A model for optimally advertising and launching a product

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Abstract

We formulate and solve a problem that combines the features of the so-called monotone follower of singular stochastic control theory with optimal stopping. In particular, we consider a stochastic system whose uncontrolled state dynamics are modelled by a general one-dimensional Itô diffusion. The aim of the problem that we solve is to maximise the utility derived from the system's state at the discretionary time when the system's control is terminated. This objective is reflected by the performance criterion that we maximise, which also penalises control expenditure as well as waiting. The model that we study is motivated by the so-called goodwill problem, a variant of which is concerned with how to optimally raise a new product's image, e.g., through advertising, and with determining the best time to launch the product in the market. In the presence of the rather general assumptions that we make, we fully characterise the optimal strategy, which can take one of three qualitatively different forms, depending on the problem data.

Keywords: singular control, monotone follower, optimal stopping, goodwill problem, real options

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1 Introduction

We consider a stochastic system whose state is modelled by the controlled one-dimensional positive Itô diffusion

$$dX_t = b(X_t)\, dt + dZ_t + \sigma(X_t)\, dW_t, \quad X_0 = x > 0, \tag{1}$$

where W is a standard one-dimensional Brownian motion, and the controlled process Z is an adapted càglàd increasing process. The objective of the optimisation problem that we solve is to maximise the performance criterion

$$J_x(Z, \tau) = \mathbb{E}\left[\int_0^\tau e^{-\Lambda_t} H(X_t)\, dt - \int_0^\tau e^{-\Lambda_t} K'(X_t) \circ\, dZ_t + e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right], \tag{2}$$

over all admissible choices of Z and all stopping times τ, where

$$\Lambda_t = \int_0^t r(X_u)\, du, \tag{3}$$

and

$$\int_0^\tau e^{-\Lambda_t} K'(X_t) \circ dZ_t = \int_0^\tau e^{-\Lambda_t} K'(X_t)\, dZ_t^{\mathrm{c}} + \sum_{0\le t\le\tau} \int_0^{\Delta Z_t} e^{-\Lambda_t} K'(X_t + s)\, ds, \tag{4}$$

in which expression, Z^{c} is the continuous part of the increasing process Z. It is worth noting that the integral given by (4), which we use to penalise control expenditure, was introduced by Zhu [32] and is now standard in the singular stochastic control literature.

This stochastic control problem is motivated by the following application that arises in the context of the so-called goodwill problem. A company considers the timing of launching a new product that they have developed. Prior to launching it in a given market, the company attribute an image to the product based on the market's attitudes to similar products, the new product's quality differences from existing products, and the company's own image in the market. We use X to model the evolution in time of the product's image. In this context, Z represents the effect of costly interventions, such as advertising, that the company can make to raise the product's image. The company's objective is to maximise their utility from launching the product minus their "dis-utility" associated with the cost of intervention and the cost of waiting. In particular, the company aims at maximising the performance index defined by (2)–(4) over all intervention strategies Z and launching times τ.

Optimal control problems addressing this type of application have attracted significant interest in the literature for about half a century. Most of the models that have been studied in this area involve deterministic control and can be traced back to Nerlove and Arrow [27] (see Buratto and Viscolani [14] and the references therein). More realistic models in which the product's image evolves randomly over time have also been proposed and studied (see Feichtinger, Hartl and Sethi [18] for a review and Marinelli [25] for some more recent references). In particular, Marinelli [25] considers extensions of the classical Nerlove and Arrow model, and studies a class of problems that involve linear dynamics of the state process, absolutely continuous control and linear or quadratic payoff functions. Also, Jack, Johnson and Zervos [19] study a related model involving singular control only, in which, the product is assumed launched at time 0 and the objective is to select an advertising strategy that maximises the expected payoff resulting from its marketing.

The problem that we solve combines the features of the so-called monotone follower of the singular stochastic control theory with optimal stopping. Singular stochastic control, which was introduced by Bather and Chernoff [7] and Beneš, Shepp and Witsenhausen [12], has a well-developed body of theory, and we do not attempt a comprehensive literature survey. Also, we refer the interested reader to Peskir and Shiryaev [28] for a recent exposition of the theory of optimal stopping. Models that combine singular control with discretionary stopping were introduced by Davis and Zervos [15] who assumed that the uncontrolled system dynamics follow a standard Brownian motion and considered quadratic cost functions. In the same context, Karatzas, Ocone, Wang and Zervos [21] solved the problem that arises if an additional finite-fuel constraint is incorporated. A problem combining the singular control of a Brownian motion with drift with optimal stopping was later studied by Ly Vath, Pham and Villeneuve [24]. More recently, Morimoto [26] studied a model similar to the one in Davis and Zervos [15] but with a controlled geometric Brownian motion instead of a controlled standard Brownian motion. Also, Bayraktar and Egami [9], motivated by issues in initial public offerings rather than the goodwill problem, solved a problem that has the same general structure as the one of the problem we consider here. These authors assumed that the uncontrolled state dynamics are given by a Brownian motion with drift added to a compound Poisson process with exponentially distributed jump sizes, and that $H(x) = 0$, $K'(x) = 1$, $r(x) = \varrho$ and $U(x) = \lambda x$ for all x, for some constants $\varrho, \lambda > 0$. It is of interest to observe that the optimal strategy derived in that paper has a qualitatively different form from the one we obtain here. In particular, reflecting the state process at a given level figures among the optimal tactics in Bayraktar and Egami [9] but is never optimal in the problem we study here (see also the discussion of our main results below).

The control of one-dimensional Itô diffusions such as the one we considered here has recently attracted considerable interest in the literature. The optimal stopping of such processes has been studied by Salminen [31], Alvarez [2, 3], Beibel and Lerche [11], Dayanik and Karatzas [17], Dayanik [16] and Lamberton and Zervos [23], among others. Also, Alvarez [1, 4], Bayraktar and Egami [8], and Jack, Johnson and Zervos [19] have studied several singular control problems, Alvarez [5], and Alvarez and Lempa [6] have studied models with impulse control, while Bayraktar and Egami [10], Pham, Ly Vath and Zhou [30] and Johnson and Zervos [20] have analysed models with sequential switching (see also Pham [29]). In the spirit of certain references in this rather incomplete list, we solve the problem we consider by constructing an appropriate solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. To the best of our knowledge, the model that we study here is the first one that combines the singular control of a general one-dimensional Itô diffusion with optimal stopping.

It turns out that the optimal strategy of the problem that we solve here may involve only a single impulse applied to the state process. In particular, the optimal strategy does not involve reflecting the state process in the boundary of a state space's subset, which characterises singular stochastic control problems. Beyond this observation, the optimal strategy can take one of three different possible forms, depending on parameter values. These forms involve combinations of the following three tactics: wait, move (i.e., advertise the product), and stop (i.e., launch the product). Specifically, it is optimal either to move and stop, or to wait and stop, or to wait, move and stop, in which list, we order the sequence of optimal tactics according to small, moderate and large values of the underlying state process X (see Theorem 1, which is our main result).

We illustrate our main result by means of several special cases. Apart from an independent interest that each of these has, they reveal that the form of the optimal strategy is dependent on the functional form of the problem data as well as on parameter values. Indeed, if the uncontrolled system dynamics are modelled by a geometric Brownian motion, then the move and stop strategy is always optimal if the terminal payoff function U is a power utility function, while the move and stop strategy is never optimal if U is the logarithmic utility function. On the other hand, if the uncontrolled system dynamics are modelled by a mean-reverting square-root process, such as the one appearing in the Cox-Ingersoll-Ross model, then the optimal strategy can take any of the three different possible forms, whether U is a power or the logarithmic utility function.

At this point, we should make a comment on two possible generalisations of the model that we study. The first one arises if we slightly relaxed our assumptions so that never advertising and/or launching the product could be optimal. We decided against including the relevant analysis partly because this case never arises in the context of the examples we consider and partly to limit the paper's size. The second generalisation arises if we assume that the product's image is only partially observable. In this case, we would have to modify our analysis by expressing the performance criterion given by (2) in terms of a sufficient statistic for the process X, the dynamics of which should satisfy appropriate filtering equations. Exploring such an interesting generalisation would go well beyond the scope of this paper, and it could be a topic for future research.

The paper is organised as follows. In Section 2, we formulate the control problem that we solve and list all of the assumptions that we make. Section 3 is concerned with the solution of the problem. In Section 4, we study a number of special cases, which illustrate the full spectrum of the possible optimal scenarios, and which show that our general assumptions are easy to verify in practice. Finally, in the Appendix, we review a number of results on the solvability of a second order linear ODE that our analysis uses.

2 Problem formulation

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a stochastic system whose uncontrolled dynamics are modelled by the Itô diffusion associated with the stochastic differential equation

$$dX_t^0 = b(X_t^0)\,dt + \sigma(X_t^0)\,dW_t, \quad X_0^0 = x > 0, \tag{5}$$

and we make the following assumption.

Assumption 1 The functions $b, \sigma :]0, \infty[\to \mathbb{R}$ are locally Lipschitz, and $\sigma^2(x) > 0$ for all $x > 0$. □

This assumption implies that (5) has a unique strong solution. It also implies that, given any $c > 0$, the scale function p_c, given by

$$p_c(c) = 0, \quad p_c'(x) = \exp\left(-2\int_c^x \frac{b(s)}{\sigma^2(s)}\,ds\right), \tag{6}$$

is well-defined, and the speed measure m_c, given by

$$m_c(dx) = \frac{2}{\sigma^2(x)p_c'(x)}\,dx,$$

is a Radon measure. Additionally, we assume that the solution of (5) is non-explosive, so that, given any initial condition x, $X_t^0 \in]0, \infty[$ for all $t \geq 0$, with probability 1 (see Karatzas and Shreve [22, Theorem 5.5.29] for appropriate necessary and sufficient analytic conditions).

Assumption 2 The Itô diffusion X^0 defined by (5) is non-explosive. □

We model the system's controlled dynamics by the SDE (1). With each admissible intervention strategy, we associate the performance criterion defined by (2)–(4).

Definition 1 The set $\mathcal{A}$ of all *admissible strategies* is the set of all pairs (Z, τ) where τ is an $(\mathcal{F}_t)$-stopping time and Z is an $(\mathcal{F}_t)$-adapted increasing càglàd process such that $Z_0 = 0$,

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t} K'(X_t) \circ dZ_t\right] < \infty \quad \text{and} \quad \mathbb{E}\left[e^{-\Lambda_\tau} U^-(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right] < \infty, \tag{7}$$

where $U^-(x) = -\min\{0, U(x)\}$. □

The objective of our control problem is to maximise J_x over all admissible strategies. Accordingly, we define the problem's value function v by

$$v(x) = \sup_{(Z,\tau)\in\mathcal{A}} J_x(Z, \tau), \quad \text{for } x > 0.$$

For our optimisation problem to be well-posed, we need additional assumptions.

Assumption 3 The discounting rate function r is absolutely continuous. Also, there exists a constant $r_0 > 0$ such that $r(x) \geq r_0$ for all $x > 0$. $\square$

Assumption 4 The functions K and U are C^2 with absolutely continuous second derivatives, and the function H is absolutely continuous. There exists a point $\beta > 0$ such that

$$K'(x) - U'(x) = \begin{cases} \leq 0, & \text{for } x < \beta, \\ \geq 0, & \text{for } x > \beta. \end{cases} \tag{8}$$

Also, the function H/r is bounded, and $K'(x)$ remains bounded as $x \downarrow 0$. $\square$

In the context of the goodwill problem that has motivated this paper, it is worth noting that (8) in this assumption has a simple economic interpretation. In view of (4), which provides the cost of an intervention strategy Z, $K'(x)\,\varepsilon$ is the cost of raising the product's image from x to $x+\varepsilon$, for small $\varepsilon > 0$. Also, $U'(x)\,\varepsilon$ is the change in the utility that the company derives if the product is launched when its image is $x+\varepsilon$ rather than x, for small $\varepsilon > 0$. In light of these observations, assumption (8) captures the idea that the marginal cost of advertising is less (resp., greater) than the marginal utility derived from the product's launch when the product's image is low (resp., high), which is a rather natural one.

In the presence of Assumption 4, we can see that, if we define

$$\Theta(x) = \begin{cases} U(\beta) - \int_x^\beta K'(s)\,ds, & \text{for } x < \beta, \\ U(x), & \text{for } x \geq \beta, \end{cases} \tag{9}$$

then the function Θ is C^1 in $]0,\infty[$ and C^2 with absolutely continuous second derivative in $]0,\beta[\,\cup\,]\beta,\infty[$, and it satisfies

$$\max\{\Theta'(x) - K'(x),\ U(x) - \Theta(x)\} = 0. \tag{10}$$

In the context of the goodwill problem, Θ would be the value function of the control problem if advertise and launch immediately were the only tactics available to the decision maker, i.e., if waiting for any amount of time were not a possibility.

We need to make additional assumptions. To this end, we consider the operator $\mathcal{L}$ acting on C^1 functions with absolutely continuous first derivatives that is defined by

$$\mathcal{L}w(x) = \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x), \tag{11}$$

and the operator $\mathcal{D}_r$ acting on absolutely continuous functions that is defined by

$$\mathcal{D}_r w(x) = \frac{r(x)w'(x) - r'(x)w(x)}{r(x)} \equiv r(x)\left(\frac{w}{r}\right)'(x). \tag{12}$$

At first glance, the conditions in the following assumption may appear involved. However, they are quite general, and, apart from a growth and an integrability condition, they have a natural economic interpretation (see the discussion below). Furthermore, they are rather easy to verify in practice, as we will see in Section 4.

Assumption 5 The function Θ satisfies

$$\lim_{x\downarrow 0}\frac{\Theta(x)}{\varphi(x)} = \lim_{x\to\infty}\frac{\Theta(x)}{\psi(x)} = 0, \tag{13}$$

where the functions φ and ψ span the solution space of the homogeneous ODE $\mathcal{L}w(x) = 0$ and satisfy (49)–(51) in the Appendix. Furthermore, Θ satisfies

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}|\mathcal{L}\Theta(X_t^0)|\,dt\right] < \infty. \tag{14}$$

There exists a point $x^* \geq 0$ such that

$$[\mathcal{L}\Theta + H](x-) = \begin{cases} > 0, & \text{for } x < x^*, \text{ if } x^* > 0, \\ \leq 0, & \text{for } x > x^*. \end{cases} \tag{15}$$

Furthermore,

$$[\mathcal{L}\Theta + H](\beta-) \geq [\mathcal{L}\Theta + H](\beta+), \tag{16}$$

$$\mathcal{D}_r[\mathcal{L}\Theta + H](x) \leq 0 \quad \text{Lebesgue-a.e. in }]0,\beta[\cup]\beta,\infty[, \tag{17}$$

$$\frac{[\mathcal{L}\Theta + H](x)}{r(x)} \text{ remains bounded as } x \downarrow 0, \tag{18}$$

$$\liminf_{x\to\infty}\,[\Theta - R_H](x) > 0, \tag{19}$$

where R_H is defined by (58) in the Appendix for $F = H$. □

The operator $\mathcal{L}$ is the infinitesimal generator of the uncontrolled diffusion X^0 killed at a rate given by the discounting rate function r. Also, as we have discussed after Assumption 4, Θ is the best value that the company can get from just advertising and launching the product, while H is the running payoff that the company accumulates by delaying the product's launch. Therefore, $[\mathcal{L}\Theta + H](x)\,\Delta t$ is the expected payoff associated with the company's waiting for a small amount of time $\Delta t > 0$ before advertising and launching. In view of this observation, (15)–(16) capture the following natural idea: if the product's image is low (resp., high), then waiting may be a good (resp., bad) choice because the product's image may improve (resp., deteriorate) due to its stochastic dynamics.

Building on the above ideas, we can view the function $[\mathcal{L}\Theta + H]/r$ as the expected rate at which the company's payoff from advertising and launching changes by delaying taking action, measured in units of time that are proportional to the discounting rate r. In light of this interpretation and the definition (12) of the operator $\mathcal{D}_r$, (17) reflects the idea that the expected rate at which the best payoff resulting from "pure" action changes by waiting is decreasing as the product's image increases. Furthermore, (18) reflects the idea that waiting cannot be associated with an infinite expected rate of improvement.

In view of (57) in the Appendix, the function R_H identifies with the expected payoff that the company face if they exert no advertising effort and they never launch the product. Combining this observation with the interpretation of the function Θ as the optimal payoff that the company can receive if advertising and launching were the only available tactics, we can see that (19) is a necessary condition for guaranteeing that waiting forever and never taking any action is not an optimal strategy.

Remark 1 The conditions (13)–(14) in the previous assumption imply that the function Θ admits the representation

$$\Theta(x) = R_{-\mathcal{L}\Theta}(x) \quad \text{for all } x > 0, \tag{20}$$

where $R_{-\mathcal{L}\Theta}$ is defined by (57) or (58) in the Appendix with $F = -\mathcal{L}\Theta$ (see also the discussion at the end of the Appendix). The boundedness of H/r (see Assumption 4) and the definition (3) of Λ imply that

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}|H(X_t^0)|\, dt\right] = -\mathbb{E}\left[\int_0^\infty \frac{|H(X_t^0)|}{r(X_t^0)}\, de^{-\Lambda_t}\right] \leq \sup_{x>0} \frac{|H(x)|}{r(x)} < \infty.$$

This observation and (56) in the Appendix imply that the function R_H given by (57)–(58) with $F = H$ is well-defined and satisfies

$$\mathcal{L}R_H(x) + H(x) = 0 \quad \text{for all } x > 0. \tag{21}$$

□

3 The solution of the control problem

In light of the general theory of stochastic optimal control and optimal stopping, we expect that the value function v of our control problem identifies with a solution w of the HJB equation

$$\max\left\{\mathcal{L}w(x) + H(x),\ w'(x) - K'(x),\ U(x) - w(x)\right\} = 0. \tag{22}$$

A function w is a solution of this equation if it is C^1 with absolutely continuous first derivative, and it satisfies

$$\mathcal{L}w(x) + H(x) \leq 0 \quad \text{Lebesgue-a.e. in }]0,\infty[,$$
$$w'(x) \leq K'(x) \quad \text{and} \quad U(x) \leq w(x) \quad \text{for all } x > 0,$$
$$[\mathcal{L}w(x) + H(x)]\,[w'(x) - K'(x)]\,[U(x) - w(x)] = 0 \quad \text{Lebesgue-a.e. in }]0,\infty[.$$

We now solve the control problem by constructing an appropriate solution of this equation. To this end, we have to consider two possibilities. The first one arises when it is optimal to move and stop immediately.

Lemma 1 *In the presence of Assumptions 1–5, the function Θ defined by (9) satisfies the HJB equation (10) if and only if $x^* = 0$, where x^* is the point in (15) of Assumption 5.*

Proof. In view of (10), we can see that Θ satisfies the HJB equation of (22) if and only if $\mathcal{L}\Theta(x) + H(x) \leq 0$ Lebesgue-a.e. in $]0, \infty[$, which is true if and only if $x^* = 0$. $\square$

The second possibility arises when waiting enters the set of optimal tactics. In this case, we postulate that it is optimal to wait for as long as the state process X takes values below a given threshold level, and move and stop as soon as the state process exceeds the threshold level. If we denote by α this threshold level, then we look for a solution w of the HJB equation (22) that satisfies the ODE $\mathcal{L}w(x) + H(x) = 0$ Lebesgue-a.e. in $]0, \alpha[$, and is such that $\max\{w'(x) - K'(x),\ U(x) - w(x)\} = 0$ for all $x \geq \alpha$. In view of (10) and (21), we therefore look for a solution of the form

$$w(x) = \begin{cases} A\psi(x) + R_H(x), & \text{for } x < \alpha, \\ \Theta(x), & \text{for } x \geq \alpha, \end{cases} \tag{23}$$

where A is an appropriate constant, ψ is as in (50)–(51), and R_H is defined by (57)–(58) with $F = H$ (see also Remark 1).

To specify the parameter A and the free-boundary point α, we postulate that w satisfies the so-called "principle of smooth fit". In particular, we assume that w is C^1 at α, which gives rise to the system of equations

$$A\psi(\alpha) + R_H(\alpha) = \Theta(\alpha) \quad \text{and} \quad A\psi'(\alpha) + R_H'(\alpha) = \Theta'(\alpha),$$

which is equivalent to

$$A = \frac{\Theta(\alpha) - R_H(\alpha)}{\psi(\alpha)} = \frac{\Theta'(\alpha) - R_H'(\alpha)}{\psi'(\alpha)}. \tag{24}$$

In view of the fact that

$$\Theta - R_H = -R_{\mathcal{L}\Theta + H}, \tag{25}$$

which follows from (20) and (62) with $F = \mathcal{L}\Theta + H$, we can check that the second identity in (24) is equivalent to $(R_{\mathcal{L}\Theta+H}/\psi)'(\alpha) = 0$. It follows that the free-boundary point α should satisfy the equation

$$q(\alpha) := \int_0^\alpha \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p_c'(s)}\, ds = 0, \tag{26}$$

because (60) in the Appendix with $F = \mathcal{L}\Theta + H$ implies the expression

$$\left(\frac{R_{\mathcal{L}\Theta+H}}{\psi}\right)'(x) = -\frac{2p_c'(x)}{\psi^2(x)} \int_0^x \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p_c'(s)}\, ds = -\frac{2p_c'(x)}{\psi^2(x)} q(x). \tag{27}$$

The following result is concerned with the solvability of this equation and with the associated solution of the HJB equation (22).

Lemma 2 *In the presence of Assumptions 1–5, equation (26) has a unique solution $\alpha > 0$ if and only if $x^* > 0$, where x^* is the point appearing in (15) of Assumption 5. In this case, $\alpha > x^*$, and the function w defined by (23), where A is given by (24), is C^1 with absolutely continuous first derivative and satisfies the HJB equation (22).*

Proof. In view of (15), we can see that the left-hand derivative $q'(x-)$ of q at $x > 0$ satisfies

$$q'(x-) = \frac{[\mathcal{L}\Theta + H](x-)\psi(x)}{\sigma^2(x)p_c'(x)} \begin{cases} > 0, & \text{for } x < x^*, \text{ if } x^* > 0, \\ \leq 0, & \text{for } x > x^*. \end{cases} \tag{28}$$

Combining this observation with the fact that $q(0) = 0$, we can see that the equation $q(\alpha) = 0$ has a unique solution $\alpha > 0$ if and only if $x^* > 0$ and $\lim_{x\to\infty} q(x) < 0$. Furthermore, this solution is such that

$$x^* < \alpha \quad \text{and} \quad q(x) = \begin{cases} > 0, & \text{for } x < \alpha, \\ < 0, & \text{for } x > \alpha. \end{cases} \tag{29}$$

To see that the inequality $\lim_{x\to\infty} q(x) < 0$ is indeed true, we first note that (27)–(28) imply that the function $R_{\mathcal{L}\Theta+H}/\psi$ is monotone as $x \to \infty$. This observation, the fact that $\lim_{x\to\infty} R_{\mathcal{L}\Theta+H}(x)/\psi(x) = 0$ (see (59) in the Appendix) and (27) imply that $\lim_{x\to\infty} q(x) < 0$ if and only if $R_{\mathcal{L}\Theta+H}(x) \equiv -[\Theta - R_H](x) < 0$ for all x sufficient large, which is true thanks to (19) in Assumption 5.

In view of the construction of w and the fact that Θ satisfies (10), we will prove that w satisfies the HJB equation (22) if we show that

$$\mathcal{L}\Theta + H(x) \leq 0 \quad \text{Lebesgue-a.e. in }]\alpha, \infty[, \tag{30}$$

$$A\psi(x) + R_H(x) \geq U(x) \quad \text{for all } x \leq \alpha, \tag{31}$$

$$A\psi'(x) + R_H'(x) \leq K'(x) \quad \text{for all } x \leq \alpha. \tag{32}$$

To this end, we note that (30) follows immediately from (15) in Assumption 5 and the first inequality in (29). To establish (31), it suffices to show that $A\psi(x) + R_H(x) \geq \Theta(x)$ for all $x < \alpha$, because $\Theta \geq U$ (see (10)). In view of (24), (25) and the fact that $\psi > 0$, we can see that this inequality is equivalent to $(R_{\mathcal{L}\Theta+H}/\psi)(x) \geq (R_{\mathcal{L}\Theta+H}/\psi)(\alpha)$ for all $x < \alpha$, which is true thanks to (27) and (29).

Finally, (32) will follow if we prove that $A\psi'(x) + R_H'(x) \leq \Theta'(x)$ for all $x < \alpha$, because $\Theta' \leq K'$ (see (10)). Combining (24) with (25) and the strict positivity of ψ', we can see that this inequality is equivalent to

$$\frac{R'_{\mathcal{L}\Theta+H}(x)}{\psi'(x)} \leq \frac{R'_{\mathcal{L}\Theta+H}(\alpha)}{\psi'(\alpha)} \quad \text{for all } x < \alpha. \tag{33}$$

Using the identity (61) in the Appendix with $F = [\mathcal{L}\Theta + H]$ and the definition (26) of q, we can see that the left-hand derivative $(R'_{\mathcal{L}\Theta+H}/\psi')'(x-)$ exists for all $x > 0$, and is given by

$$\left(\frac{R'_{\mathcal{L}\Theta+H}}{\psi'}\right)'(x-) = \frac{2r(x)p_c'(x)}{[\sigma(x)\psi'(x)]^2}\left[2q(x) - \frac{[\mathcal{L}\Theta + H](x-)}{r(x)}\frac{\psi'(x)}{p_c'(x)}\right]. \tag{34}$$

Furthermore, recalling that the function $\mathcal{L}\Theta + H$ is absolutely continuous in $]0, \beta[\cup]\beta, \infty[$, we can use the integration by parts formula, the expression (55) in the Appendix and the definition (12) of the operator $\mathcal{D}_r$ to calculate

$$\begin{aligned}
&\frac{[\mathcal{L}\Theta + H](x-)}{r(x)} \frac{\psi'(x)}{p_c'(x)} \\
&= \frac{[\mathcal{L}\Theta + H](x_0-)}{r(x_0)} \frac{\psi'(x_0)}{p_c'(x_0)} + \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)} \frac{\psi'(\beta)}{p_c'(\beta)} \mathbf{1}_{[x_0,x[}(\beta) \\
&\quad + \int_{x_0}^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p_c'(s)} \mathbf{1}_{[x_0,x]\setminus\{\beta\}}(s)\, ds + 2\int_{x_0}^x \frac{[\mathcal{L}\Theta + H](s)\psi(s)}{\sigma^2(s)p_c'(s)}\, ds. \qquad (35)
\end{aligned}$$

The limits (54) in the Appendix and (18) in Assumption 5 imply that

$$\lim_{x_0 \downarrow 0} \frac{[\mathcal{L}\Theta + H](x_0)}{r(x_0)} \frac{\psi'(x_0)}{p_c'(x_0)} = 0.$$

In light of (15)–(17) in Assumption 5, we can use the monotone convergence theorem and this observation to pass to the limit $x_0 \downarrow 0$ in (35) to obtain

$$\begin{aligned}
\frac{[\mathcal{L}\Theta + H](x-)}{r(x)} \frac{\psi'(x)}{p_c'(x)} &= \int_0^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p_c'(s)} \mathbf{1}_{[0,x]\setminus\{\beta\}}(s)\, ds + 2q(x) \\
&\quad + \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)} \frac{\psi'(\beta)}{p_c'(\beta)} \mathbf{1}_{]0,x[}(\beta).
\end{aligned}$$

This calculation and (34) imply that

$$\begin{aligned}
\left(\frac{R'_{\mathcal{L}\Theta+H}}{\psi'}\right)'(x-) = -\frac{2r(x)p_c'(x)}{[\sigma(x)\psi'(x)]^2} \Bigg(&\int_0^x \frac{\mathcal{D}_r[\mathcal{L}\Theta + H](s)\psi'(s)}{r(s)p_c'(s)} \mathbf{1}_{[0,x]\setminus\{\beta\}}(s)\, ds \\
&+ \frac{[\mathcal{L}\Theta + H](\beta+) - [\mathcal{L}\Theta + H](\beta-)}{r(\beta)} \frac{\psi'(\beta)}{p_c'(\beta)} \mathbf{1}_{]0,x[}(\beta) \Bigg) \geq 0,
\end{aligned}$$

the inequality following thanks to (16) and (17) in Assumption 5. It follows that the function $R'_{\mathcal{L}\Theta+H}/\psi'$ is increasing, which establishes (33). □

We can now prove our main result.

Theorem 1 *Consider the stochastic control problem formulated in Section 2 and suppose that Assumptions 1–5 hold true. The optimal strategy takes the form of one of the following mutually exclusive cases, which are characterised by the point $\beta > 0$ appearing in (8) of Assumption 4, the point $x^* \geq 0$ appearing in (15) of Assumption 5, and the solution $\alpha > x^*$ of equation (26):*
(I) *If $x^* = 0$, then it is optimal to move and stop, and the optimal strategy is given by $\tau^* = 0$ and $Z_t^* = (\beta - x)^+ \mathbf{1}_{]0,\infty[}(t)$.*

(II) *If $x^* > 0$ and $\alpha < \beta$, then it is optimal to wait, move and stop, and the optimal strategy is given by $\tau^* = \inf\{t \geq 0 \mid \; X^0_t \geq \alpha\}$ and $Z^*_t = (\beta - \alpha \vee x)^+ \mathbf{1}_{]\tau^*, \infty[}(t)$.*
(III) *If $x^* > 0$ and $\alpha \geq \beta$, then it is optimal to wait and stop, and the optimal strategy is given by $\tau^* = \inf\{t \geq 0 \mid \; X^0_t \geq \alpha\}$ and $Z^* \equiv 0$.*
In the first case, the value function v identifies with the function Θ defined by (9), while, in cases (II) and (III), the value function v identifies with the function w constructed in Lemma 2.

Proof. Throughout the proof, we consider the solution w of the HJB equation (22) that is as in Lemma 1 or in Lemma 2, depending on whether $x^* = 0$ or not, and we fix any initial condition $x > 0$ and any admissible strategy $(Z, \tau) \in \mathcal{A}$. Also, we consider the local martingale defined by $M_T = \int_0^T e^{-\Lambda_t} \sigma(X_t) w'(X_t) \, dW_t$, and we let (τ_n) be any localising sequence of $(\mathcal{F}_t)$-stopping times such that $\tau_n \leq n$, for all $n \geq 1$. Using Itô's formula and the fact that $\Delta X_t = \Delta Z_t$, we calculate

$$
\begin{aligned}
e^{-\Lambda_{\tau \wedge \tau_n}} w(X_{\tau \wedge \tau_n +}) = w(x) &+ \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} \mathcal{L} w(X_t) \, dt + \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} w'(X_t) \, dZ^{\mathrm{c}}_t \\
&+ \sum_{0 \leq t \leq \tau \wedge \tau_n} e^{-\Lambda_t} \left[w(X_t + \Delta Z_t) - w(X_t) \right] + M_{\tau \wedge \tau_n},
\end{aligned}
$$

where the operator $\mathcal{L}$ is defined by (11) and Z^{c} is the continuous part of the process Z. In view of (4) and the fact that w satisfies the HJB equation (22), we can therefore see that

$$
\begin{aligned}
&\int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} H(X_t) \, dt - \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t + e^{-\Lambda_\tau} U(X_{\tau +}) \mathbf{1}_{\{\tau \leq \tau_n\}} \\
&\quad = w(x) + e^{-\Lambda_\tau} \left[U(X_{\tau +}) - w(X_{\tau +}) \right] \mathbf{1}_{\{\tau \leq \tau_n\}} - e^{-\Lambda_{\tau_n}} w(X_{\tau_n +}) \mathbf{1}_{\{\tau_n < \tau\}} \\
&\qquad + \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} \left[\mathcal{L} w(X_t) + H(X_t) \right] dt + \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} \left[w'(X_t) - K'(X_t) \right] dZ^{\mathrm{c}}_t \\
&\qquad + \sum_{0 \leq t \leq \tau \wedge \tau_n} e^{-\Lambda_t} \int_0^{\Delta Z_t} \left[w'(X_t + s) - K'(X_t + s) \right] ds + M_{\tau \wedge \tau_n} \\
&\quad \leq w(x) + e^{-\Lambda_{\tau_n}} w^-(X_{\tau_n +}) \mathbf{1}_{\{\tau_n < \tau\}} + M_{\tau \wedge \tau_n},
\end{aligned}
\tag{36}
$$

where $w^-(x) = -\min\{0, w(x)\}$. Taking expectation, we obtain

$$
\begin{aligned}
\mathbb{E}\biggl[\int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} H(X_t) \, dt - \int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t + e^{-\Lambda_\tau} U(X_{\tau +}) \mathbf{1}_{\{\tau < \tau_n\}} \biggr] \\
\leq w(x) + \mathbb{E}\left[e^{\Lambda_{\tau_n}} w^-(X_{\tau_n +}) \mathbf{1}_{\{\tau_n < \tau\}} \right].
\end{aligned}
\tag{37}
$$

The assumption that H/r is bounded, the fact that the process E defined by $E_t = -\exp(-\Lambda_t)$ is increasing and the dominated convergence theorem imply that

$$
\lim_{n \to \infty} \mathbb{E}\left[\int_0^{\tau \wedge \tau_n} e^{-\Lambda_t} H(X_t) \, dt \right] = \lim_{n \to \infty} \mathbb{E}\left[\int_0^{\tau \wedge \tau_n} \frac{H(X_t)}{r(X_t)} \, dE_t \right] = \mathbb{E}\left[\int_0^{\tau} e^{-\Lambda_t} H(X_t) \, dt \right], \tag{38}
$$

while the monotone convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau\wedge\tau_n} e^{-\Lambda_t} K'(X_t) \circ dZ_t\right] = \mathbb{E}\left[\int_0^{\tau} e^{-\Lambda_t} K'(X_t) \circ dZ_t\right].$$

The admissibility condition (7) and the monotone convergence theorem imply that

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau\leq\tau_n\}}\right] = \mathbb{E}\left[e^{-\Lambda_\tau} U(X_{\tau+}) \mathbf{1}_{\{\tau<\infty\}}\right]. \tag{39}$$

Also, since w^- is bounded, which follows from the inequality $w \geq \Theta$ and the fact that Θ is bounded from below (see (9) and the last claim in Assumption 4), we can use the dominated convergence theorem and Assumption 3 to obtain

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\Lambda_{\tau_n}} w^-(X_{\tau_n+}) \mathbf{1}_{\{\tau_n<\tau\}}\right] = 0.$$

In view of these observations, we can pass to the limit as $n \to \infty$ in (37) to obtain $J_x(Z, \tau) \leq w(x)$, which implies that $v(x) \leq w(x)$.

In each of the cases (I)–(III) in the theorem's statement, we can check that the strategy (Z^*, τ^*) is admissible in the sense of Definition 1 because the process Z^* has at most one jump and because $U(X^*_{\tau^*}) = U(\alpha) \in \mathbb{R}$. Furthermore, we can check that (36) and (37) both hold with equality, which, combined with (38)–(39), implies that $J_x(Z^*, \tau^*) = w(x)$. This conclusion and the inequality $v(x) \leq w(x)$, which we have established above, imply that $v(x) = w(x)$ and that (Z^*, τ^*) is optimal. □

4 Special cases

We now consider a number of special cases that arise when the uncontrolled system's dynamics are modelled by a geometric Brownian motion (Section 4.1) or by a mean-reverting square-root process such as the one in the Cox-Ingersoll-Ross interest rate model (Section 4.2). In these special cases, we assume that

$$H(x) = -\gamma, \quad K'(x) = \kappa \quad \text{and} \quad r(x) = \varrho \quad \text{for all } x > 0,$$

where $\gamma \geq 0$ and $\kappa, \varrho > 0$ are constants. Also, we assume that the terminal payoff function U is a power utility function, given by

$$U(x) = \frac{x^p}{p} \quad \text{for all } x > 0, \tag{40}$$

for some $p \in]0, 1[$, in which case, the function Θ defined by (9) takes the form

$$\Theta(x) = \begin{cases} \frac{1-p}{p}\kappa^{-\frac{p}{1-p}} + \kappa x, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta, \\ \frac{x^p}{p}, & \text{for } x \geq \kappa^{-\frac{1}{1-p}} \equiv \beta, \end{cases} \tag{41}$$

or the logarithmic utility function, namely

$$U(x) = \ln x \quad \text{for all } x > 0, \tag{42}$$

in which case,

$$\Theta(x) = \begin{cases} \kappa x - 1 - \ln \kappa, & \text{for } x < \kappa^{-1} \equiv \beta, \\ \ln x, & \text{for } x > \kappa^{-1} \equiv \beta. \end{cases} \tag{43}$$

It is straightforward to verify that these choices satisfy all of the conditions appearing in Assumptions 3 and 4.

4.1 Geometric Brownian motion

Suppose that X^0 is a geometric Brownian motion, so that

$$dX_t^0 = bX_t^0 \, dt + \sigma X_t^0 \, dW_t, \quad X_0^0 = x > 0,$$

for some constants b and $\sigma \neq 0$, and assume that $\varrho > b$. In this case, Assumptions 1 and 2 both hold true, and it is a standard exercise to verify that, if we choose $c = 1$, then

$$\varphi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p_c'(x) = x^{n+m-1}, \tag{44}$$

for some appropriate constants $m < 0 < n$. Also, it is well-known that $n > 1$ if and only if $\varrho > b$, in which case,

$$\mathbb{E}\left[\int_0^\infty e^{-\varrho t} X_t^0 \, dt\right] = \frac{x}{\varrho - b} < \infty. \tag{45}$$

Since there exists a constant $C_1 > 0$ such that $|\mathcal{L}\Theta(x)| \leq C_1(1+x)$ for all $x > 0$, whether Θ is given by (41) or (43), it follows from (44)–(45) that conditions (13) and (14) in Assumption 5 hold true. Also, we can use (57) in the Appendix with $F = H \equiv -\gamma$ to calculate $R_H = -\gamma/\varrho$, which implies that (19) in Assumption 5 is satisfied, whether Θ is given by (41) or (43).

In the following two subsections, we show that the choices for the problem data that we have made satisfy the remaining conditions (15)–(18) in Assumption 5, and we discuss the possible forms that the optimal strategy takes.

4.1.1 Power utility function *U*

If the terminal payoff function U is the power utility function given by (40), then we can check that the function Θ defined by (41) satisfies

$$[\mathcal{L}\Theta + H](x) = \left\{ \begin{array}{ll} -(\varrho - b)\kappa x - \varrho \frac{1-p}{p} \kappa^{-\frac{p}{1-p}} - \gamma, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta \\ -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right] x^p - \gamma, & \text{for } x > \kappa^{-\frac{1}{1-p}} \equiv \beta \end{array} \right\} < 0,$$

where the inequality follows from the assumption that $\varrho > b$ and the fact that $p \in]0,1[$. It follows that (15) is satisfied with $x^* = 0$ and that (18) holds true. We can also calculate

$$\begin{aligned}[\mathcal{L}\Theta + H](\beta-) &= -\left[\frac{\varrho}{p} - b\right] \kappa^{-\frac{p}{1-p}} - \gamma \\ &> -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right] \kappa^{-\frac{p}{1-p}} - \gamma = [\mathcal{L}\Theta + H](\beta+),\end{aligned}$$

which establishes (16), and

$$\mathcal{D}_r[\mathcal{L}\Theta + H](x) = \left\{\begin{array}{ll} -(\varrho - b)\kappa, & \text{for } x < \kappa^{-\frac{1}{1-p}} \equiv \beta \\ -\left[(1-p)\frac{1}{2}\sigma^2 + \frac{\varrho}{p} - b\right] p x^{-(1-p)}, & \text{for } x > \kappa^{-\frac{1}{1-p}} \equiv \beta \end{array}\right\} < 0,$$

which implies that (17) is also true. Finally, the fact that $x^* = 0$ puts us in the context of case (I) of Theorem 1, so the move-and-stop strategy is the optimal strategy.

4.1.2 Logarithmic utility function U

If the terminal payoff function U is the logarithmic utility function given by (42), then we can check that the function Θ defined by (43) satisfies

$$[\mathcal{L}\Theta + H](x) = \begin{cases} -(\varrho - b)\kappa x + \varrho \ln \kappa + \varrho - \gamma, & \text{for } x < \kappa^{-1} \equiv \beta, \\ -\varrho \ln x - \frac{1}{2}\sigma^2 + b - \gamma, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases}$$

$$[\mathcal{L}\Theta + H](\beta-) = \varrho \ln \kappa + b - \gamma > \varrho \ln \kappa - \frac{1}{2}\sigma^2 + b - \gamma = [\mathcal{L}\Theta + H](\beta+),$$

$$\mathcal{D}_r[\mathcal{L}\Theta + H](x) = \left\{\begin{array}{ll} -(\varrho - b)\kappa, & \text{for } x < \kappa^{-1} \equiv \beta \\ -\varrho x^{-1}, & \text{for } x > \kappa^{-1} \equiv \beta \end{array}\right\} < 0.$$

These calculations imply that (17)–(18) hold true, and that (15) is satisfied with

$$x^* = \begin{cases} \frac{\varrho + \varrho \ln \kappa - \gamma}{(\varrho - b)\kappa}, & \text{if } \varrho \ln \kappa < -b + \gamma, \\ \beta, & \text{if } -b + \gamma \le \varrho \ln \kappa \le \frac{1}{2}\sigma^2 - b + \gamma, \\ \exp\left(\frac{-\frac{1}{2}\sigma^2 + b - \gamma}{\varrho}\right), & \text{if } \frac{1}{2}\sigma^2 - b + \gamma < \varrho \ln \kappa. \end{cases}$$

In this case $x^* > 0$, so "waiting" belongs to the set of optimal tactics. To obtain the free-boundary point $\alpha > 0$ that determines the waiting region, we use (26) and (44) to calculate

$$q(\alpha) = \begin{cases} -\frac{(\varrho - b)\kappa}{\sigma^2 (1-m)\alpha^m} \left[\alpha - \frac{(1-m)(\varrho + \varrho \ln \kappa - \gamma)}{-m(\varrho - b)\kappa}\right], & \text{if } \alpha \le \kappa^{-1} \equiv \beta, \\ \frac{\varrho}{\sigma^2 m \alpha^m} \left[\ln \alpha + \frac{1}{m} - \frac{\varrho + \varrho \ln \kappa - \gamma}{\varrho} + (\alpha\kappa)^m \left(\ln \kappa - \frac{1}{m} - \frac{m(\varrho - b)}{(1-m)\varrho}\right)\right], & \text{if } \alpha > \kappa^{-1} \equiv \beta. \end{cases}$$

From these calculations, it follows that the unique solution $\alpha > 0$ of the equation $q(\alpha) = 0$ is strictly less than $\beta \equiv \kappa^{-1}$ if and only if

$$\varrho \ln \kappa < \frac{-m}{-m+1}(\varrho - b) - \varrho + \gamma. \tag{46}$$

In light of this analysis, we can see that the optimal strategy takes one of the following forms. If the parameter values are such that (46) is true, then we are in the context of case (II) of Theorem 1, and the wait-move-and-stop strategy is optimal. Otherwise, we are in the context of case (III) of Theorem 1, and the wait-and-stop strategy is optimal.

4.2 Mean-reverting square-root process

Suppose that X^0 is a mean-reverting square-root process, so that

$$dX_t^0 = \zeta(\vartheta - X_t^0)\, dt + \sigma\sqrt{X_t^0}\, dW_t, \quad X_0^0 = x > 0,$$

for some constants $\zeta, \vartheta, \sigma > 0$, and assume that $\zeta\vartheta - \frac{1}{2}\sigma^2 > 0$, which is a necessary and sufficient condition for X^0 to be non-explosive. In this context, Assumptions 1 and 2 are plainly satisfied. Also, the functions φ and ψ identify with confluent hypergeometric functions (see Jack, Johnson and Zervos [19, Section 5.2] for precise expressions) and ψ has exponential growth as x tends to ∞. Also, the calculation

$$\mathbb{E}\left[\int_0^\infty e^{-\varrho t} X_t^0\, dt\right] = \int_0^\infty e^{-\varrho t}\left[\vartheta + (x - \vartheta)e^{-\zeta t}\right] dt = \frac{\zeta\vartheta + \varrho x}{\varrho(\zeta + \varrho)} < \infty$$

is a standard exercise in financial mathematics. This calculation implies that Θ satisfies (14) in Assumption 5 because there exists a constant $C_2 > 0$ such that $|\mathcal{L}\Theta(x)| \leq C_2(1 + x)$ for all $x > 0$, whether Θ is given by (41) or (43) (see also (47) and (48) below). Such a bound of Θ also implies that (13) in Assumption 5 holds true because $\lim_{x\downarrow 0} \varphi(x) = \infty$ and $\psi(x)$ has exponential growth as x tends to ∞. Furthermore, the fact that $R_H \equiv -\gamma/\varrho$, which follows from (57), implies that (19) in Assumption 5 holds true, whether Θ is given by (41) or (43).

In the following two subsections, we verify that conditions (15)–(18) of Assumption 5 are satisfied as well, and we discuss the possible forms that the optimal strategy takes.

4.2.1 Power utility function U

If the terminal payoff function U is the power utility function given by (40), then we can check that the function Θ defined by (41) satisfies

$$[\mathcal{L}\Theta + H](x) = \begin{cases} -(\varrho+\zeta)\kappa x + \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}} - \gamma, & \text{for } x < \kappa^{\frac{-1}{1-p}} \equiv \beta, \\ \left[\zeta\vartheta - \frac{1}{2}\sigma^2(1-p)\right] x^{-(1-p)} - \left(\zeta + \frac{\varrho}{p}\right) x^p - \gamma, & \text{for } x > \kappa^{\frac{-1}{1-p}} \equiv \beta, \end{cases} \tag{47}$$

$$\begin{aligned} [\mathcal{L}\Theta + H](\beta-) &= -\left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} + \zeta\vartheta\kappa - \gamma \\ &> -\left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} + \left[\zeta\vartheta - \frac{1}{2}(1-p)\sigma^2\right]\kappa - \gamma = [\mathcal{L}\Theta + H](\beta+) \end{aligned}$$

$$\begin{aligned} &\mathcal{D}_r[\mathcal{L}\Theta + H](x) \\ &= \left.\begin{cases} -(\varrho+\zeta)\kappa, & \text{for } x < \kappa^{\frac{-1}{1-p}} \equiv \beta \\ -\left[\zeta\vartheta - \frac{1}{2}\sigma^2(1-p)\right](1-p)x^{-(2-p)} - \left(\zeta + \frac{\varrho}{p}\right) p x^{-(1-p)}, & \text{for } x > \kappa^{\frac{-1}{1-p}} \equiv \beta \end{cases}\right\} < 0, \end{aligned}$$

where the inequality follows from the assumption that $\zeta\vartheta - \frac{1}{2}\sigma^2 > 0$ and the fact that $p \in]0,1[$. These calculations imply immediately that (17)–(18) hold true. Also, these calculations imply that there exists a unique point x^* such that (15) in Assumption 5 is true. In particular,

$$\begin{aligned} x^* &= 0, \quad \text{if } \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}} \le \gamma, \\ x^* &\in]0,\beta[, \quad \text{if } \zeta\vartheta\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} < \gamma < \zeta\vartheta\kappa - (1-p)\frac{\varrho}{p}\kappa^{\frac{-p}{1-p}}, \\ x^* &= \beta, \quad \text{if } \left[\zeta\vartheta - \frac{1}{2}(1-p)\sigma^2\right]\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}} \le \gamma \le \zeta\vartheta\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}}, \\ x^* &> \beta, \quad \text{if } \gamma < \left[\zeta\vartheta - \frac{1}{2}(1-p)\sigma^2\right]\kappa - \left(\zeta + \frac{\varrho}{p}\right)\kappa^{\frac{-p}{1-p}}. \end{aligned}$$

In view of Lemmas 1 and 2, we conclude that, in the special case of the general problem that we consider here, the optimal strategy can take the form of any of the cases (I)–(III) of Theorem 1, depending on parameter values.

4.2.2 Logarithmic utility function U

If the terminal payoff function U is the logarithmic utility function given by (42), then we can calculate

$$[\mathcal{L}\Theta + H](x) = \begin{cases} -(\zeta+\varrho)\kappa x + \zeta\vartheta\kappa + \varrho\ln\kappa + \varrho - \gamma, & \text{for } x < \kappa^{-1} \equiv \beta, \\ \left[\zeta\vartheta - \frac{1}{2}\sigma^2\right] x^{-1} - \zeta - \varrho\ln x - \gamma, & \text{for } x > \kappa^{-1} \equiv \beta, \end{cases} \tag{48}$$

$$[\mathcal{L}\Theta + H](\beta-) = -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa - \gamma > -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa - \frac{1}{2}\sigma^2\kappa - \gamma = [\mathcal{L}\Theta + H](\beta+),$$

$$\mathcal{D}_r[\mathcal{L}\Theta + H](x) = \left.\begin{cases} -(\zeta+\varrho)\kappa, & \text{for } x < \kappa^{-1} \equiv \beta \\ -\left[\zeta\vartheta - \frac{1}{2}\sigma^2\right] x^{-2} - \varrho x^{-1}, & \text{for } x > \kappa^{-1} \equiv \beta \end{cases}\right\} < 0,$$

where the inequality follows from the assumption that $\zeta\vartheta - \frac{1}{2}\sigma^2 > 0$. These calculations imply immediately that (17)–(18) are satisfied and that there exists a unique point x^* such that (15) is true. In particular,

$$\begin{aligned} x^* &= 0, \quad \text{if } \zeta\vartheta\kappa + \varrho\ln\kappa + \varrho \le \gamma, \\ x^* &\in]0,\beta[, \quad \text{if } -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa < \gamma < \zeta\vartheta\kappa + \varrho\ln\kappa + \varrho, \\ x^* &= \beta, \quad \text{if } -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa - \frac{1}{2}\sigma^2\kappa \le \gamma \le -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa, \\ x^* &> \beta, \quad \text{if } \gamma < -\zeta + \zeta\vartheta\kappa + \varrho\ln\kappa - \frac{1}{2}\sigma^2\kappa. \end{aligned}$$

As in the previous case, the optimal strategy can be as in any of the cases (I)–(III) of Theorem 1, depending on parameter values.

Appendix: a second order linear ODE

In this section, we review a range of results regarding the solvability of a second order linear ODE on which part of our analysis has been based. All of the claims that we do not prove here are standard, and can be found in various forms in several references (e.g., see Borodin and Salminen [13, Chapter II]).

In the presence of Assumptions 1, 2 and 3, the general solution of the second-order linear homogeneous ODE

$$\mathcal{L}w(x) \equiv \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x) = 0, \quad \text{for } x > 0,$$

is given by

$$w(x) = A\varphi(x) + B\psi(x),$$

for some constants $A, B \in \mathbb{R}$. The functions φ and ψ are C^2,

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{49}$$
$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0, \tag{50}$$
$$\lim_{x \downarrow 0} \varphi(x) = \lim_{x \to \infty} \psi(x) = \infty. \tag{51}$$

In this context, φ and ψ are unique, modulo multiplicative constants. To simplify the notation we assume, without loss of generality, that $\varphi(c) = \psi(c) = 1$, where $c > 0$ is the same constant as the one that we used in the definition (6) of the scale function p_c. Also, these functions satisfy

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp_c'(x), \tag{52}$$

where $C := [\psi'(c) - \varphi'(c)] > 0$. Furthermore, the identity

$$\varphi''(x)\psi'(x) - \varphi'(x)\psi''(x) = \frac{2Cr(x)}{\sigma^2(x)} p_c'(x), \tag{53}$$

follows immediately from the fact that φ and ψ satisfy the ODE $\mathcal{L}f(x) = 0$ and (52).

Combining the inequalities

$$0 < \frac{\varphi(x)\psi'(x)}{Cp_c'(x)} < 1 \quad \text{and} \quad 0 < -\frac{\varphi'(x)\psi(x)}{Cp_c'(x)} < 1,$$

which follow from (49)–(50) and (52), with (51), we can see that

$$\lim_{x \downarrow 0} \frac{\psi'(x)}{p_c'(x)} = \lim_{x \to \infty} \frac{\varphi'(x)}{p_c'(x)} = 0. \tag{54}$$

Also, the calculation

$$\frac{d}{dx}\left(\frac{1}{p_c'(x)}\right) = \frac{2b(x)}{\sigma^2(x)p_c'(x)},$$

and the fact that ψ satisfies the ODE $\mathcal{L}w(x) = 0$, imply that

$$\frac{d}{dx}\left(\frac{\psi'(x)}{p_c'(x)}\right) = \frac{2}{\sigma^2(x)p_c'(x)}\left[\frac{1}{2}\sigma^2(x)\psi''(x) + b(x)\psi'(x)\right] = \frac{2r(x)\psi(x)}{\sigma^2(x)p_c'(x)}. \tag{55}$$

Now, we consider any Borel measurable function F such that

$$\int_0^x \frac{|F(s)|\psi(s)}{\sigma^2(s)p_c'(s)}\,ds + \int_x^\infty \frac{|F(s)|\varphi(s)}{\sigma^2(s)p_c'(s)}\,ds < \infty \quad \text{for all } x > 0.$$

A function F satisfies this integrability condition if and only if

$$\mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}|F(X_t^0)|\,dt\right] < \infty \tag{56}$$

for every initial condition $x > 0$ of the SDE (5). Given such F, the function R_F defined by

$$R_F(x) = \mathbb{E}\left[\int_0^\infty e^{-\Lambda_t}F(X_t^0)\,dt\right], \quad \text{for } x > 0, \tag{57}$$

admits the analytic representation

$$R_F(x) = \frac{2}{C}\varphi(x)\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds + \frac{2}{C}\psi(x)\int_x^\infty \frac{F(s)\varphi(s)}{\sigma^2(s)p_c'(s)}\,ds, \tag{58}$$

and satisfies the ODE $\mathcal{L}R_F(x) + F(x) = 0$, Lebesgue-a.e., as well as

$$\lim_{x\downarrow 0}\frac{|R_F(x)|}{\varphi(x)} = \lim_{x\to\infty}\frac{|R_F(x)|}{\psi(x)} = 0. \tag{59}$$

In view of (52)–(53) and (58), we can calculate

$$\left(\frac{R_F}{\psi}\right)'(x) = \frac{R_F'(x)\psi(x) - R_F(x)\psi'(x)}{\psi^2(x)} = -\frac{2p_c'(x)}{\psi^2(x)}\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds, \tag{60}$$

and we can check that the function R_F'/ψ' is absolutely continuous with derivative

$$\left(\frac{R_F'}{\psi'}\right)'(x) = \frac{4r(x)p_c'(x)}{[\sigma(x)\psi'(x)]^2}\int_0^x \frac{F(s)\psi(s)}{\sigma^2(s)p_c'(s)}\,ds - \frac{2F(x)}{\sigma^2(x)\psi'(x)}. \tag{61}$$

Noting that $-\mathcal{L}R_F = F$, we can see that, if $R_{-\mathcal{L}F}$ (resp., $R_{\mathcal{L}F}$) is defined as in (57)–(58) with $-\mathcal{L}F$ (resp., $\mathcal{L}F$) in the place of F, then

$$R_F = R_{-\mathcal{L}R_F} = -R_{\mathcal{L}R_F}. \tag{62}$$

Also, if Θ is a C^1 function with absolutely continuous first derivative that satisfies (13) and (14) then Θ satisfies (20).

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