, we can use the second limit in (24), the upper bound in (22), (53) and (55) to calculate

$$\begin{aligned} 0 = \lim_{y \uparrow y_\infty^*} \int_{\alpha(y)}^{\infty} H(u,y)\, m(du) &\leq \lim_{y \uparrow y_\infty^*} \int_{x^*(y)}^{\infty} H(u,y)\, m(du) \\ &\leq C_1 \lim_{y \uparrow y_\infty^*} \big(1 + y^{-C_1}\big) \int_{x^*(y)}^{\infty} \big(1 + u^{C_1}\big)\, m(du) = 0. \end{aligned}$$

Therefore, $\lim_{y \uparrow y_\infty^*} \alpha(y) = \infty$ because $\lim_{y \uparrow y_\infty^*} H(u,y) < 0$ for all $u < \lim_{y \uparrow y_\infty^*} x^*(y) = \infty$.

Differentiating the identities in (52) and (53) with respect to y, we obtain

$$\beta'(y) = -\frac{\sigma^2\big(\beta(y)\big)p'\big(\beta(y)\big)}{2H\big(\beta(y), y\big)} \int_0^{\beta(y)} \frac{\partial H}{\partial y}(u,y)\, m(du) > 0 \quad \text{for all } y \in (y_0^*, y^\dagger) \tag{140}$$

and

$$\alpha'(y) = \frac{\sigma^2\big(\alpha(y)\big)p'\big(\alpha(y)\big)}{2H\big(\alpha(y), y\big)} \int_{\alpha(y)}^{\infty} \frac{\partial H}{\partial y}(u,y)\, m(du) > 0 \quad \text{for all } y \in (y^\dagger, y_\infty^*). \tag{141}$$

We can see the inequalities here once we combine the strict positivity of p' with (17) in Assumption 2 and the observations that

$$H\big(\beta(y), y\big) > 0 \text{ for all } y \in (y_0^*, y^\dagger) \quad \text{and} \quad H\big(\alpha(y), y\big) < 0 \text{ for all } y \in (y^\dagger, y_\infty^*),$$

which follow from (52), (53) and (20) in Assumption 2. □

Proof of Lemma 4. Combining (20) in Assumption 2 with (52) and (54), we can see that, for all $y \in (y_0^*, y^\dagger)$ and $z \in (0, x^*(y))$, there exists a unique point $L(z,y) \in (x^*(y), \beta(y))$ such that

$$\int_z^x H(u,y)\, m(du) \begin{cases} < 0, & \text{if } x \in \big(z, L(z,y)\big), \\ = 0, & \text{if } x = L(z,y), \\ > 0, & \text{if } x > L(z,y). \end{cases} \tag{142}$$

The resulting function $L(\cdot, y)$ satisfies

$$\lim_{z \downarrow 0} L(z,y) = \beta(y) \quad \text{and} \quad \lim_{z \uparrow x^*(y)} L(z,y) = x^*(y). \tag{143}$$

Similarly, we can combine (20) in Assumption 2 with (53) and (55) to see that, for all $y \in (y^\dagger, y_\infty^*)$ and $z \in (\alpha(y), x^*(y))$, there exists a unique point $L(z,y) \in (x^*(y), \infty)$ such that (142) is true. In this case,

$$\lim_{z \downarrow \alpha(y)} L(z,y) = \infty \quad \text{and} \quad \lim_{z \uparrow x^*(y)} L(z,y) = x^*(y). \tag{144}$$

Furthermore, (20) and (26) imply that, for $y = y^\dagger$ and all $z \in (0, x^*(y^\dagger)$, there exists a unique point $L(z, y^\dagger) \in (x^*(y^\dagger), \infty)$ such that (142) is true, in which case,

$$\lim_{z \downarrow 0} L(z, y^\dagger) = \infty \quad \text{and} \quad \lim_{z \uparrow x^*(y^\dagger)} L(z, y^\dagger) = x^*(y^\dagger). \tag{145}$$

For future reference, we note that differentiation of the identity in (142) yields

$$\frac{\partial L}{\partial z}(z, y) = \frac{H(z, y)\sigma^2\big(L(z, y)\big)p'\big(L(z, y)\big)}{H\big(L(z, y), y\big)\sigma^2(z)p'(z)} \tag{146}$$

and

$$\frac{\partial L}{\partial y}(z, y) = -\frac{\sigma^2\big(L(z, y)\big)p'\big(L(z, y)\big)}{2H\big(L(z, y), y\big)} \int_z^{L(z,y)} \frac{\partial H}{\partial y}(u, y)\, m(du) > 0, \tag{147}$$

the inequality following because $\partial H/\partial y < 0$ (see (17) in Assumption 2), p' is strictly positive and $H(z, y) > 0$ for all $z > x^*(y)$. Also, we note that for no values of z, y other than the ones we have considered above does there exist a point $L(z, y) > z$ satisfying (142).

Given $y \in (y_0^*, y_\infty^*)$, the analysis thus far implies that the system of equations (58) and (61) has a unique solution $\big(F(y), G(y)\big)$ such that $F(y) < G(y)$ if and only if there exists a unique point $F(y)$ such that

$$F(y) \in \big(0, x^*(y)\big) \text{ if } y \in (y_0^*, y^\dagger], \quad F(y) \in \big(\alpha(y), x^*(y)\big) \text{ if } y \in (y^\dagger, y_\infty^*) \tag{148}$$

and

$$\int_{F(y)}^{L(F(y),y)} p(u)H(u, y)\, m(du) = K^+ - K^-, \tag{149}$$

in which case, $G(y) = L(F(y), y) \in (x^*(y), \infty)$ and (68)–(70) are true.

Using (146), we calculate

$$\frac{\partial}{\partial z}\left(\int_z^{L(z,y)} p(u)H(u, y)\, m(du)\right) = \frac{2H(z, y)}{\sigma^2(z)p'(z)}\big[p\big(L(z, y)\big) - p(z)\big] < 0,$$

the inequality following because p is strictly increasing, $z < L(z, y)$ and $H(z, y) < 0$. In view of this result and (143)–(145), we can see that, given $y \in (y_0^*, y_\infty^*)$, there exists $F(y)$ satisfying (148)–(149) if and only if

$$\text{(65) is true if } y = y^\dagger,$$

$$Q_\beta(y) > 0 \text{ if } y \in (y_0^*, y^\dagger) \quad \text{and} \quad Q_\alpha(y) > 0 \text{ if } y \in (y^\dagger, y_\infty^*),$$

where Q_β, Q_α are defined by (63)–(64). It follows that we will establish all our claims regarding the solvability of the system of equations (58) and (61) if we show that, if (65) is false, then $Q_\beta(y) < 0$ for all $y \in (y_0^*, y^\dagger)$ and $Q_\alpha(y) < 0$ for all $y \in (y^\dagger, y_\infty^*)$, while, if (65) is true, then there exist unique $\underline{y}_F \in [y_0^*, y^\dagger)$ and $\overline{y}_G \in (y^\dagger, y_\infty^*]$ such that the inequalities (66)–(67) hold.

If $Q_\beta(y) = \infty$ for all $y \in (y^\dagger, y^*_\infty)$, then Q_β plainly satisfies (66) for $\underline{y}_F = y^*_0$. Given any y in the open set in which Q_β is real-valued, we use (140), the fact that the scale function p is strictly increasing and (17) in Assumption 2 to calculate

$$Q'_\beta(y) = \int_0^{\beta(y)} \bigl[p(u) - p\bigl(\beta(y)\bigr)\bigr] \frac{\partial H}{\partial y}(u, y)\, m(du) > 0 \quad \text{for all } y \in (y^*_0, y^\dagger).$$

In view of this calculation, we can see that Q_β is increasing, which implies that, if (65) is false, then $Q_\beta(y) < 0$ for all $y \in (y^*_0, y^\dagger)$, while, if (65) is true, then there exists a unique $\underline{y}_F \in [y^*_0, y^\dagger)$ such Q_β satisfies the inequalities in (66).

If $Q_\alpha(y) = \infty$ for all $y \in (y^\dagger, y^*_\infty)$, then Q_α satisfies (67) for $\overline{y}_G = y^*_\infty$. Given any y in the open set in which Q_α is real-valued, we use (141), the fact that the scale function p is strictly increasing and (17) in Assumption 2 to calculate

$$Q'_\alpha(y) = \int_{\alpha(y)}^{\infty} \bigl[p(u) - p\bigl(\alpha(y)\bigr)\bigr] \frac{\partial H}{\partial y}(u, y)\, m(du) < 0.$$

This calculation implies that Q_α is decreasing. Therefore, if (65) is false, then $Q_\alpha(y) < 0$ for all $y \in (y^\dagger, y^*_\infty)$, while, if (65) is true, then there exists a unique $\overline{y}_G \in (y^\dagger, y^*_\infty]$ such Q_α satisfies (67).

We assume that (65) is true in what follows. To prove that the functions F and G are C^1 and strictly increasing, we differentiate (149) with respect to y and use (146) and (147) to obtain

$$\begin{aligned} F'(y) &= \frac{\sigma^2\bigl(F(y)\bigr)p'\bigl(F(y)\bigr) \int_{F(y)}^{L(F(y),y)} \bigl[p\bigl(L(F(y), y)\bigr) - p(u)\bigr] \frac{\partial H}{\partial y}(u, y)\, m(du)}{2H\bigl(F(y), y\bigr)\bigl[p\bigl(L(F(y), y)\bigr) - p\bigl(F(y)\bigr)\bigr]} \\ &= \frac{\sigma^2\bigl(F(y)\bigr)p'\bigl(F(y)\bigr)}{2H\bigl(F(y), y\bigr)} \int_{F(y)}^{L(F(y),y)} \left[1 + \frac{p\bigl(F(y)\bigr) - p(u)}{p\bigl(L(F(y), y)\bigr) - p\bigl(F(y)\bigr)}\right] \frac{\partial H}{\partial y}(u, y)\, m(du) \\ &> 0, \end{aligned} \tag{150}$$

the inequality following because $\partial H/\partial y < 0$, p is strictly increasing and $H(z, y) < 0$ for all $z < x^*(y)$. Also, we calculate

$$\begin{aligned} G'(y) &= \frac{\partial L}{\partial z}(F(y), y)F'(y) + \frac{\partial L}{\partial y}(F(y), y) \\ &= -\frac{\sigma^2\bigl(L(F(y), y)\bigr)p'\bigl(L(F(y), y)\bigr) \int_{F(y)}^{L(F(y),y)} \bigl[p(u) - p\bigl(F(y)\bigr)\bigr] \frac{\partial H}{\partial y}(u, y)\, m(du)}{2H\bigl(L(F(y), y), y\bigr)\bigl[p\bigl(L(F(y), y)\bigr) - p\bigl(F(y)\bigr)\bigr]} \\ &= \frac{\sigma^2\bigl(G(y)\bigr)p'\bigl(G(y)\bigr)}{2H\bigl(G(y), y\bigr)} \int_{F(y)}^{G(y)} \left[\frac{p\bigl(G(y)\bigr) - p(u)}{p\bigl(G(y)\bigr) - p\bigl(F(y)\bigr)} - 1\right] \frac{\partial H}{\partial y}(u, y)\, m(du) \\ &> 0. \end{aligned} \tag{151}$$

To complete the proof, we still need to show (71)–(74). To this end, we combine the fact that $F(y)$ and $G(y)$ satisfy the system of equations (58) and (61) with (52) and (66) to see that $\lim_{y\downarrow\underline{y}_F} F(y)=0$ and $\lim_{y\downarrow\underline{y}_F} G(y)=\beta(\underline{y}_F)$ if $\underline{y}_F > y_0^*$. Similarly, we use (53) and (67) to see that $\lim_{y\uparrow\overline{y}_G} F(y)=\alpha(y)$ and $\lim_{y\uparrow\overline{y}_G} G(y)=\infty$ if $\overline{y}_G < y_\infty^*$. On the other hand, if $\underline{y}_F = y_0^*$, then the inequalities $F(y)<G(y)<\beta(y)$ and the fact that $\lim_{y\downarrow y_0^*}\beta(y)=0$ (see (56) in Lemma 3) imply that $\lim_{y\downarrow\underline{y}_F} F(y)=\lim_{y\downarrow\underline{y}_F} G(y)=0$, while, if $\overline{y}_G = y_\infty^*$, then the inequalities $\alpha(y)<F(y)<G(y)$ and the fact that $\lim_{y\uparrow y_\infty^*}\alpha(y)=\infty$ imply that $\lim_{y\uparrow\overline{y}_G} F(y)=\lim_{y\uparrow\overline{y}_G} G(y)=\infty$.

If $\underline{y}_F > y_0^*$, then we combine the limits $\lim_{y\downarrow\underline{y}_F} F(y)=0$ and $\lim_{y\downarrow\underline{y}_F} G(y)=\beta(\underline{y}_F)$ with the fact that $\lim_{x\downarrow 0} p(x)=-\infty$ and (151) to obtain

$$\lim_{y\downarrow\underline{y}_F} G'(y) = -\frac{\sigma^2\big(\beta(\underline{y}_F)\big)p'\big(\beta(\underline{y}_F)\big)}{2H\big(\beta(\underline{y}_F),\underline{y}_F\big)}\int_0^{\beta(\underline{y}_F)}\frac{\partial H}{\partial y}(u,\underline{y}_F)\,m(du) \overset{(140)}{=} \beta'(y).$$

Finally, if $\overline{y}_G < y_\infty^*$, then we can use the limits $\lim_{y\uparrow\overline{y}_G} F(y)=\alpha(y)$, $\lim_{y\uparrow\overline{y}_G} G(y)=\infty$, the fact that $\lim_{x\to\infty} p(x)=\infty$ and (150) to calculate

$$\lim_{y\uparrow\overline{y}_G} F'(y) = \frac{\sigma^2\big(\alpha(\overline{y}_G)\big)p'\big(\alpha(\overline{y}_G)\big)}{2H\big(\alpha(\overline{y}_G),\overline{y}_G\big)}\int_{\alpha(\overline{y}_G)}^{\infty}\frac{\partial H}{\partial y}(u,\overline{y}_G)\,m(du) \overset{(141)}{=} \alpha'(y),$$

and the proof is complete. □

Proof of Proposition 5. First, we note that the existence and the claimed properties of the functions F and G follow from Lemmas 3 and 4 and (75)–(77). Also, we can check that the C^1 continuity of the functions p', F and G, the limits (71)–(74) in Lemma 4 and the definition of v_y imply that this function is $C^{2,1}$.

By construction, the function w as well as its derivatives w_y and w_{xy} are well-defined and continuous (see (43)–(46)). Using the definition of w and (44), (46), we can see that, given any $(x,y)\in\mathcal{I}$,

$$w_x(x,y) = v_x\big(x,G^{-1}(x)\big) + \big[v_y\big(x,G^{-1}(x)\big) - K^+\big]\frac{dG^{-1}}{dx}(x) = v_x\big(x,G^{-1}(x)\big) \tag{152}$$

and

$$w_{xx}(x,y) = v_{xx}\big(x,G^{-1}(x)\big) + v_{xy}\big(x,G^{-1}(x)\big)\frac{dG^{-1}}{dx}(x) = v_{xx}\big(x,G^{-1}(x)\big). \tag{153}$$

Similarly we can show that

$$w_x(x,y) = v_x\big(x,F^{-1}(x)\big) \quad\text{and}\quad w_{xx}(x,y) = v_{xx}\big(x,F^{-1}(x)\big) \quad \text{for all } (x,y)\in\mathcal{D}.$$

In light of these considerations, we deduce the $C^{2,1}$ continuity of w.

By construction, we will prove that w satisfies the HJB equation (30) if we show that

$$K^- \le w_y(x,y) \le K^+ \quad \text{for all } (x,y)\in\mathcal{C} \tag{154}$$

and

$$\frac{1}{2}\sigma^2(x)w_{xx}(x,y) + b(x)w_x(x,y) + h(x,y) \le 0 \quad \text{for all } (x,y)\in\mathcal{D}\cup\mathcal{I}. \tag{155}$$

In view of (152) and (153), we can calculate

$$\begin{aligned}
&\frac{1}{2}\sigma^2(x)w_{xx}(x,y) + b(x)w_x(x,y) + h(x,y)\\
&\qquad = \frac{1}{2}\sigma^2(x)v_{xx}\big(x, G^{-1}(x)\big) + b(x)v_x\big(x, G^{-1}(x)\big) + h(x,y)\\
&\qquad = -\int_y^{G^{-1}(x)} H(x,s)\, ds\\
&\qquad \le 0 \qquad \text{for all } (x,y) \in \mathcal{I},
\end{aligned}$$

the inequality following because $H(x,y) > 0$ for all $(x,y) \in \mathcal{I}$ (see (20) in Assumption 2 and (68)–(69) in Lemma 4). Similarly, we can see that

$$\frac{1}{2}\sigma^2(x)w_{xx}(x,y) + b(x)w_x(x,y) + h(x,y) = \int_{F^{-1}(x)}^{y} H(x,s)\, ds \le 0 \quad \text{for all } (x,y) \in \mathcal{D},$$

and (155) has been established.

If $\underline{y}_G < \underline{y}_F$ and $\mathcal{C}_2 \neq \emptyset$, then we combine (49) with the inequalities (52) in Lemma 3 and the fact that $G(y) = \beta(y)$ for all $y \in (\underline{y}_G, \underline{y}_F]$ to see that $w_y(x,y) < K^+$ for all $(x,y) \in \mathcal{C}_1$, and that $K^- \le w_y(x,y)$ for all $(x,y) \in \mathcal{C}_1$ if and only if

$$K^+ + \int_0^{\beta(y)} p'(s) \int_0^s H(u,y)\, m(du)\, ds \ge K^- \quad \text{for all } y \in (y_0^*, \underline{y}_F].$$

In view of the identity (122), this inequality is equivalent to $Q_\beta(y) \le 0$ for all $y \in (y_0^*, \underline{y}_F]$, where Q_β is defined by (63), which is true thanks to (66) in Lemma 4. It follows that (154) is satisfied in $\mathcal{C}_1$. If $\overline{y}_G < \overline{y}_F$ and $\mathcal{C}_3 \neq \emptyset$, then we can see that (154) is satisfied in $\mathcal{C}_3$ using the same arguments with (51), (53), (67) and (123).

Finally, if $\underline{y}_F < \overline{y}_G$ and $\mathcal{C}_2 \neq \emptyset$, then (70) in Lemma 4 and the first expression in (62) imply that $w_y(x,y) \le K^+$ for all $(x,y) \in \mathcal{C}_2$, while (70) and the second expression in (62) imply that $w_y(x,y) \ge K^-$ for all $(x,y) \in \mathcal{C}_2$. □

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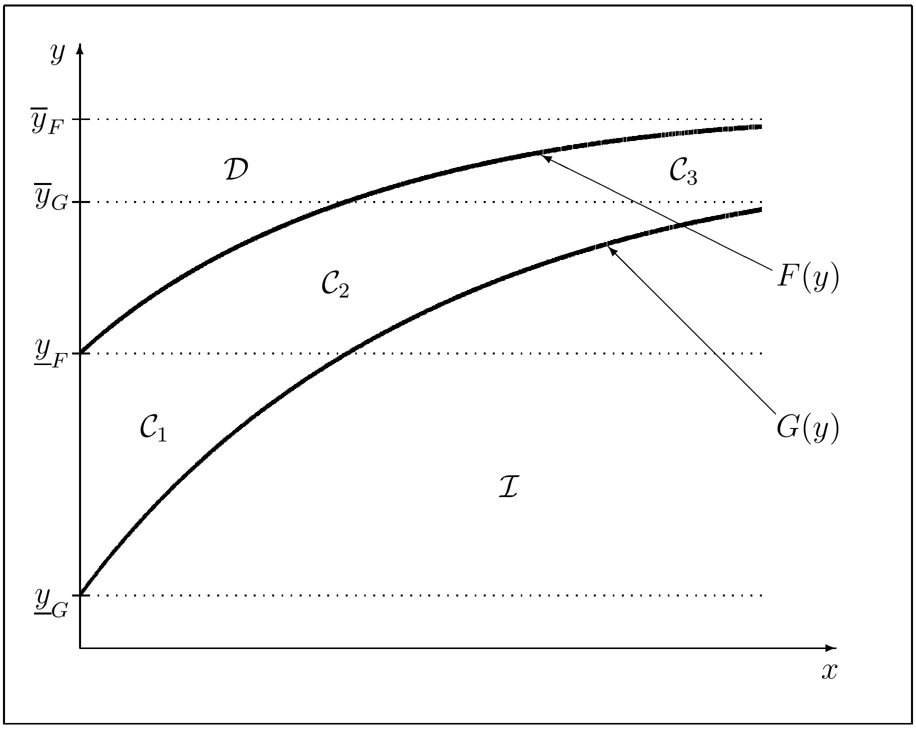

Figure 1. Graph of the functions F and G in the general context.

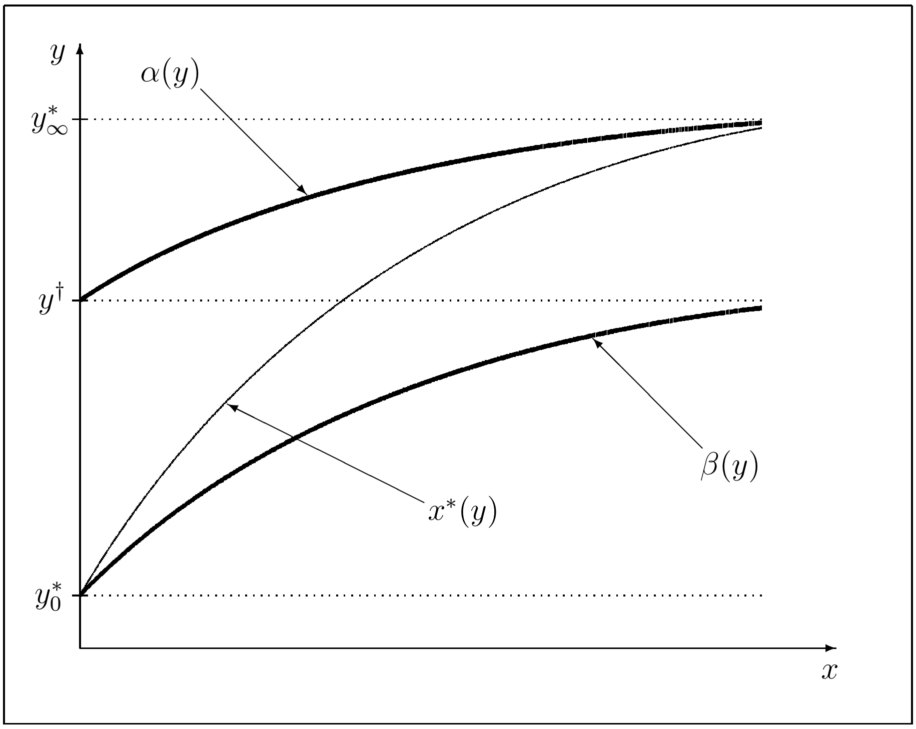

Figure 2. Graph of the functions α and β in the general context.

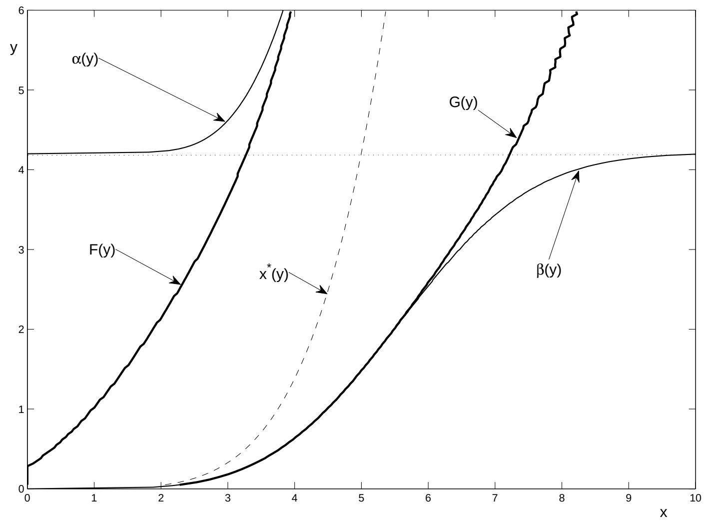

Figure 3. Simulation results for the special case considered in Section 5.1 with data given by (105).

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