A model for reversible investment capacity expansion*

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Abstract

We consider the problem of determining the optimal investment level that a firm should maintain in the presence of random price and/or demand fluctuations. We model market uncertainty by means of a geometric Brownian motion, and we consider general running payoff functions. Our model allows for capacity expansion as well as for capacity reduction, with each of these actions being associated with proportional costs. The resulting optimisation problem takes the form of a singular stochastic control problem that we solve explicitly. We illustrate our results by means of the so-called Cobb-Douglas production function. The problem that we study presents a model, the associated Hamilton-Jacobi-Bellman equation of which admits a classical solution that conforms with the underlying economic intuition but does not necessarily identify with the corresponding value function, which may be identically equal to ∞. Thus, our model provides a situation that highlights the need for rigorous mathematical analysis when addressing stochastic optimisation applications in finance and economics, as well as in other fields.

1 Introduction

We consider the problem of determining in a dynamical way the optimal capacity level of a given investment project operating within a random economic environment. In particular, we consider an investment project that yields payoff at a rate that is dependent on its

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installed capacity level and on an underlying economic indicator such as the price of or the demand for the project's unique output commodity, which we model by a geometric Brownian motion. The project's capacity level can be increased or decreased at any time and at given proportional costs. The objective is to determine the project's capacity level that maximises the associated expected, discounted payoff flow.

Irreversible capacity expansion models have attracted considerable interest in the literature, e.g., see Davis, Dempster, Sethi and Vermes [DDSV87] (see also Davis [D93]), Kobila [K93], Øksendal [Ø00], Wang [W03], Chiarolla and Haussmann [CH05], Bank [B05], and references therein. Recently, Bentolila and Bertola [BB90], and Abel and Eberly [AE96] considered models involving both expansion and reduction of a project's capacity level. These authors assume that the rate at which the project yields payoff is modelled by a constant elasticity Cobb-Douglas production function. Our model considers much more general running payoff functions that include the whole family of the Cobb-Douglas production functions as special cases, and allow for the situation where a running cost proportional to the project's installed capacity (reflecting, e.g., labour costs) is also included (see Examples 1 and 2). Also, Guo and Pham [GP05] consider a related partially reversible investment model with entry decisions and a general running payoff function. The model that these authors consider is fundamentally different from the ones considered by Bentolila and Bertola [BB90], and Abel and Eberly [AE96], or the one that we study here because, e.g., it is one-dimensional instead of two-dimensional.

Our analysis, which leads to results of an explicit analytic nature, involves the derivation of tight conditions for the project's value function to be finite. The fact that simple choices for the project's running payoff function lead to unique solutions to the associated free-boundary problem that conform with standard economic intuition but are associated with value functions that are identically equal to infinity presents a most interesting feature of our analysis (see Remark 3; also, note that this pathological situation does not arise in the context of the special cases studied by Bentolila and Bertola [BB90], and by Abel and Eberly [AE96]). Indeed, this possibility stresses the fact that treating optimisation models related to investment decision making in a "formal" way, which is often the case in the economics literature, can lead to erroneous conclusions and can suggest the adoption of potentially disastrous policies.

The paper is organised as follows. Section 2 is concerned with a rigorous formulation of the investment decision model that we study. In Section 3, we derive tight sufficient conditions, which conform with economic intuition, for the associated optimisation problem to possess a finite value function. Assumptions 1 and 2 summarise all of the assumptions that we make about the problem data in the paper. We also establish a number of estimates that we use in our subsequent analysis. Section 4 is concerned with the proof of a verification theorem that provides sufficient conditions for the value function of our control problem to be identified with a solution to the associated dynamic programming or Hamilton-Jacobi-Bellman equation. In Section 5, we solve the optimisation problem considered. Finally, we illustrate our results by a number of examples in Section 6.

2 Problem formulation

We fix a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by P-negligible sets, and carrying a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We denote by $\mathcal{A}$ the family of all càglàd, $(\mathcal{F}_t)$-adapted, increasing processes ξ such that $\xi_0 = 0$.

We consider an investment project that produces a given commodity, and we assume that the project's capacity, namely its rate of output, can be controlled at any given time. We denote by Y_t the project's capacity at time t, and we model cumulative capacity increases (resp., decreases) by a process $\xi^+ \in \mathcal{A}$ (resp., $\xi^- \in \mathcal{A}$). In particular, given any times $0 \leq s \leq t$, $\xi^+_{t+} - \xi^+_s$ and $\xi^-_{t+} - \xi^-_s$ are the total capacity increase and decrease, respectively, incurred by the project management's decisions during the time interval $[s, t]$. The project's capacity process Y is therefore given by

$$Y_t = y + \xi^+_t - \xi^-_t, \quad Y_0 = y \geq 0, \tag{1}$$

where $y \geq 0$ is the project's initial capacity. Note that project's capacity process Y is a finite variation process because it is the difference of two increasing processes. Also, the assumptions that the processes $\xi^\pm$ are càglàd and $\xi^\pm_0 = 0$ imply that $Y_0 = y$. We make the assumption that the project's management controls only the project's capacity level. Accordingly, we denote by Π_y the set of all admissible decision strategies, which is defined by

$$\Pi_y = \left\{ (\xi^+, \xi^-) : \ \xi^+, \xi^- \in \mathcal{A}, \text{ and } Y_t \geq 0, \text{ for all } t \geq 0 \right\}.$$

We assume that all randomness associated with the project's operation can be captured by a state process X that satisfies the SDE

$$dX_t = bX_t \, dt + \sqrt{2}\sigma X_t \, dW_t, \quad X_0 = x > 0, \tag{2}$$

for some constants b and σ. In practice, X_t can be the price of one unit of the output commodity or an economic indicator reflecting, e.g., the output commodity's demand, at time t.

To simplify the notation, we define

$$\mathcal{S} = \left\{ (x, y) \in \mathbb{R}^2 : \ x > 0, \ y \geq 0 \right\},$$

so that $\mathcal{S}$ is the set of all possible initial conditions.

With each decision policy $(\xi^+, \xi^-) \in \Pi_y$ we associate the performance criterion

$$J_{x,y}(\xi^+, \xi^-) = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t) \, dt - K^+ \int_{[0,\infty[} e^{-rt} \, d\xi^+_t - K^- \int_{[0,\infty[} e^{-rt} \, d\xi^-_t \right], \tag{3}$$

where $h : \mathcal{S} \to \mathbb{R}$ is a given function, and $r > 0$ and K^+, K^- are constants. Here, h models the running payoff resulting from the project's operation, and K^+ (resp., K^-) models the costs associated with increasing (resp., decreasing) the project's capacity level.

As it stands in (3), the performance index $J_{x,y}$ is not necessarily well-defined because the random variable inside the expectation may not be integrable or even well-defined. To address this issue, we define

$$U_T = \int_0^T e^{-rt} h(X_t, Y_t)\, dt - K^+ \int_{[0,T]} e^{-rt}\, d\xi_t^+ - K^- \int_{[0,T]} e^{-rt}\, d\xi_t^-, \quad \text{for } T \geq 0. \tag{4}$$

In the next section (see Lemma 4, in particular), we are going to impose assumptions on h such that U_T is well-defined, for all $T > 0$, and *either*

$$U_\infty = \lim_{T\to\infty} U_T \text{ exists in } \mathbb{R},\ P\text{-a.s.}, \quad \text{and} \quad U_\infty \in L^1(\Omega, \mathcal{F}, P), \tag{5}$$

in which case, we naturally define

$$J_{x,y}(\xi^+, \xi^-) = E\left[U_\infty\right], \tag{6}$$

as in (3), *or* there exists an $(\mathcal{F}_t)$-adapted process Z such that

$$U_T \leq Z_T,\ \text{for all } T \geq 0, \quad \text{and} \quad \limsup_{T\to\infty} E\left[Z_T\right] = -\infty, \tag{7}$$

in which case, we define

$$J_{x,y}(\xi^+, \xi^-) = -\infty. \tag{8}$$

The objective is to maximise the performance index $J_{x,y}$ thus defined over all admissible decision strategies $(\xi^+, \xi^-) \in \Pi_y$. The value function of the resulting optimisation problem is defined by

$$v(x, y) = \sup_{(\xi^+,\xi^-)\in\Pi_y} J_{x,y}(\xi^+, \xi^-). \tag{9}$$

3 Assumptions and preliminary estimates

The purpose of this section is to establish conditions on the problem data under which our control problem is well-posed and its value function is finite, and to prove certain estimates that we will need. Before we address these issues, we first discuss an ODE that will play an instrumental role in the solution of our control problem.

Every solution of the homogeneous ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - rw(x) = 0$$

is given by

$$u(x) = Ax^n + Bx^m,$$

for some $A, B \in \mathbb{R}$. Here, the constants $m < 0 < n$ are the solutions of the quadratic equation

$$\sigma^2\lambda^2 + (b - \sigma^2)\lambda - r = 0, \tag{10}$$

given by

$$m, n = \frac{-(b - \sigma^2) \pm \sqrt{(b - \sigma^2)^2 + 4\sigma^2 r}}{2\sigma^2}. \tag{11}$$

Now, let $k :]0, \infty[\to \mathbb{R}$ be any measurable function such that

$$E\left[\int_0^\infty e^{-rt}\, |k(X_t)|\, dt\right] < \infty, \quad \text{for all } x > 0. \tag{12}$$

This integrability condition is equivalent to

$$\int_0^x s^{-m-1}|k(s)|\, ds + \int_x^\infty s^{-n-1}|k(s)|\, ds < \infty, \quad \text{for all } x > 0,$$

and the function $R^{[k]} :]0, \infty[\to \mathbb{R}$ defined by

$$R^{[k]}(x) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}k(s)\, ds + x^n \int_x^\infty s^{-n-1}k(s)\, ds\right] \tag{13}$$

is a special solution to the non-homogeneous ODE

$$\sigma^2 x^2 u''(x) + bxu'(x) - rw(x) + k(x) = 0, \tag{14}$$

and satisfies

$$R^{[k]}(x) = E\left[\int_0^\infty e^{-rt}k(X_t)\, dt\right]. \tag{15}$$

Furthermore,

$$\text{if } k \text{ is increasing, then } R^{[k]} \text{ is increasing,} \tag{16}$$

and

$$\text{if } k \text{ is increasing, then } \lim_{x \downarrow 0} \frac{k(x)}{r} \geq 0 \;\Leftrightarrow\; \lim_{x \downarrow 0} R^{[k]}(x) \geq 0. \tag{17}$$

All of these results are proved in Knudsen, Meister and Zervos [KMZ98]. For future reference, we also note that, given any $\lambda \in \mathbb{R}$,

$$E\left[\int_0^\infty e^{-rt} X_t^\lambda \, dt\right] = x^\lambda \int_0^\infty e^{\left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right]t} E\left[e^{-\sigma^2\lambda^2 t + \sqrt{2}\sigma\lambda W_t}\right] dt$$
$$= \begin{cases} \infty, & \text{if } \lambda \leq m \text{ or } \lambda \geq n, \\ -x^\lambda / \left[\sigma^2\lambda^2 + (b-\sigma^2)\lambda - r\right], & \text{if } \lambda \in]m, n[. \end{cases} \tag{18}$$

We are going to need the following estimate that is related with the definitions above.

Lemma 1 *Given any $\lambda \in]0, n[$, there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$E\left[e^{-rt}\bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda e^{-\varepsilon_1 t} \quad \textit{and} \quad E\left[\sup_{t\geq 0} e^{-rt}\bar{X}_t^\lambda\right] \leq \frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_2} x^\lambda,$$

where $\bar{X}_t = \sup_{s\leq t} X_s$.

Proof. Since n is the positive solution of the quadratic equation (10), it follows that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

$$r - \varepsilon_1 > 0 \quad \text{and} \quad \sigma^2\lambda^2 + (b-\sigma^2)\lambda - (r - \varepsilon_1) = -\varepsilon_2.$$

Given such parameters, we define

$$V = \sup_{t\geq 0}\left[-\frac{\sigma^2\lambda^2 + \varepsilon_2}{\sqrt{2}|\sigma|\lambda} t + W_t\right],$$

we calculate

$$\begin{aligned} e^{-rt}\bar{X}_t^\lambda &= x^\lambda e^{-\varepsilon_1 t} e^{-(r-\varepsilon_1)t} \sup_{s\leq t} \exp\left((r-\varepsilon_1)s - (\sigma^2\lambda^2 + \varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\right) \\ &= x^\lambda e^{-\varepsilon_1 t} \sup_{s\leq t}\left[\exp\left(-(r-\varepsilon_1)(t-s)\right) \exp\left(-(\sigma^2\lambda^2+\varepsilon_2)s + \sqrt{2}\sigma\lambda W_s\right)\right] \\ &\leq x^\lambda e^{-\varepsilon_1 t} e^{\sqrt{2}|\sigma|\lambda V}, \end{aligned}$$

and we observe that

$$\sup_{t\geq 0} e^{-rt}\bar{X}_t^\lambda \leq x^\lambda e^{\sqrt{2}|\sigma|\lambda V}.$$

Since V is exponentially distributed with parameter $2\left(\sigma^2\lambda^2 + \varepsilon_2\right) / \left(\sqrt{2}|\sigma|\lambda\right)$ (see Karatzas and Shreve [KS88, Exercise 3.5.9]), the two bounds follow by a simple integration. □

The following assumptions on the data of the control problem formulated in Section 2 will ensure that the associated free-boundary problem has a unique solution that conforms with economical intuition.

Assumption 1 $r > 0$, and the function h is C^3 and satisfies

$$\int_0^x s^{-m-1}|h(s,y)|\, ds + \int_x^\infty s^{-n-1}|h(s,y)|\, ds < \infty,$$

for all $(x, y) \in \mathcal{S}$. If we define

$$H(x,y) = h_y(x,y), \quad \text{for } x, y > 0, \tag{19}$$

then, given any $y > 0$,

$$H_x(x,y) > 0, \text{ for all } x > 0, \quad \text{and} \quad \lim_{x\to\infty} H(x,y) = \infty, \tag{20}$$

and, given any $x > 0$,

$$H_y(x,y) < 0, \quad \text{for all } y > 0. \tag{21}$$

Also, $K^+ + K^- > 0$, and

$$\int_0^x s^{-m-1}\left[|H(s,y)| + |H_y(s,y)|\right] ds + \int_x^\infty s^{-n-1}\left[|H(s,y)| + |H_y(s,y)|\right] ds < \infty,$$

for all $x, y > 0$. □

It is worth observing that (20) and (21) in this assumption have a natural economic interpretation. Indeed, we can think of $H(x,y)\Delta y$ as the *additional* running payoff that we are faced with if we increase the project's capacity level from y to $y+\Delta y$, for small Δy, and the underlying state process X assumes the value x. In view of this observation, (20) reflects the idea that, given y, a small amount of extra capacity should be associated with increasing values of additional running payoff as the value of x, which, e.g., models the price of or the demand for the project's output commodity, is increasing. Similarly, (21) reflects the fact that, for a given value x of the underlying state process, the extra running payoff resulting from a small amount of additional capacity is decreasing as the level of the already installed capacity y increases. Also, the assumption that $K^+ + K^- > 0$, which is an indispensable one, is a most realistic one. Indeed, the inequality $K^+ + K^- < 0$ gives rise to the unrealistic scenario where the project's management can realise arbitrarily high profits by just sequentially increasing and then decreasing the project's capacity by the same amount sufficiently fast.

At this point, we should also observe that (20) and (21) in Assumption 1 exclude the special case that arises when the running payoff function h does not depend on the capacity level y, i.e., when $h(x,y) = \tilde{h}(x)$, for some function $\tilde{h}$. In this case, it is plainly optimal to never change the project's capacity level. However, the qualitative nature of this strategy is fundamentally different from any of the forms that our analysis allows for the optimal strategy to have, which is reflected in our assumptions.

The following additional assumptions will ensure that the value function of the control problem considered is finite and identifies with the solution of the associated Hamilton-Jacobi-Bellman equation. Apart from (26), which can be justified by straightforward economics considerations such as the ones discussed above, the conditions in the assumption are of a technical nature.

Assumption 2 $K^+ > 0$, and there exist constants

$$\alpha > 0, \ \beta \in]0,1[, \ \vartheta \in]0, K^+ \wedge (K^+ + K^-) \wedge n[\text{ and } C > 0, \tag{22}$$

where $n > 0$ is as in (11), such that

$$\frac{\alpha}{1-\beta} < n, \tag{23}$$

$$-C(1+y) \leq h(x,y) \leq C(1 + x^{n-\vartheta} + x^\alpha y^\beta) + r(K^+ - \vartheta)y, \quad \text{for all } (x,y) \in \mathcal{S}, \tag{24}$$

$$-C \leq H(x,y) \equiv h_y(x,y) \leq \beta C x^\alpha y^{-(1-\beta)} + r(K^+ - \vartheta), \quad \text{for all } x, y > 0. \tag{25}$$

Also,

$$h_x(x,y) \geq 0, \quad \text{for all } (x,y) \in \mathcal{S}. \tag{26}$$

□

Remark 1 Note that we could have replaced the upper bound in (25) by

$$H(x,y) \leq \begin{cases} C\left(1 + x^\alpha y^{-(1-\beta)}\right), & \text{for all } x > 0 \text{ and } y < y_1, \\ \beta C x^\alpha y^{-(1-\beta)} + r(K^+ - \vartheta), & \text{for all } x > 0 \text{ and } y \geq y_1, \end{cases}$$

for some constant $y_1 > 0$. Depending on the problem data, such a significant relaxation could result in optimal policies such as the one depicted by Figure 5 that would enrich qualitatively the class of optimal capacity control strategies (see also Example 3 in Section 6). However, we decided against such a relaxation because this would complicate both the presentation and the analysis of our results. □

Example 1 A choice for the running payoff function h that has been widely considered in the literature is the so-called Cobb-Douglas production function given by

$$h(x,y) = x^\alpha y^\beta, \quad \text{for some constants } \alpha > 0 \text{ and } \beta \in]0,1[. \tag{27}$$

It is straightforward to verify that this function satisfy all of our assumptions if and only if the parameters α and β satisfy the inequality (23). □

Example 2 A choice for the running payoff function h that is a variation of the Cobb-Douglas function and incorporates a running cost proportional to the project's installed capacity is given by

$$h(x,y) = (x+\eta)^\alpha (y+\zeta)^\beta - Ky, \quad \text{for some constants } \alpha, \beta, \eta, \zeta, K > 0. \tag{28}$$

This choice satisfies our assumptions if and only if

$$\alpha, \beta \in]0,1[, \quad \frac{\alpha}{1-\beta} < n \quad \text{and} \quad \beta \eta^\alpha \zeta^{-(1-\beta)} < K + rK^+. \tag{29}$$

To see this claim, fix any $\vartheta > 0$ such that

$$\alpha < n - \vartheta \quad \text{and} \quad \beta \eta^\alpha \zeta^{-(1-\beta)} < K + r(K^+ - \vartheta),$$

and observe that there exist constants $\Gamma_1, \Gamma_2, \Gamma_3 > 1$ such that

$$(x+\eta)^\alpha \leq \Gamma_1 (1 + x^\alpha), \quad (y+\zeta)^\beta \leq \Gamma_2 (1 + y^\beta) \quad \text{and} \quad \Gamma_1 \Gamma_2 y^\beta < \Gamma_3 + r(K^+ - \vartheta) y,$$

because $\alpha, \beta \in]0,1[$. In view of these inequalities, we can see that

$$\begin{aligned} h(x,y) &\leq \Gamma_1 \Gamma_2 \left(1 + x^\alpha + x^\alpha y^\beta\right) + \Gamma_1 \Gamma_2 y^\beta \\ &\leq \Gamma_1 \Gamma_2 \Gamma_3 \left(1 + x^\alpha + x^\alpha y^\beta\right) + r(K^+ - \vartheta) y, \end{aligned}$$

and check that Assumption 1, and (23), (24) and (26) in Assumption 2 all hold true. To verify (25) in Assumption 2, we note that, given a constant $C > 1$,

$$\frac{\partial}{\partial x} \left[H(x,y) - \beta C x^\alpha y^{-(1-\beta)} \right] < 0$$

is equivalent to

$$\left(\frac{x}{x+\eta} \right)^{1-\alpha} < C \left(\frac{y+\zeta}{y} \right)^{1-\beta},$$

which is true for all $x, y > 0$. It follows that (25) is satisfied if it is true for $x = 0$, i.e., if

$$\beta \eta^\alpha (y+\zeta)^{-(1-\beta)} \leq K + r(K^+ - \vartheta), \quad \text{for all } y \geq 0,$$

which is true when the associated parameters satisfy (29).

To see that, if the last inequality in (29) is not true, then the upper bound in (25) does not hold, we argue by contradiction. Indeed, if there are constants $C, \vartheta > 0$ such that (25) is satisfied, then we can pass to the limit as $x \downarrow 0$ to obtain

$$\beta \eta^\alpha (y+\zeta)^{-(1-\beta)} \leq K + r(K^+ - \vartheta), \quad \text{for all } y > 0.$$

However, this inequality cannot be true for all $y > 0$ if the last inequality in (29) above does not hold. $\square$

It is a straightforward exercise to show that the bounds in (24)–(25) imply the following estimates.

Lemma 2 *With reference to the notation in (13), the bounds provided by (24) and (25) in Assumption 2 imply that there exists a constant $C_1 > 0$ such that*

$$-C_1(1+y) \le R^{[h(\cdot,y)]}(x) \le C_1\left(1 + y + x^{n-\vartheta} + x^\alpha y^\beta\right), \quad \textit{for all } (x,y) \in \mathcal{S},$$
$$-C_1 \le R^{[H(\cdot,y)]}(x) \le C_1\left(1 + x^\alpha y^{-(1-\beta)}\right), \quad \textit{for all } (x,y) \in \mathcal{S}.$$

As we have remarked above, bounds such as the ones appearing in Assumption 2 are essential for the value function to be finite. Indeed, we can prove the following result.

Lemma 3 *Consider the control problem formulated in Section 2 that arises if the running payoff function h is defined by (27) in Example 1, and suppose that $\frac{\alpha}{1-\beta} > n > \alpha$. Then, under any well-posed definition of the performance index $J_{x,y}$ that is consistent with (3), $v(x,y) = \infty$, for every initial condition $(x,y) \in \mathcal{S}$.*

Proof. Consider the strategy defined by

$$\tilde{\xi}_t^+ = \bar{X}_t^{(n-\alpha)/\beta} \quad \text{and} \quad \tilde{\xi}_t^- = 0, \quad \text{for all } t \ge 0, \tag{30}$$

where $\bar{X}_t = \sup_{s \le t} X_s$. With regard to (18), we can see that this strategy is associated with

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha \tilde{Y}_t^\beta \, dt\right] \ge E\left[\int_0^\infty e^{-rt} X_t^n \, dt\right] = \infty. \tag{31}$$

Now, let us assume that $\frac{\alpha}{1-\beta} > n > \alpha$. If we define $\lambda = \frac{n-\alpha}{\beta} > 0$, then such an assumption implies $\lambda < n$. In view of this observation, we can use the first estimate in Lemma 1, the monotone convergence theorem and the integration by parts formula to see that the strategy given by (30) satisfies

$$\begin{aligned}
E\left[\int_{[0,\infty[} e^{-rt} \, d\tilde{\xi}_t^+\right] &= \lim_{T\to\infty} E\left[r\int_0^T e^{-rt}\tilde{\xi}_t^+ \, dt + e^{-rT}\tilde{\xi}_{T+}^+\right] \\
&= \lim_{T\to\infty}\left(r\int_0^T E\left[e^{-rt}\bar{X}_t^\lambda\right] dt + E\left[e^{-rT}\bar{X}_T^\lambda\right]\right) \\
&\le r\frac{\sigma^2\lambda^2 + \varepsilon_2}{\varepsilon_1\varepsilon_2} x^\lambda \\
&< \infty.
\end{aligned}$$

However, this calculation, (30) and (31) imply that

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha \tilde{Y}_t^\beta \, dt - \int_{[0,\infty[} e^{-rt} \, d\tilde{\xi}_t^+ - \int_{[0,\infty[} e^{-rt} \, d\tilde{\xi}_t^-\right]$$

is well-defined and equal to ∞, which proves the result. $\square$

We can now prove that our assumptions are sufficient for the optimisation problem considered to be well-posed and for its value function to be finite.

Lemma 4 *Suppose that the running payoff function h satisfies (24) in Assumption 2 and that $K^+, K^+ + K^- > 0$. Given any initial condition $(x,y) \in \mathcal{S}$, (5)–(8) provide a well-posed definition of the performance criterion $J_{x,y}$, and the following statements hold true:*

(a) *Given any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$, $J_{x,y}(\xi^+, \xi^-) \in \mathbb{R}$ if and only if*

$$E\left[\int_0^\infty e^{-rt} Y_t \, dt + K^+ \int_{[0,\infty[} e^{-rt} \, d\xi_t^+ + |K^-| \int_{[0,\infty[} e^{-rt} \, d\xi_t^-\right] < \infty. \tag{32}$$

(b) *Condition (32) implies*

$$\liminf_{T\to\infty} e^{-rT} E\left[Y_{T+}\right] = 0. \tag{33}$$

(c) *$v(x,y) \in \mathbb{R}$.*

Proof. Fix any initial condition $(x,y) \in \mathcal{S}$ and any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$. Since ξ^+, ξ^- are increasing càglàd processes with $\xi_0^+ = \xi_0^- = 0$, we can use the integration by parts formula to calculate

$$\begin{aligned} -K^+ \int_{[0,T]} e^{-rt} \, d\xi_t^+ - K^- \int_{[0,T]} e^{-rt} \, d\xi_t^- \\ = -r \int_0^T e^{-rt} \left[K^+ \xi_t^+ + K^- \xi_t^-\right] dt - e^{-rT} \left[K^+ \xi_{T+}^+ + K^- \xi_{T+}^-\right]. \end{aligned} \tag{34}$$

With regard to (1) and the inequality $K^+ + K^- > 0$, we can see that

$$-K^+ \xi_t^+ - K^- \xi_t^- \leq -K^+ \left(\xi_t^+ - \xi_t^-\right) = -K^+ Y_t + K^+ y, \tag{35}$$

which, combined with (34), implies

$$-K^+ \int_{[0,T]} e^{-rt} \, d\xi_t^+ - K^- \int_{[0,T]} e^{-rt} \, d\xi_t^- \leq -rK^+ \int_0^T e^{-rt} Y_t \, dt - e^{-rT} K^+ Y_{T+} + K^+ y. \tag{36}$$

However, this inequality and (24) in Assumption 2 imply that the random variables U_T defined by (4) satisfy

$$
\begin{aligned}
U_T &\leq K^+ y + \int_0^T e^{-rt} \left[h(X_t, Y_t) - rK^+ Y_t\right] dt \\
&\leq K^+ y + C \int_0^T e^{-rt} \left(1 + X_t^{n-\vartheta}\right) - \hat{Z}_T,
\end{aligned} \tag{37}
$$

where

$$
\hat{Z}_T = \int_0^T e^{-rt} \left[r\vartheta Y_t - C X_t^{\alpha} Y_t^{\beta}\right] dt, \quad \text{for } T \geq 0.
$$

With reference to (18), we note that

$$
\begin{aligned}
I_1(x) &:= E\left[C \int_0^\infty e^{-rt} \left(1 + X_t^{n-\vartheta}\right) dt\right] \\
&= \frac{C}{r} - \frac{C x^{n-\vartheta}}{\sigma^2(n-\vartheta)^2 + (b-\sigma^2)(n-\vartheta) - r} \in \,]0, \infty[.
\end{aligned} \tag{38}
$$

Now, suppose that the strategy strategy $(\xi^+, \xi^-) \in \Pi_y$ is associated with

$$
E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] = \infty. \tag{39}
$$

With regard to (23) in Assumption 2 and (18), we observe that

$$
I_2(x) := E\left[\int_0^\infty e^{-rt} X_t^{\alpha/(1-\beta)} \, dt\right] < \infty. \tag{40}
$$

Therefore, given any constant $\mu > 0$,

$$
E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right] \leq \mu I_2(x) < \infty. \tag{41}
$$

It follows that (39) is true if and only if

$$
E\left[\int_0^\infty e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right] = \infty. \tag{42}
$$

Now, let any $\mu > 0$ such that $r\vartheta - C\mu^{-(1-\beta)} > 0$, where the constants $\vartheta, C > 0$ and $\beta \in]0, 1[$ are as in Assumption 2, and note that

$$
\begin{aligned}
E\left[\hat{Z}_T\right] \geq &- C\mu^{\beta} E\left[\int_0^T e^{-rt} X_t^{\alpha/(1-\beta)} \mathbf{1}_{\left\{Y_t < \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right] \\
&+ \left(r\vartheta - C\mu^{-(1-\beta)}\right) E\left[\int_0^T e^{-rt} Y_t \mathbf{1}_{\left\{Y_t \geq \mu X_t^{\alpha/(1-\beta)}\right\}} \, dt\right].
\end{aligned}
$$

In view of (41)–(42) and the monotone convergence theorem, the right hand side of this inequality tends to ∞ as $T \to \infty$, which implies that $\lim_{T\to\infty} E[\hat{Z}_T] = \infty$. However, this conclusion, (37) and (38) imply that there exists a process Z such that (7) is satisfied and, therefore, $J_{x,y}(\xi^+, \xi^-) = -\infty$.

To proceed further, let us assume that

$$E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty, \tag{43}$$

which is necessary for condition (32) to be satisfied. Since Y is a finite variation process, its sample paths can have at most countable discontinuities. Using Fubini's theorem, we can see that this observation and (43) imply

$$\int_0^\infty e^{-rt} E\left[Y_{t+}\right] dt = E\left[\int_0^\infty e^{-rt} Y_{t+} \, dt\right] = E\left[\int_0^\infty e^{-rt} Y_t \, dt\right] < \infty,$$

which proves that (32) implies (33), and establishes part (b) of the lemma.

Now, using Hölder's inequality, we calculate

$$E\left[\int_0^\infty e^{-rt} X_t^\alpha Y_t^\beta \, dt\right] \leq I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t \, dt\right]\right)^\beta < \infty, \tag{44}$$

where $I_2(x)$ is given by (40). This inequality, (38), (43) and the bounds in (24) in Assumption 2 imply

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} \left|h(X_t, Y_t)\right| dt\right] &\leq E\left[\int_0^\infty e^{-rt} \left[C\left(1 + X_t^{n-\vartheta} + X_t^\alpha Y_t^\beta\right) + \left\{r(K^+ - \vartheta) \vee C\right\} Y_t\right] dt\right] \\ &< \infty, \end{aligned}$$

which, combined with the dominated convergence theorem, implies

$$\lim_{T\to\infty} E\left[\int_0^T e^{-rt} h(X_t, Y_t) \, dt\right] = E\left[\int_0^\infty e^{-rt} h(X_t, Y_t) \, dt\right] \in \mathbb{R}. \tag{45}$$

This observation gives rise to two possibilities. The first one is associated with the inequality

$$E\left[\int_{[0,\infty[} e^{-rt} \, d\xi_t^+ + \int_{[0,\infty[} e^{-rt} \, d\xi_t^-\right] < \infty.$$

In this case, $\lim_{T\to\infty} U_T$ exists, P-a.s., and belongs to $L^1(\Omega, \mathcal{F}, P)$, so $J_{x,y}(\xi^+, \xi^-)$ is finite and is given by (6). The second possibility is associated with

$$E\left[\int_{[0,\infty[} e^{-rt} \, d\xi_t^+ + \int_{[0,\infty[} e^{-rt} \, d\xi_t^-\right] = \infty,$$

which, combined with (43) and (33), implies

$$E\left[\int_{[0,\infty[} e^{-rt}\, d\xi_t^+\right] = E\left[\int_{[0,\infty[} e^{-rt}\, d\xi_t^-\right] = \infty. \tag{46}$$

If $K^- < 0$, then we can use (1) and the integration by parts formula to calculate

$$\begin{aligned} K^- \int_{[0,T]} e^{-rt}\, d\xi_t^- &= K^- \int_{[0,T]} e^{-rt}\, d\xi_t^+ + |K^-| \int_{[0,T]} e^{-rt}\, dY_t \\ &= K^- \int_{[0,T]} e^{-rt}\, d\xi_t^+ + r|K^-| \int_0^T e^{-rt} Y_t\, dt + |K^-| e^{-rT} Y_{T+} - |K^-| y \\ &\geq K^- \int_{[0,T]} e^{-rt}\, d\xi_t^+ - |K^-| y, \end{aligned}$$

which implies

$$E\left[K^+ \int_{[0,T]} e^{-rt}\, d\xi_t^+ + K^- \int_{[0,T]} e^{-rt}\, d\xi_t^-\right] \geq (K^+ + K^-) E\left[\int_{[0,T]} e^{-rt}\, d\xi_t^+\right] - |K^-| y.$$

This inequality, the assumption that $K^+ + K^- > 0$, (46) and the monotone convergence theorem imply

$$\lim_{T\to\infty} E\left[K^+ \int_{[0,T]} e^{-rt}\, d\xi_t^+ + K^- \int_{[0,T]} e^{-rt}\, d\xi_t^-\right] = \infty \tag{47}$$

On the other hand, if $K^- \geq 0$, then (46) plainly implies (47). However, (45) and (47) imply that $\lim_{T\to\infty} E[U_T] = -\infty$, so (7) is satisfied for $Z = U$ and $J_{x,y}(\xi^+, \xi^-) = -\infty$.

The analysis above establishes the well-posedness of the definition of $J_{x,y}$ given by (5)–(8) as well as parts (a) and (b) of the lemma. To prove part (c) of the lemma, we first note that the first bound in Lemma 2 and (18) imply

$$R^{[h(\cdot,y)]}(x) = E\left[\int_0^\infty e^{-rt} h(X_t, y)\, dt\right] \in \mathbb{R}.$$

However, this shows that our performance criterion is finite for the strategy that involves no capacity changes at any time, which proves that $v(x,y) > -\infty$. To show that $v(x,y) < \infty$, consider any admissible decision strategy $(\xi^+, \xi^-) \in \Pi_y$ such that $J_{x,y}(\xi^+, \xi^-) > -\infty$. With reference to (43) and (44),

$$\begin{aligned} &E\left[\int_0^\infty e^{-rt} \left[r\vartheta Y_t - C X_t^\alpha Y_t^\beta\right] dt\right] \\ &\qquad \geq r\vartheta E\left[\int_0^\infty e^{-rt} Y_t\, dt\right] - C I_2^{1-\beta}(x) \left(E\left[\int_0^\infty e^{-rt} Y_t\, dt\right]\right)^\beta \\ &\qquad \geq -\frac{(1-\beta) r\vartheta}{\beta} \left(\frac{\beta C}{r\vartheta}\right)^{1/(1-\beta)} I_2(x), \quad \text{for all } T > 0, \end{aligned} \tag{48}$$

the second inequality following because, given any constants $\kappa, \lambda > 0$ and $\beta \in]0,1[$,

$$\kappa Q - \lambda Q^\beta \geq -\frac{(1-\beta)\kappa}{\beta}\left(\frac{\beta\lambda}{\kappa}\right)^{1/(1-\beta)}, \quad \text{for all } Q \geq 0,$$

in particular, for $Q = E\left[\int_0^\infty e^{-rt} Y_t \, dt\right]$. However, (37), (38) and (48) imply

$$J_{x,y}(\xi^+, \xi^-) \leq I_1(x) + K^+ y + \frac{(1-\beta)r\vartheta}{\beta}\left(\frac{\beta C}{r\vartheta}\right)^{1/(1-\beta)} I_2(x),$$

which proves that $v(x,y) < \infty$ because the right hand side of this inequality is finite and independent of ξ^+ and ξ^-. □

4 The Hamilton-Jacobi-Bellman (HJB) equation

The problem described in the previous section has the structure of a singular stochastic control problem. With regard to standard theory of singular control, we expect that its value function can be identified with a solution $w : \mathcal{S} \to \mathbb{R}$ to the HJB quasi-variational inequalities

$$\max\bigl\{\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y), \\ w_y(x,y) - K^+, \; -w_y(x,y) - K^-\bigr\} = 0, \quad x, y > 0, \quad (49)$$

$$\max\bigl\{\sigma^2 x^2 w_{xx}(x,0) + bxw_x(x,0) - rw(x,0) + h(x,0), \; w_y(x,0) - K^+\bigr\} = 0, \quad x > 0, \quad (50)$$

where $w_y(x,0) := \lim_{y \downarrow 0} w_y(x,y)$.

To obtain some qualitative understanding of the origins of this equation, we observe that, at time 0, the project's management has to choose between three options. The first one is to wait for a short time Δt, and then continue optimally. With respect to Bellman's principle of optimality, this option is associated with the inequality

$$v(x,y) \geq E\left[\int_0^{\Delta t} e^{-rt} h(X_t, y) \, dt + e^{-r\Delta t} v(X_{\Delta t}, y)\right].$$

Applying Itô's formula to the second term in the expectation, and dividing by Δt before letting $\Delta t \downarrow 0$, we obtain

$$\sigma^2 x^2 v_{xx}(x,y) + bxv_x(x,y) - rv(x,y) + h(x,y) \leq 0. \quad (51)$$

The second option is to increase capacity immediately by $\varepsilon > 0$, and then continue optimally. This action is associated with the inequality

$$v(x,y) \geq v(x, y+\varepsilon) - K^+\varepsilon.$$

Rearranging terms and letting $\varepsilon \downarrow 0$, we obtain

$$v_y(x, y) - K^+ \leq 0. \tag{52}$$

Assuming that $y > 0$, the final option is to decrease capacity immediately by $\varepsilon > 0$, and then continue optimally. This option yields the inequality

$$v(x, y) \geq v(x, y - \varepsilon) - K^- \varepsilon,$$

which, in the limit as $\varepsilon \downarrow 0$, implies

$$-v_y(x, y) - K^- \leq 0. \tag{53}$$

Since these three are the only options available, we expect that one of them should be optimal, so that one of the inequalities (51)–(53) should hold with equality if $y > 0$, while, one of the inequalities (51)–(52) should hold with equality if $y = 0$. However, this observation combined with (51)–(53) implies that the value function v should identify with a solution w to (49)–(50).

The following result is concerned with *sufficient* conditions under which the value function v of the control problem considered identifies with a solution to (49)–(50). We impose some of these conditions, (58)–(59) in particular, which are not standard in similar "verification" theorems, with a hindsight relative to our analysis in the next section.

Theorem 5 *Suppose that the running payoff function h satisfies (24) in Assumption 2 and that $K^+, K^+ + K^- > 0$. Also, assume that the HJB equation (49)–(50) has a C^2 solution $w : \mathcal{S} \to \mathbb{R}$ such that*

$$-C_2 \left(1 + y + x^{\alpha/(1-\beta)}\right) \leq w(x, y), \quad \textit{for all } (x, y) \in \mathcal{S}, \tag{54}$$

for some constant $C_2 > 0$. The following statements hold true:

(a) *$v(x, y) \leq w(x, y)$, for all initial conditions $(x, y) \in \mathcal{S}$.*

(b) *Given any initial condition $(x, y) \in \mathcal{S}$, suppose that there exists a decision strategy $(\xi^{o+}, \xi^{o-}) \in \Pi_y$ such that, if Y^o is the associated capacity process, then*

$$(X_t, Y_t^o) \in \left\{(x, y) \in \mathcal{S} : \ \sigma^2 x^2 w_{xx}(x, y) + bx w_x(x, y) - rw(x, y) + h(x, y) = 0\right\}, \tag{55}$$

Lebesgue-a.e., P-a.s.,

$$\int_{[0,T]} e^{-rs} \left[w_y(X_t, Y_t) - K^+\right] d\xi_s^{o+} = 0, \quad \textit{for all } T \geq 0, \ P\textit{-a.s.}, \tag{56}$$

$$\int_{[0,T]} e^{-rs} \left[w_y(X_t, Y_t) + K^-\right] d\xi_s^{o-} = 0, \quad \textit{for all } T \geq 0, \ P\textit{-a.s.}, \tag{57}$$

and

$$Y_t^{\mathrm{o}} + X_t^\alpha (Y_t^{\mathrm{o}})^\beta + \xi_t^{\mathrm{o}+} \le C_3(y)\left(1 + \bar{X}_t^{n-\varepsilon_3}\right), \quad \text{for all } t \ge 0,\ P\text{-a.s.}, \tag{58}$$

$$w(X_t, Y_t^{\mathrm{o}}) \le C_3(y)\left(1 + \bar{X}_t^{n-\varepsilon_3}\right), \quad \text{for all } t \ge 0,\ P\text{-a.s.}, \tag{59}$$

where $\bar{X}_t = \sup_{s\le t} X_s$, $\varepsilon_3 \in]0, \vartheta[$ *is a constant, and* $C_3(y) > 0$ *is a constant depending on the initial condition* y *only. Then* $v(x,y) = w(x,y)$ *and* $(\xi^{\mathrm{o}+}, \xi^{\mathrm{o}-})$ *is the optimal strategy.*

Proof. (a) Fix any initial condition (x,y) and any admissible strategy $(\xi^+, \xi^-) \in \Pi_y$ such that $J_{x,y}(\xi^+, \xi^-) > -\infty$, so that $J_{x,y}(\xi^+, \xi^-) = E[U_\infty]$ (see (5)–(6)). Using Itô's formula and the fact that X has continuous sample paths, we obtain

$$\begin{aligned} e^{-rT} w(X_T, Y_{T+}) = w(x,y) &+ \int_0^T e^{-rt}\left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t)\right] dt \\ &+ \int_{[0,T]} e^{-rt}\left[w_y(X_t, Y_t)\, d\xi_t^+ - w_y(X_t, Y_t)\, d\xi_t^-\right] + M_T \\ &+ \sum_{0\le t\le T} e^{-rt}\left[w(X_t, Y_{t+}) - w(X_t, Y_t) - w_y(X_t, Y_t)\Delta Y_t\right], \end{aligned}$$

where

$$M_T = \sqrt{2}\sigma \int_0^T e^{-rt} X_t w_x(X_t, Y_t)\, dW_t, \quad T \ge 0. \tag{60}$$

Recalling the definition of U_T in (4), this implies

$$\begin{aligned} &U_T + e^{-rT} w(X_T, Y_{T+}) \\ &\quad = w(x,y) + \int_0^T e^{-rt}\left[\sigma^2 X_t^2 w_{xx}(X_t, Y_t) + bX_t w_x(X_t, Y_t) - rw(X_t, Y_t) + h(X_t, Y_t)\right] dt \\ &\qquad + \int_{[0,T]} e^{-rt}\left[w_y(X_t, Y_t) - K^+\right] d\left(\xi^+\right)_t^{\mathrm{c}} + \int_{[0,T]} e^{-rt}\left[-w_y(X_t, Y_t) - K^-\right] d\left(\xi^-\right)_t^{\mathrm{c}} \\ &\qquad + M_T + \sum_{0\le t\le T} e^{-rt}\left[w(X_t, Y_{t+}) - w(X_t, Y_t) - K^+\Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t > 0\}} \\ &\qquad + \sum_{0\le t\le T} e^{-rt}\left[w(X_t, Y_{t+}) - w(X_t, Y_t) + K^-\Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t < 0\}}. \end{aligned}$$

Observing that

$$\begin{aligned} &\left[w(X_t, Y_{t+}) - w(X_t, Y_t) - K^+\Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t > 0\}} = \mathbf{1}_{\{\Delta Y_t > 0\}} \int_0^{\Delta Y_t} \left[w_y(X_t, Y_t + u) - K^+\right] du, \\ &\left[w(X_t, Y_{t+}) - w(X_t, Y_t) + K^-\Delta Y_t\right] \mathbf{1}_{\{\Delta Y_t < 0\}} \\ &\qquad\qquad = \mathbf{1}_{\{\Delta Y_t < 0\}} \int_0^{|\Delta Y_t|} \left[-w_y\left(X_t, Y_t - |\Delta Y_t| + u\right) - K^-\right] du, \end{aligned}$$

we can see that, since w satisfies the HJB equation (49)–(50),

$$U_T + e^{-rT} w(X_T, Y_{T+}) \le w(x,y) + M_T. \tag{61}$$

Now, in view of (36) and the assumption $K^+ > 0$,

$$-e^{-rT} Y_{T+} \ge -\int_{[0,T]} e^{-rt}\, d\xi_t^+ - \frac{|K^-|}{K^+} \int_{[0,T]} e^{-rt}\, d\xi_t^- - y,$$

which, combined with assumption (54), implies

$$e^{-rT} w(X_T, Y_{T+}) \ge -C_{21} \left(1 + \int_{[0,T]} e^{-rt}\, d\xi_t^+ + \int_{[0,T]} e^{-rt}\, d\xi_t^- + e^{-rT} X_T^{\alpha/(1-\beta)} \right),$$

for some constant $C_{21} = C_{21}(y) > 0$. Combining this inequality with

$$\int_0^T e^{-rt} h(X_t, Y_t)\, dt \ge -C \int_0^T e^{-rt} Y_t\, dt - \frac{C}{r} \left(1 - e^{-rT}\right),$$

which follows from (24) in Assumption 2, we can see that (61) implies

$$\inf_{T \ge 0} M_T \ge -C_{22} \left(1 + \int_0^\infty e^{-rt} Y_t\, dt + \int_{[0,\infty[} e^{-rt}\, d\xi_t^+ + \int_{[0,\infty[} e^{-rt}\, d\xi_t^- + \sup_{T \ge 0} e^{-rT} \bar{X}_T^{\alpha/(1-\beta)} \right),$$

where $C_{22} = C_{22}(x,y) > 0$ is a constant and $\bar{X}_t = \sup_{s \le t} X_s$. Recalling the assumption that $\frac{\alpha}{1-\beta} \in]0, n[$, we can see that the second bound in Lemma 1 and (32) in Lemma 4 imply that the random variable on the right hand side of this inequality has finite expectation. It follows that the stochastic integral M defined by (60) is a supermartingale, and therefore, $E\left[M_T\right] \le 0$, for all $T > 0$. Taking expectations in (61), we therefore obtain

$$E\left[U_T\right] \le w(x,y) + e^{-rT} E\left[-w(X_T, Y_{T+})\right]. \tag{62}$$

Furthermore, since

$$U_T \ge -C_{22} \left(1 + \int_0^\infty e^{-rt} Y_t\, dt + \int_{[0,\infty[} e^{-rt}\, d\xi_t^+ + \int_{[0,\infty[} e^{-rt}\, d\xi_t^- \right), \quad \text{for all } T \ge 0,$$

and the random variable on the right hand side of this inequality has finite expectation, Fatou's lemma implies

$$J_{x,y}(\xi^+, \xi^-) \le \liminf_{T \to \infty} E\left[U_T\right], \tag{63}$$

while (54) implies

$$\begin{aligned} \liminf_{T\to\infty} e^{-rT} E\left[-w(X_T, Y_{T+})\right] \le{}& \lim_{T\to\infty} e^{-rT} C_2 + C_2 \liminf_{T\to\infty} e^{-rT} E\left[Y_{T+}\right] \\ &+ C_2 \lim_{T\to\infty} e^{-rT} E\left[\bar{X}_T^{\alpha/(1-\beta)}\right] \\ ={}& 0, \end{aligned} \tag{64}$$

the equality being true thanks to the first bound in Lemma 1 and (33). However, (62)–(64) imply that $J_{x,y}(\xi^+, \xi^-) \leq w(x, y)$, which establishes part (a) of the theorem.

(b) If (ξ^{o+}, ξ^{o-}) is as in the statement of the theorem, then we can see that the monotone convergence theorem, the integration by parts formula, (58) and the first estimate in Lemma 1 imply

$$
\begin{aligned}
E\bigg[\int_0^\infty & e^{-rt} Y_t^o \, dt + \int_{[0,\infty[} e^{-rt} \, d\xi_t^{o+}\bigg] \\
&= \lim_{T\to\infty} E\left[\int_0^T e^{-rt} Y_t^o \, dt + r \int_0^T e^{-rt} \xi_t^{o+} \, dt + e^{-rT} \xi_{T+}^{o+}\right] \\
&\leq (1+r) C_3(y) \left(\frac{1}{r} + \int_0^\infty e^{-rt} E\left[\bar{X}_t^{n-\varepsilon_3}\right] dt\right) + \lim_{T\to\infty} e^{-rT} E\left[\bar{X}_T^{n-\varepsilon_3}\right] \\
&< \infty,
\end{aligned}
$$

which, combined with (1), implies that (32) in Lemma 4 is satisfied, and, therefore,

$$
J_{x,y}(\xi^{o+}, \xi^{o-}) = E\left[\lim_{T\to\infty} U_T^o\right] \in \mathbb{R}, \tag{65}
$$

where U^o is defined as in (4). Furthermore, we can verify that (61) holds with equality, i.e.,

$$
U_T^o + e^{-rT} w(X_T, Y_{T+}^o) = w(x, y) + M_T^o, \tag{66}
$$

where the stochastic integral M^o is defined as in (60). In view of (24) in Assumption 2 and (58), there exist constants $C_{31} > 0$ and $C_{32} = C_{32}(y) > 0$ such that

$$
\begin{aligned}
\sup_{T\geq 0} \int_0^T e^{-rt} h(X_t, Y_t^o) \, dt &\leq C_{31} \left(1 + \int_0^\infty e^{-rt} \left[X_t^{n-\vartheta} + X_t^\alpha (Y_t^o)^\beta + Y_t^o\right] dt\right) \\
&\leq C_{32} \left(1 + \int_0^\infty e^{-rt} \bar{X}_t^{n-\varepsilon_3} \, dt\right).
\end{aligned} \tag{67}
$$

With reference to (1), the assumption $K^+ + K^- > 0$, the integration by parts formula and (58), we can see that there exists a constant $C_{33} = C_{33}(y) > 0$ such that

$$
\begin{aligned}
\sup_{T\geq 0}\bigg(-K^+ \int_{[0,T]} & e^{-rt} \, d\xi_t^{o+} - K^- \int_{[0,T]} e^{-rt} \, d\xi_t^{o-}\bigg) \\
&\leq \sup_{T\geq 0} K^- \left(\int_{[0,T]} e^{-rt} \, d\xi_t^{o+} - \int_{[0,T]} e^{-rt} \, d\xi_t^{o-}\right) \\
&\leq |K^-| \sup_{T\geq 0} \int_{[0,T]} e^{-rt} \, dY_t^o \\
&\leq |K^-| \sup_{T\geq 0} e^{-rT} Y_{T+}^o + r|K^-| \int_0^\infty e^{-rt} Y_t^o \, dt \\
&\leq |K^-| \sup_{T\geq 0} e^{-rT} Y_{T+}^o + C_{33} \left(1 + \int_0^\infty e^{-rt} \bar{X}_t^{n-\varepsilon_3} \, dt\right).
\end{aligned} \tag{68}
$$

Moreover, (58)–(59) imply

$$\sup_{T\geq 0} e^{-rT} Y^{\mathrm{o}}_{T+} + \sup_{T\geq 0} e^{-rT} w(X_T, Y^{\mathrm{o}}_{T+}) \leq 2C_3(y) \left(1 + \sup_{T\geq 0} e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right). \tag{69}$$

Now, (18) implies

$$E\left[\int_0^\infty e^{-rt} \bar{X}_t^{n-\varepsilon_3}\, dt\right] < \infty, \tag{70}$$

while the second estimate in Lemma 1 implies

$$E\left[\sup_{T\geq 0} e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right] < \infty. \tag{71}$$

However, (66) and the estimates (67)–(71) imply that $E\left[\sup_{T\geq 0} M^{\mathrm{o}}_T\right] < \infty$, which proves that the stochastic integral M^{o} is a submartingale. Taking expectations in (66), we therefore obtain

$$E\left[U^{\mathrm{o}}_T\right] \geq w(x,y) + e^{-rT} E\left[-w(X_T, Y^{\mathrm{o}}_T)\right]. \tag{72}$$

Furthermore, the estimates (67)–(71) imply that the random variables U^{o}_T, indexed by $T \geq 0$, are all bounded from above by a random variable with finite expectation. This observation, (65) and Fatou's lemma imply

$$J_{x,y}(\xi^{\mathrm{o}+}, \xi^{\mathrm{o}-}) \geq \limsup_{T\to\infty} E\left[U^{\mathrm{o}}_T\right]. \tag{73}$$

Finally, (59) and the first estimate in Lemma 1 imply

$$\begin{aligned}\limsup_{T\to\infty} e^{-rT} E\left[-w(X_T, Y^{\mathrm{o}}_T)\right] &\geq -\lim_{T\to\infty} C_3(y) \left(e^{-rT} + E\left[e^{-rT} \bar{X}_T^{n-\varepsilon_3}\right]\right) \\ &= 0,\end{aligned}$$

which, combined with (72) and (73), implies $J_{x,y}(\xi^{\mathrm{o}+}, \xi^{\mathrm{o}-}) \geq w(x,y)$. However, this inequality and part (a) of this theorem complete the proof. □

5 The solution of the control problem

We can now derive an explicit solution to the control problem formulated in Section 2 by constructing an appropriate solution w to the HJB equation (49)–(50). With respect to the heuristic arguments in Section 4 that led to the derivation of this equation, we start by conjecturing that the optimal strategy is characterised by three disjoint open subsets of $]0,\infty[\,\times\mathbb{R}_+$: the "wait" region $\mathcal{W}$ where (51) holds with equality, the "investment" region

$\mathcal{I}$ where (52) holds with equality, and the "disinvestment" region $\mathcal{D}$ where (53) holds with equality. Also, we conjecture that each of the regions $\mathcal{W}$, $\mathcal{I}$, $\mathcal{D}$ is connected. In particular, we expect that, depending on the problem data, the optimal strategy can take any of the forms depicted by Figures 1–4. Note that one can envisage other possibilities such as the one depicted by Figure 5. However, our assumptions do not allow for the optimality of such other cases under any admissible choice of the problem data (see also Remark 1 in Section 3 and Example 3 in Section 6).

With regard to Figures 1–4, we denote by $\mathbb{F}$ and $\mathbb{G}$ the boundaries separating the regions $\mathcal{D}$, $\mathcal{W}$ and $\mathcal{W}$, $\mathcal{I}$, respectively, so that

$$\mathbb{F} = \overline{\mathcal{D}} \cap \overline{\mathcal{W}} \quad \text{and} \quad \mathbb{G} = \overline{\mathcal{W}} \cap \overline{\mathcal{I}},$$

where $\overline{\mathcal{W}}$, $\overline{\mathcal{I}}$ and $\overline{\mathcal{D}}$ are the closures of $\mathcal{W}$, $\mathcal{I}$ and $\mathcal{D}$ in $\mathbb{R}_+^2$, respectively. Furthermore, we define

$$y^* = \inf \{ y \geq 0 : \text{ there exists } x > 0 \text{ such that } (x, y) \in \mathbb{F} \}, \tag{74}$$

with the usual convention that $\inf \emptyset = \infty$. We will prove that

$$\begin{array}{c} \text{there exists an increasing function } G : \, [0, \infty[\to \, [0, \infty[\text{ such that} \\ \mathbb{G} = \{ (G(y), y) : \; y \geq 0 \}, \end{array} \tag{75}$$

and, if $y^* < \infty$, then

$$\begin{array}{c} \text{there exists an increasing function } F : \, [y^*, \infty[\to [0, \infty[\text{ such that} \\ \mathbb{F} \cap (\mathbb{R}_+ \setminus \{0\})^2 = \{ (F(y), y) : \; y > y^* \}. \end{array} \tag{76}$$

Given such a characterisation of $\mathbb{F}$ and $\mathbb{G}$,

$$\begin{aligned} \overline{\mathcal{W}} = & \left\{ (x, y) \in \mathbb{R}_+^2 : \; y \leq y^* \text{ and } x \in [0, G(y)] \right\} \\ & \cup \left\{ (x, y) \in \mathbb{R}_+^2 : \; y > y^* \text{ and } x \in [F(y), G(y)] \right\}, \\ \overline{\mathcal{I}} = & \left\{ (x, y) \in \mathbb{R}_+^2 : \; G(y) \leq x \right\}, \end{aligned}$$

while, if $y^* < \infty$, then

$$\overline{\mathcal{D}} = \left\{ (x, y) \in \mathbb{R}_+^2 : \; y \geq y^* \text{ and } x \in \, [0, F(y)] \right\}.$$

In view of this structure, it is worth noting that, if $y^* = 0$ and $0 < F(0) < G(0)$ (see Figure 3), then $\{ (x, 0) : \; x < G(0) \} \subset \mathcal{W}$, so that the segment $]0, F(0)]$ is part of the "wait" region $\mathcal{W}$.

Inside the region $\mathcal{W}$, w satisfies the differential equation

$$\sigma^2 x^2 w_{xx}(x, y) + bx w_x(x, y) - rw(x, y) + h(x, y) = 0. \tag{77}$$

In view of the discussion regarding the solvability of (14) in Section 3, every solution to this equation is given by

$$w(x,y) = A(y)x^n + B(y)x^m + R(x,y), \tag{78}$$

for some functions A and B. Here, the constants $m < 0 < n$ are given by (11), while the function $R \equiv R^{[h(\cdot,y)]}$ is given by

$$R(x,y) = \frac{1}{\sigma^2(n-m)}\left[x^m \int_0^x s^{-m-1}h(s,y)\,ds + x^n \int_x^\infty s^{-n-1}h(s,y)\,ds\right]. \tag{79}$$

For $y \in [0, y^*] \cap \mathbb{R}$, we must have $B(y) = 0$. This choice is supported by the heuristic observation that, for fixed capacity level $y \geq 0$, the problem's value function should remain bounded as the value x of the underlying state process tends to 0. Also, it eventually turns out that (58)–(59) in the verification Theorem 5 cannot be satisfied if $B(y) \neq 0$. To determine $A(y)$ and $G(y)$ when $y \in [0, y^*] \cap \mathbb{R}$, we postulate that $w(\cdot, y)$ is C^2 at the free-boundary point $G(y)$. In particular, we postulate that

$$\lim_{x\uparrow G(y)} w_y(x,y) = \lim_{x\downarrow G(y)} w_y(x,y) \quad \text{and} \quad \lim_{x\uparrow G(y)} w_{yx}(x,y) = \lim_{x\downarrow G(y)} w_{yx}(x,y). \tag{80}$$

Since w satisfies

$$w_y(x,y) = K^+, \quad \text{for } (x,y) \in \mathcal{I}, \tag{81}$$

which implies

$$w_{xy}(x,y) = 0, \quad \text{for } (x,y) \in \mathcal{I}, \tag{82}$$

this requirement yields the system of equations

$$A'(y)G^n(y) = K^+ - R_y(G(y), y), \tag{83}$$

$$A'(y)G^n(y) = -\frac{1}{n}G(y)R_{xy}(G(y), y). \tag{84}$$

Equating the right-hand sides of these equations and using the definition of R in (79), we obtain

$$G^m(y)\int_0^{G(y)} s^{-m-1}H(s,y)\,ds - \sigma^2 n K^+ = 0, \tag{85}$$

where H is the function defined by (19). Using the identity $\sigma^2 mn = -r$, which follows from the definition of the constants m, n in (11), we can see that $G(y)$ should satisfy

$$q(G(y), y) = 0, \tag{86}$$

where

$$q(x,y) = \int_0^x s^{-m-1}\left[H(s,y) - rK^+\right] ds, \quad (x,y) \in \mathcal{S}. \tag{87}$$

Furthermore, adding (83) and (84) side by side and using (79) and (85), we obtain

$$\begin{aligned} A'(y) &= \frac{1}{2} G^{-n}(y) \left[K^+ - R_y(G(y),y) - \frac{1}{n} G(y) R_{xy}(G(y),y) \right] \\ &= -\frac{1}{\sigma^2(n-m)} \int_{G(y)}^{\infty} s^{-n-1}\left[H(s,y) - rK^+\right] ds. \end{aligned} \tag{88}$$

The following result, whose proof is developed in the Appendix, is concerned with the solvability of equation (86).

Lemma 6 *Suppose that Assumption 1 is true. Given any $y \geq 0$, the equation $q(x,y) = 0$ has a unique solution $x = x(y) > 0$ if and only if $\inf_{x>0} H(x,y) < rK^+$. If we define*

$$\tilde{y}_* = \inf\left\{ y \geq 0 : \inf_{x>0} H(x,y) < rK^+ \right\}, \tag{89}$$

then equation (86) defines uniquely a function $\tilde{G} :]\tilde{y}_*, \infty[\to]0, \infty[$ that is C^1, strictly increasing, and satisfies

$$H(\tilde{G}(y), y) - rK^+ > 0, \quad \text{for all } y > \tilde{y}_*. \tag{90}$$

Furthermore, if (25) in Assumption 2 is also true, then $\tilde{y}_* = 0$ and

$$C_4^{-\frac{1-\beta}{\alpha}} y^{\frac{1-\beta}{\alpha}} \leq \tilde{G}(y), \text{ for all } y \geq 0 \quad \Leftrightarrow \quad \tilde{G}^{[-1]}(x) \leq C_4 x^{\frac{\alpha}{1-\beta}}, \text{ for all } x \geq \tilde{G}(0), \tag{91}$$

where $\tilde{G}(0) := \lim_{y \downarrow 0} \tilde{G}(y)$, $\tilde{G}^{[-1]} : [\tilde{G}(0), \infty[\to \mathbb{R}_+$ is the inverse function of $\tilde{G}$, and $C_4 > 0$ is a constant.

Now, let us consider the case where $\mathcal{D} \neq \emptyset$ and the point y^* defined by (74) is finite (see Figures 2–4). For $y > y^*$, w is given by (77) for x such that $(x,y) \in \mathcal{W}$, by (81) for x such that $(x,y) \in \mathcal{I}$, and by

$$w_y(x,y) = -K^-, \tag{92}$$

for x such that $(x,y) \in \mathcal{D}$. Plainly, C^2 continuity of w inside $\mathcal{D}$ implies

$$w_{xy}(x,y) = 0, \quad \text{for } (x,y) \in \mathcal{D}. \tag{93}$$

To determine $A(y)$, $B(y)$, $F(y)$ and $G(y)$, we postulate that $w(\cdot, y)$ is C^2 at both of the free-boundary points $F(y)$ and $G(y)$. With regard to (78), (81)–(82), (92)–(93), the definition (79) of $R(x, y)$ and the identity $\sigma^2 mn = -r$, this requirement yields

$$A'(y) = -\frac{1}{\sigma^2(n-m)} \int_{F(y)}^{\infty} s^{-n-1} \left[H(s, y) + rK^-\right] ds, \tag{94}$$

$$A'(y) = -\frac{1}{\sigma^2(n-m)} \int_{G(y)}^{\infty} s^{-n-1} \left[H(s, y) - rK^+\right] ds, \tag{95}$$

$$B'(y) = -\frac{1}{\sigma^2(n-m)} \int_{0}^{F(y)} s^{-m-1} \left[H(s, y) + rK^-\right] ds, \tag{96}$$

$$B'(y) = -\frac{1}{\sigma^2(n-m)} \int_{0}^{G(y)} s^{-m-1} \left[H(s, y) - rK^+\right] ds, \tag{97}$$

where H is defined by (19). These calculations imply that the points $F(y)$ and $G(y)$ should satisfy the system of equations

$$f(F(y), G(y), y) = 0, \tag{98}$$

$$g(F(y), G(y), y) = 0, \tag{99}$$

where

$$f(x_1, x_2, y) = \int_0^{x_1} s^{-m-1} \left[H(s, y) + rK^-\right] ds - \int_0^{x_2} s^{-m-1} \left[H(s, y) - rK^+\right] ds, \tag{100}$$

$$g(x_1, x_2, y) = \int_{x_1}^{\infty} s^{-n-1} \left[H(s, y) + rK^-\right] ds - \int_{x_2}^{\infty} s^{-n-1} \left[H(s, y) - rK^+\right] ds. \tag{101}$$

In the Appendix, we prove the following result that is concerned with the solvability of the system of equations (98) and (99).

Lemma 7 *Suppose that Assumption 1 holds. Given $y \geq 0$, the system of equations (98) and (99) has a unique solution $(x_1, x_2) = (x_1(y), x_2(y))$ such that $0 < x_1 < x_2$ if and only if $\inf_{x>0} H(x, y) < -rK^-$. Moreover, if we define*

$$\bar{y}^* = \inf \left\{ y \geq 0 : \inf_{x>0} H(x, y) < -rK^- \right\}, \tag{102}$$

with the usual convention that $\inf \emptyset = \infty$, then, if $\bar{y}^* < \infty$, the system of equations (98) and (99) defines uniquely two functions $\bar{F}, \bar{G} :]\bar{y}^*, \infty[\to]0, \infty[$ that are C^1, strictly increasing, and satisfy $\bar{F}(y) < \bar{G}(y)$, for all $y > \bar{y}^*$,

$$\bar{F}(\bar{y}^*) := \lim_{y \downarrow \bar{y}^*} \bar{F}(y) = 0, \quad \text{if } \bar{y}^* > 0, \tag{103}$$

$$\bar{F}(0) := \lim_{y \downarrow 0} \bar{F}(y) \leq \lim_{y \downarrow 0} \bar{G}(y) =: \bar{G}(0), \quad \text{if } \bar{y}^* = 0, \tag{104}$$

$$H(\bar{F}(y), y) + rK^- < 0 \quad \text{and} \quad H(\bar{G}(y), y) - rK^+ > 0, \quad \text{for all } y > \bar{y}^*. \tag{105}$$

Furthermore, if (25) in Assumption 2 also holds, then

$$C_4^{-\frac{1-\beta}{\alpha}} y^{\frac{1-\beta}{\alpha}} \leq \bar{G}(y), \text{ for all } y \geq \bar{y}^* \;\Leftrightarrow\; \bar{G}^{[-1]}(x) \leq C_4 x^{\frac{\alpha}{1-\beta}}, \text{ for all } x \geq \bar{G}(\bar{y}^*), \tag{106}$$

where $\bar{G}^{[-1]} : [\bar{G}(0), \infty[\to \mathbb{R}_+$ *is the inverse function of* $\bar{G}$ *and the constant* $C_4 > 0$ *is the same constant as in Lemma 6.*

In light of the results above, and in the presence of (25) in Assumption 2, $\tilde{y}_* = \infty$, where $\tilde{y}_*$ is defined by (89), and the point $\bar{y}^*$ defined by (102) identifies with the point y^* in (74). Also, the functions $F : [y^*, \infty[\to [0, \infty[$ and $G : [0, \infty[\to [0, \infty[$ separating the three possible regions, as conjectured in (75)–(76), are given by

$$F = \bar{F}, \text{ if } y^* < \infty, \tag{107}$$

$$G = \tilde{G}, \text{ if } y^* = \infty, \quad \text{and} \quad G(y) = \begin{cases} \tilde{G}(y), & \text{for } y \in [0, y^*], \\ \bar{G}(y), & \text{for } y > y^*, \end{cases} \text{ if } y^* < \infty, \tag{108}$$

where $\tilde{G}$ is as in Lemma 6, $\bar{F}$, $\bar{G}$ are as in Lemma 7, and $y^* \equiv \bar{y}^*$, where $\bar{y}^*$ is given by (102).

The results above determine completely the boundaries of the three possible regions. To specify w inside the "wait" region $\mathcal{W}$, we still have to solve (88) and (94)–(97). To this end, it is straightforward to see that, if the associated integrals are finite, then the function

$$A(y) = \frac{1}{\sigma^2(n-m)} \int_y^\infty \int_{G(u)}^\infty s^{-n-1} \left[H(s,u) - rK^+\right] ds\, du > 0, \quad y \geq 0, \tag{109}$$

satisfies (88) as well as (94) and (95). In this expression, the inequality follows thanks to (90) or the second inequality in (105), depending on the case, and the assumption that $H(\cdot, y)$ is increasing. It is worth noting that adding a constant on the right hand side of (109) would yield a further solution to (88). However, it turns out that (109) gives the *only* solution of (88) that renders w compatible with the requirements of the verification theorem that we proved in Section 4.

If $y^* < \infty$, then

$$B(y) = -\frac{1}{\sigma^2(n-m)} \int_{y^*}^y \int_0^{F(u)} s^{-m-1} \left[H(s,u) + rK^-\right] ds\, du > 0, \quad y > y^*, \tag{110}$$

satisfies (96) or (97). Here, the positivity of B follows from the first inequality in (105) and the assumption that $H(\cdot, y)$ is increasing. As above, we have set a possible additive constant to zero because for *no other* choice can the resulting function w be identified with the value function of the control problem.

With reference to (81), w must satisfy

$$w(x,y) = w(x, G^{[-1]}(x)) - K^+ \left(G^{[-1]}(x) - y\right), \quad \text{for } (x,y) \in \mathcal{I},$$

where $G^{[-1]} : [G(0), \infty[\to \mathbb{R}_+$ is the inverse function of G. Also, if $\mathcal{D} \neq \emptyset$, then (92) implies that w should satisfy

$$w(x, y) = w(x, \Phi(x)) - K^-(y - \Phi(x)), \quad \text{for } (x, y) \in \mathcal{D},$$

where the function $\Phi :]0, \infty[\to \mathbb{R}_+$ is defined by

$$\Phi(x) = \begin{cases} F^{[-1]}(x), & \text{if } x \geq F(y^*), \\ 0, & \text{if } y^* = 0 \text{ and } F(0) > x, \end{cases} \tag{111}$$

in which expression, $F^{[-1]} : [F(y^*), \infty[\to \mathbb{R}_+$ is the inverse function of F. Summarising, we have two possibilities. If the point $y^* \equiv \bar{y}^*$ as in (74) or (102) is equal to ∞, then

$$w(x, y) = \begin{cases} A(y)x^n + R(x, y), & \text{for } (x, y) \text{ such that } 0 < x \leq G(y), \\ w(x, G^{[-1]}(x)) - K^+(G^{[-1]}(x) - y), & \text{for } (x, y) \text{ such that } G(y) < x. \end{cases} \tag{112}$$

On the other hand, if $y^* < \infty$, then

$$w(x, y) = \begin{cases} w(x, \Phi(x)) - K^-(y - \Phi(x)), & \text{for } (x, y) \text{ s.t. } y > y^*, \ x < F(y), \\ A(y)x^n + R(x, y), & \text{for } (x, y) \text{ s.t. } y \in [0, y^*] \cap \mathbb{R}, \ x \leq G(y), \\ A(y)x^n + B(y)x^m + R(x, y), & \text{for } (x, y) \text{ s.t. } y > y^*, \ F(y) \leq x \leq G(y), \\ w(x, G^{[-1]}(x)) - K^+(G^{[-1]}(x) - y), & \text{for } (x, y) \text{ s.t. } G(y) < x. \end{cases} \tag{113}$$

It is worth noting that, if $y^* = 0$ and $F(0) > 0$, then (78) and (110) imply

$$w(x, 0) = A(0)x^n + R(x, 0), \quad \text{for } 0 < x \leq G(0),$$

which is consistent with the associated expression resulting from (113).

The next result, which we prove in the Appendix, is concerned with proving that the construction above indeed provides a solution to the HJB equation (49)–(50), as well as with certain estimates that we will need.

Lemma 8 *Suppose that Assumptions 1 and 2 hold. The function w given by (112)–(113), where F, G and A, B are as in (107), (108) and (109), (110), respectively, is C^2 and satisfies the HJB equation (49)–(50). Also, w satisfies*

$$w(x, y) \leq C_5 \left(1 + y + G^{n-\varepsilon_4}(y) + G^{\alpha}(y)y^{\beta} + x^{n-\varepsilon_4}\right), \quad \textit{for all } (x, y) \in \mathcal{S}, \tag{114}$$

for some constants $C_5 > 0$ and $\varepsilon_4 \in]0, n[$, as well as (54) in the verification Theorem 5.

Remark 2 A careful inspection of the proof of this result reveals that, had we perturbed the expressions on the right hand sides of (109) and (110) by additive constants, we would still have obtained a further solution to the HJB equation (49)–(50). However, such a solution would not satisfy an estimate such as the one provided by (114) that plays a fundamental role in the proof of the verification Theorem 5. □

We can now prove the main result of the paper.

Theorem 9 *Consider the capacity control problem formulated in Section 2, and suppose that Assumptions 1 and 2 hold. The value function v identifies with the function w given by (112)–(113), where F, G and A, B are as in (107), (108) and (109), (110), respectively. The optimal capacity process Y^{o} reflects the joint process (X, Y^{o}) along the boundaries G and F in the positive and in the negative y-direction, respectively, and can be constructed as follows.*

(a) *If $y^* = \infty$, then Y^{o} is given by*

$$Y_t^{\mathrm{o}} = y\mathbf{1}_{\{t\leq\tau_0\}} + G^{[-1]}\left(\sup_{s\leq t} X_s\right)\mathbf{1}_{\{\tau_0<t\}},$$

where $\tau_0 = \inf\{t \geq 0 : \; X_t \geq G(y)\}$ and $G^{[-1]} : [G(0), \infty[\to \mathbb{R}_+$ is the inverse function of G.

(b) *If $y^* < \infty$, we first define*

$$\hat{y} = \begin{cases} \Phi(x), & \text{if } y > \Phi(x), \\ y, & \text{otherwise}, \end{cases} \qquad \tau_0 = \inf\{t \geq 0 : \; X_t \geq G(\hat{y})\}$$

and

$$Y_t^{(1)} = y\mathbf{1}_{\{t=0\}} + \hat{y}\mathbf{1}_{\{0<t\leq\tau_0\}} + G^{[-1]}\left(\sup_{s\leq t} X_s\right)\mathbf{1}_{\{\tau_0<t\}},$$

where Φ is defined by (111). We then define recursively the $(\mathcal{F}_t)$-stopping times τ_n and the processes $Y^{(n)}$ by

$$\tau_{2k+1} = \inf\left\{t > 0 : \; X_t < \hat{F}\left(Y_t^{(2k+1)}\right)\right\},$$

$$Y_t^{(2k+2)} = Y_t^{(2k+1)}\mathbf{1}_{\{t\leq\tau_{2k+1}\}} + \Phi\left(\inf_{\tau_{2k+1}<s\leq t} X_s\right)\mathbf{1}_{\{\tau_{2k+1}<t\}},$$

for $k = 0, 1, \ldots$, where

$$\hat{F}(y) = \begin{cases} 0, & \text{if } y < y^*, \\ F(y), & \text{if } y \geq y^*, \end{cases}$$

and by

$$\tau_{2k} = \inf\left\{t > 0 : \; X_t > G\left(Y_t^{(2k)}\right)\right\},$$

$$Y_t^{(2k+1)} = Y_t^{(2k)}\mathbf{1}_{\{t\leq\tau_{2k}\}} + G^{[-1]}\left(\sup_{\tau_{2k}<s\leq t} X_s\right)\mathbf{1}_{\{\tau_{2k}<t\}},$$

for $k = 1, 2, \ldots$. *The optimal capacity process* $Y^{\rm o}$ *is given by* $Y_t^{\rm o} = Y_t^{(n)}$, *for* $t < \tau_n$ *and* $n \geq 1$.

Proof. In view of Lemma 8, we only have to show that the process $Y^{\rm o}$ satisfies (55)–(59) in the verification Theorem 5. To this end, we first make the following comments on the construction of $Y^{\rm o}$. If $y^* = \infty$, then the boundary F does not exist and $Y^{\rm o} = Y^{(1)}$ is all we need because it reflects the joint process $(X, Y^{(1)})$ along the boundary G in the positive y-direction. On the other hand, if $y^* < \infty$, then the boundary F becomes part of the picture and we need to define $Y^{\rm o}$ in a recursive way. If the initial condition (x, y) is in the interior of the "disinvestment" region $\mathcal{D}$, then the process $Y^{(1)}$ has a jump of size $-(y - \Phi(x))$ at time 0, which instantaneously repositions the joint process $(X, Y^{(1)})$ in the closure of the "wait" region $\mathcal{W}$. Similarly, if the initial condition (x, y) is in the interior of the "investment" region $\mathcal{I}$, then the process $Y^{(1)}$ has a jump of size $G^{[-1]}(x) - y$ at time 0, which instantaneously repositions $(X, Y^{(1)})$ in the closure of the "wait" region. After time 0, the process $Y^{(1)}$ reflects the joint process $(X, Y^{(1)})$ along the boundary G in the positive y-direction, and $(X, Y^{(1)})$ enters the interior of the "disinvestment" region $\mathcal{D}$ after time τ_1 with positive probability. The process $Y^{(2)}$ is the same as $Y^{(1)}$ up to time τ_1, $Y_{\tau_1}^{(2)} \equiv Y_{\tau_1}^{(1)} > y^*$ and $X_{\tau_1} = F(Y_{\tau_1}^{(2)})$. Beyond time τ_1, $Y^{(2)}$ reflects the joint process $(X, Y^{(2)})$ along the boundary F in the negative y-direction. As a result, the process $(X, Y^{(2)})$ is kept outside the interior of $\mathcal{I} \cup \mathcal{D}$ at all times up to τ_2, after which time, it enters the interior of the "investment" region $\mathcal{I}$ with positive probability. The process $Y^{(3)}$ is the same as $Y^{(2)}$ up to time τ_2 and $X_{\tau_2} = G(Y_{\tau_2})$. After τ_2, $Y^{(3)}$ reflects $(X, Y^{(3)})$ along the boundary G in the positive y-direction. It follows that the process $(X, Y^{(3)})$ does not enter the interior of $\mathcal{I} \cup \mathcal{D}$ up to time τ_3. Iterating this construction, which ensures that $Y_t^{(n)} = Y_t^{(n+1)}$, for all $t \in [0, \tau_{n+1}]$ and $n \geq 1$, and observing that $\lim_{n\to\infty} \tau_n = \infty$, we can see that $Y_t^{\rm o}$ is defined for all $t \geq 0$ and that (55) is satisfied. Also, if $\xi^{{\rm o}+}$ and $\xi^{{\rm o}-}$ are the increasing processes providing the minimal decomposition of $Y^{\rm o}$ into $Y^{\rm o} = y + \xi^{{\rm o}+} - \xi^{{\rm o}-}$, then both of (56) and (57) hold.

To proceed further, we note that the construction of $Y^{\rm o}$ implies

$$Y_t^{\rm o} \leq y\mathbf{1}_{\{\bar{X}_t\leq G(y)\}} + G^{[-1]}(\bar{X}_t)\mathbf{1}_{\{\bar{X}_t>G(y)\}}, \tag{115}$$

where $\bar{X}_t = \sup_{s\leq t} X_s$. Combining this inequality with the definition (108) of G and the estimates in (91) and (106), we can see that

$$Y_t^{\rm o} \leq y\mathbf{1}_{\{\bar{X}_t\leq G(y)\}} + C_4\bar{X}_t^{\alpha/(1-\beta)}\mathbf{1}_{\{\bar{X}_t>G(y)\}} \tag{116}$$

and

$$\xi_t^{\mathrm{o}+} \leq C_4 \bar{X}_t^{\alpha/(1-\beta)}. \tag{117}$$

Now, we can use (116), the observation that

$$G(Y_t^{\mathrm{o}}) \leq G(y)\mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + \bar{X}_t \mathbf{1}_{\{\bar{X}_t > G(y)\}},$$

which follows immediately from (115), to see that, e.g.,

$$\begin{aligned} G^\alpha\left(Y_t^{\mathrm{o}}\right)\left(Y_t^{\mathrm{o}}\right)^\beta &\leq G^\alpha(y) y^\beta \mathbf{1}_{\{\bar{X}_t \leq G(y)\}} + C_4^\beta \bar{X}_t^{\alpha/(1-\beta)} \mathbf{1}_{\{\bar{X}_t > G(y)\}} \\ &\leq G^\alpha(y) y^\beta + C_4^\beta \bar{X}_t^{\alpha/(1-\beta)}. \end{aligned}$$

In view of this and similar calculations involving the other terms, as well as the estimate (114) and the fact that $\alpha < \frac{\alpha}{1-\beta} < n$ (see Assumption 2), we can conclude that (116)–(117) imply that the estimates (58)–(59) hold true, and the proof is complete. □

6 Examples

We can illustrate our main results by means of the special cases that we now consider.

Corollary 10 *Suppose that h is given by (27) in Example 1, and $K^+, K^+ + K^- > 0$. If $\frac{\alpha}{1-\beta} < n$, then $v < \infty$, while, if $\frac{\alpha}{1-\beta} > n > \alpha$, then $v \equiv \infty$, where n is the positive solution of (10). In the former case, the following hold true:*

(a) If $K^- \geq 0$, then $y^* = \infty$,

$$G(y) = \left[-\frac{rK^+(\alpha - m)}{m\beta}\right]^{1/\alpha} y^{(1-\beta)/\alpha}, \tag{118}$$

and the optimal strategy can be depicted by Figure 1.

(b) If $K^- < 0$, then $y^* = 0$ and

$$\lim_{y \downarrow 0} F(y) = \lim_{y \downarrow 0} G(y) = 0. \tag{119}$$

and the optimal strategy can be depicted by Figure 4.

Proof. As we have observed in Example 1, Assumptions 1 and 2 are satisfied and $v < \infty$ if and only if $\frac{\alpha}{1-\beta} < n$. Also, if $\frac{\alpha}{1-\beta} > n > \alpha$, then we have proved in Lemma 3 that $v \equiv \infty$.

The condition distinguishing the two cases follows from a simple inspection of (102), while showing (118) involves elementary calculations. To see (119), we observe that the system of equations (100)–(101), which specifies F and G, is equivalent to

$$\frac{\beta}{\alpha - m} y^{-(1-\beta)} \left[G^{\alpha-m}(y) - F^{\alpha-m}(y)\right] = -\frac{r}{m}\left[K^+ G^{-m}(y) + K^- F^{-m}(y)\right], \tag{120}$$

$$\frac{\beta}{n - \alpha} y^{-(1-\beta)} \left[G^{\alpha-n}(y) - F^{\alpha-n}(y)\right] = \frac{r}{n}\left[K^+ G^{-n}(y) + K^- F^{-n}(y)\right]. \tag{121}$$

Since $m < 0 < \alpha, 1-\beta$ and F, G are increasing, the right hand side of (120) remains bounded as $y \downarrow 0$, and $\lim_{y \downarrow 0} y^{-(1-\beta)} = \infty$. It follows that (120) cannot be true unless (119) is satisfied, and the proof is complete. □

Remark 3 In the context of the special case considered in Corollary 10, it is worth noting that the solution w to the HJB equation (49)–(50) that we have constructed following intuition based on economical considerations is finite for all $\alpha \in]0, n[$ and $\beta \in]0, 1[$. Had we adopted a *formal* approach, this observation would have suggested the adoption of the capacity expansion strategy that keeps the process (X, Y) inside the "wait" region $\mathcal{W}$ that is determined by the functions F and G provided by the unique solution to the associated free-boundary problem. However, such a formal approach would have lead us to wrong conclusions because

$$w(x, y) < \infty = v(x, y), \quad \text{for all } (x, y) \in \mathcal{S},$$

if $\frac{\alpha}{1-\beta} > n$. □

Remark 4 In the special case of Corollary 10 arising when $\alpha = 1 - \beta$ and $K^- < 0$, we can verify that (120) and (121) are satisfied by the functions

$$F(y) = \kappa y \quad \text{and} \quad G(y) = \nu y, \quad \text{for } y \geq 0,$$

where κ and ν are constants satisfying the system of algebraic equations

$$\frac{1-\alpha}{\alpha - m}\left[\nu^{\alpha-m} - \kappa^{\alpha-m}\right] = -\frac{r}{m}\left[K^+\nu^{-m} + K^-\kappa^{-m}\right],, \tag{122}$$

$$\frac{1-\alpha}{n-\alpha}\left[\nu^{-(n-\alpha)} - \kappa^{-(n-\alpha)}\right] = \frac{r}{n}\left[K^+\nu^{-n} + K^-\kappa^{-n}\right]. \tag{123}$$

Abel and Eberly [AE96] considered this special case with $r > b$, which satisfies our assumptions thanks to the equivalence $r > b \Leftrightarrow n > 1$, and have proved that the system of equations (122)–(123) has a unique solution such that $0 < \kappa < \nu$. □

The following special case follows from our general results and (29).

Corollary 11 *Suppose that $K^+, -K^-, K^+ + K^- > 0$, consider the running payoff function h given by (28) in Example 2, and assume that the associated parameters satisfy (29). The following cases hold true:*

(a) If $-rK^- \in \left]\left(\beta\eta^\alpha\zeta^{-(1-\beta)} - K\right) \vee 0, rK^+\right[$, then $y^* = 0$, $0 < \lim_{y\downarrow 0} F(y) < \lim_{y\downarrow 0} G(y)$, and the optimal strategy can be depicted by Figure 3.

(b) If $\beta\eta^\alpha\zeta^{-(1-\beta)} > K$ and $-rK^- \in \left]0, \beta\eta^\alpha\zeta^{-(1-\beta)} - K\right[$, then

$$y^* = \left(\frac{\beta\eta^\alpha}{K - rK^-}\right)^{\frac{1}{1-\beta}} - \zeta > 0,$$

$\lim_{y\downarrow y^*} F(y) = 0$, $\lim_{y\downarrow 0} G(y) > 0$, and the optimal strategy can be depicted by Figure 2.

We conclude with the following example that does not satisfy the requirements imposed on the problem data by Assumptions 1 and 2.

Example 3 Suppose that the running payoff function h is given by $h(x,y) = (x+\eta)^\alpha y^\beta$, for some constants $\eta > 0$ and $\alpha, \beta \in]0,1[$ such that $\frac{\alpha}{1-\beta} < n$. Using the same arguments as the ones in Example 2, we can check that Assumption 1, and (23), (24) and (26) in Assumption 2 all hold true. However, this payoff function does not satisfy the upper bound required by (25) in Assumption 2. Furthermore, if we assume that $K^+, -K^-, K^+ + K^- > 0$, then we can check that the points y_* and y^* defined as in Lemma 6 and Lemma 7 are given by

$$0 < y_* = \left(\frac{\beta\eta^\alpha}{rK^+}\right)^{\frac{1}{1-\beta}} < \left(\frac{\beta\eta^\alpha}{-rK^-}\right)^{\frac{1}{1-\beta}} = y^*.$$

It follows that, at least formally, this example provides a case in which a strategy such as the one depicted by Figure 5 is optimal.

Appendix: Proof of selected results

Proof of Lemma 6. Suppose that (20) in Assumption 1 is satisfied. Fix any $y \geq 0$, and suppose that $\inf_{x>0} H(x,y) - rK^+ \geq 0$. In this case, $H(x,y) - rK^+ > 0$, for all $x > 0$, because $H(\cdot, y)$ is a strictly increasing function. This implies that $q(x,y) > 0$, for all $x > 0$, and, therefore, the equation $q(x,y) = 0$ has no solution $x > 0$.

Now, fix any $y \geq 0$, and assume that $\inf_{x>0} H(x,y) < rK^+$. Recalling the assumption that $H(\cdot, y)$ is strictly increasing, we define

$$x^\dagger = x^\dagger(y) := \inf\left\{x > 0 : \; H(x,y) - rK^+ > 0\right\} > 0,$$

and we observe that

$$\frac{\partial}{\partial x} q(x,y) = x^{-m-1}\left[H(x,y) - rK^+\right] \begin{cases} < 0, & \text{for all } x \in]0, x^\dagger[, \\ > 0, & \text{for all } x > x^\dagger. \end{cases} \tag{124}$$

Combining the fact that $q(\cdot, y)$ is strictly decreasing in $]0, x^\dagger[$ and strictly increasing in $]x^\dagger, \infty[$, with $q(0,y) = 0$, we can see that $q(x,y) < 0$, for all $x \leq x^\dagger$. In particular, $q(x^\dagger, y) < 0$. Therefore, if $q(x,y) = 0$ has a solution $x > 0$ then this must satisfy $x > x^\dagger$. Also, given that it exists, this solution is unique because $q(\cdot, y)$ is strictly increasing in $]x^\dagger, \infty[$. To prove that the required solution indeed exists, it suffices to show that $\lim_{x\to\infty} q(x,y) = \infty$. The assumption that $\lim_{x\to\infty} H(x,y) = \infty$ implies that, given any constant $M > 0$, there exists $\gamma > x^\dagger$ such that $H(x,y) - rK^+ \geq M$, for all $x \geq \gamma$. However, given any such choice of these constants, we calculate

$$\begin{aligned}\lim_{x\to\infty} q(x,y) &= \lim_{x\to\infty}\left[q(\gamma,y) + \int_\gamma^x s^{-m-1}\left[H(s,y) - rK^+\right] ds\right] \\ &\geq \lim_{x\to\infty}\left[q(\gamma,y) + \frac{M}{m}\gamma^{-m} - \frac{M}{m}x^{-m}\right] = \infty.\end{aligned}$$

If (21) in Assumption 1 also holds and the point $\tilde{y}_*$ defined as in (89) is finite, then $\inf_{x>0} H(x,y) < rK^+$, for all $y > \tilde{y}_*$. It follows that equation (86) defines uniquely a continuous function $\tilde{G} :]\tilde{y}_*, \infty[\to]0, \infty[$. Moreover, the arguments above regarding the solvability of $q(x,y) = 0$ imply (90).

To see that $\tilde{G}$ is C^1 and strictly increasing, we differentiate $q(\tilde{G}(y), y) = 0$ with respect to y to obtain

$$\tilde{G}'(y) = -\tilde{G}^{m+1}(y) \left[H(\tilde{G}(y), y) - rK^+ \right]^{-1} \int_0^{\tilde{G}(y)} s^{-m-1} H_y(s,y)\, ds > 0, \tag{125}$$

for all $y > \tilde{y}_*$. The inequality here follows thanks to (90) and (21) in Assumption 1.

Now, suppose that (25) in Assumption 2 also holds and observe that this implies

$$\inf_{x>0} H(x,y) < rK^+, \quad \text{for all } y > 0.$$

However, this inequality implies that $\tilde{y}_* = 0$. Finally, with regard to (25) in Assumption 2 and (124) above, we calculate

$$\frac{\partial}{\partial x} q(x,y) \leq x^{-m-1} \left[\beta C x^{\alpha} y^{-(1-\beta)} - r\vartheta \right].$$

Combining this inequality with $q(0,y) = 0$, we can see that, given any $y > 0$, $\tilde{G}(y)$ is greater than or equal to the strictly positive solution of the equation

$$\int_0^z s^{-m-1} \left[\beta C s^{\alpha} y^{-(1-\beta)} - r\vartheta \right] ds = 0,$$

which yields

$$\tilde{G}(y) \geq \left(-\frac{r\vartheta(\alpha - m)}{\beta C m} \right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \quad \text{for all } y > 0.$$

However, this implies (91). □

Proof of lemma 7. Suppose that Assumption 1 holds. We develop the proof in a number of steps.

Step 1. To study the solvability of the system of equations (98) and (99), we first prove that (98) defines uniquely a mapping $L : (\mathbb{R}_+ \setminus \{0\})^2 \to]0, \infty[$ such that

$$f(x_1, L(x_1,y), y) = 0 \quad \text{and} \quad L(x_1,y) > x_1. \tag{126}$$

To this end, fix any $x_1 > 0$, $y > 0$, and observe that

$$f(x_1, x_1, y) = -\frac{1}{m} r \left(K^+ + K^- \right) x_1^{-m} > 0. \tag{127}$$

Given $M > 0$, observe that the assumption that $\lim_{x\to\infty} H(x,y) = \infty$, for all $y > 0$, implies that there exists a constant $\gamma > x_1$ such that $H(x,y) - rK^+ \geq M$, for all $x \geq \gamma$. For such a choice of parameters, since $m < 0$, we calculate

$$\begin{aligned}
\lim_{x_2\to\infty} f(x_1,x_2,y) &= \lim_{x_2\to\infty} \Bigg[-\int_{x_1}^{\gamma} s^{-m-1} \left[H(s,y) - rK^+\right] ds \\
&\qquad - \int_{\gamma}^{x_2} s^{-m-1} \left[H(s,y) - rK^+\right] ds - \frac{r}{m} \left(K^+ + K^-\right) x_1^{-m} \Bigg] \\
&\leq \lim_{x_2\to\infty} \left[f(x_1,\gamma,y) - M \int_{\gamma}^{x_2} s^{-m-1}\, ds \right] \\
&= \lim_{x_2\to\infty} \left[f(x_1,\gamma,y) - \frac{M}{m}\gamma^{-m} + \frac{M}{m} x_2^{-m} \right] \\
&= -\infty.
\end{aligned} \tag{128}$$

Also, it is straightforward to calculate

$$\frac{\partial f}{\partial x_2}(x_1,x_2,y) = -x_2^{-m-1} \left[H(x_2,y) - rK^+\right] \begin{cases} > 0, & \text{for all } x_2 \in]0, x^\dagger[, \\ < 0, & \text{for all } x_2 > x^\dagger, \end{cases} \tag{129}$$

where

$$x^\dagger = x^\dagger(y) := \inf\left\{ x > 0 : \; H(x,y) - rK^+ > 0 \right\}.$$

Combining the fact that $f(x_1,\cdot,y)$ is strictly increasing in the interval $[x_1, x^\dagger[$, if $x_1 < x^\dagger$, and strictly decreasing in the interval $]x^\dagger \vee x_1, \infty[$, with (128) and (127), we can conclude that the equation $f(x_1,x_2,y) = 0$ has a unique solution $x_2 = L(x_1,y)$ which satisfies (126) as well as

$$H(L(x_1,y),y) - rK^+ > 0. \tag{130}$$

For future reference, we also note that differentiation of $f(x_1, L(x_1,y), y) = 0$ with respect to x_1 yields

$$\frac{\partial}{\partial x_1} L(x_1,y) = \frac{x_1^{-m-1} \left[H(x_1,y) + rK^-\right]}{L^{-m-1}(x_1,y) \left[H(L(x_1,y),y) - rK^+\right]}, \tag{131}$$

while differentiation of $f(x_1, L(x_1,y), y) = 0$ with respect to y gives

$$\frac{\partial}{\partial y} L(x_1,y) = -L^{m+1}(x_1,y) \left[H(L(x_1,y),y) - rK^+\right]^{-1} \int_{x_1}^{L(x_1,y)} s^{-m-1} H_y(s,y)\, ds. \tag{132}$$

Step 2. To prove that the system of equations (98) and (99) has a unique solution (x_1,x_2) such that $0 < x_1 < x_2$ we have to show that there exists a unique $x_1 > 0$ such that $g(x_1, L(x_1,y), y) = 0$. To this end, we first observe that the calculation

$$g(x_1, L(x_1,y), y) = \int_{x_1}^{L(x_1,y)} s^{-n-1} \left[H(s,y) - rK^+\right] + \frac{1}{n} r \left(K^+ + K^-\right) x_1^{-n}$$

and the assumptions $\lim_{x\to\infty} H(x,y) = \infty$, $K^+ + K^- > 0$ imply that

$$\text{there exists a constant } N > 0 \text{ such that } g(x_1, L(x_1,y), y) > 0, \text{ for all } x_1 \geq N. \tag{133}$$

Now, with regard to (131), we calculate

$$\frac{\partial}{\partial x_1} g(x_1, L(x_1,y), y) = x_1^{-m-1} \left[L^{m-n}(x_1,y) - x_1^{m-n}\right] \left[H(x_1,y) + rK^-\right]. \tag{134}$$

Since $L(x_1,y) > x_1$ and $m < n$, $L^{m-n}(x_1,y) - x_1^{m-n} < 0$. Therefore, if $\inf_{x>0} H(x,y) \geq -rK^-$, then $g(\cdot, L(\cdot,y), y)$ is decreasing, which, combined with (133), implies that the equation $g(x_1, L(x_1,y), y) = 0$ cannot have a solution $x_1 > 0$. Therefore, we must have $\inf_{x>0} H(x,y) < -rK^-$. Assuming that this condition holds, we recall that $H(\cdot, y)$ is strictly increasing, we define

$$x^\ddagger = x^\ddagger(y) := \inf\left\{x > 0 : \; H(x,y) + rK^- > 0\right\},$$

and we observe that

$$g(\cdot, L(\cdot,y), y) \text{ is strictly increasing in }]0, x^\ddagger[\text{ and strictly decreasing in }]x^\ddagger, \infty[. \tag{135}$$

Furthermore, under this condition, there exist $\varepsilon > 0$ and $\delta < x^\ddagger$ such that $H(x_1,y) + rK^- \leq -\varepsilon$, for all $x_1 \leq \delta$. For such a choice of parameters, we calculate

$$\begin{aligned}
&\lim_{x_1\downarrow 0} \int_{x_1}^{\infty} s^{-n-1} \left[H(s,y) + rK^-\right] ds \\
&\qquad \leq \lim_{x_1\downarrow 0} \left[\frac{\varepsilon}{n}\delta^{-n} - \frac{\varepsilon}{n} x_1^{-n} + \int_{\delta}^{\infty} s^{-n-1}\left[H(s,y) + rK^-\right] ds\right] \\
&\qquad = -\infty.
\end{aligned} \tag{136}$$

In view of this, (130), and the assumption that $H(\cdot, y)$ is increasing,

$$\begin{aligned}
&\lim_{x_1\downarrow 0} g(x_1, L(x_1,y), y) \\
&\quad = \lim_{x_1\downarrow 0} \left[\int_{x_1}^{\infty} s^{-n-1}\left[H(s,y) + rK^-\right] ds - \int_{L(x_1,y)}^{\infty} s^{-n-1}\left[H(s,y) - rK^+\right] ds\right] \\
&\quad \leq \lim_{x_1\downarrow 0} \int_{x_1}^{\infty} s^{-n-1}\left[H(s,y) + rK^-\right] ds \\
&\quad = -\infty.
\end{aligned} \tag{137}$$

However, combining (133), (135) and (137), we can see that the equation $g(x_1, L(x_1,y), y) = 0$ has a unique solution $x_1 > 0$, which also satisfies

$$H(x_1,y) + rK^- < 0. \tag{138}$$

Step 3. Summarising the analysis above, under the assumption that the point $\bar{y}^*$ defined as in (102) is finite, the system of equations (98) and (99) defines uniquely two continuous functions $\bar{F}, \bar{G} :]\bar{y}^*, \infty[\to]0, \infty[$ that satisfy $\bar{F}(y) < \bar{G}(y)$, for all $y > \bar{y}^*$, as well as (105). Also, (103)–(104) follow from a simple continuity argument combining the definition of $\bar{y}^*$ and (138).

Step 4. Now, assuming that $\bar{y}^* < \infty$, we consider any point $y > \bar{y}^*$. Differentiating the equation $g(\bar{F}(y), L(\bar{F}(y), y), y) = 0$ with respect to y, using (131), and observing that $\bar{G}(y) = L(\bar{F}(y), y)$, we calculate

$$\begin{aligned}\bar{F}'(y) = &-\bar{F}^{m+1}(y)\bar{G}^{-n}\left[\bar{G}^{-(n-m)}(y) - \bar{F}^{-(n-m)}(y)\right]^{-1}\left[H(\bar{F}(y), y) + rK^-\right]^{-1} \\ &\times \int_{\bar{F}(y)}^{\bar{G}(y)} \left[\left(\frac{\bar{G}(y)}{s}\right)^n - \left(\frac{\bar{G}(y)}{s}\right)^m\right]\frac{1}{s}H_y(s, y)\, ds > 0, \end{aligned} \tag{139}$$

the inequality following thanks to assumption (21), the first inequality in (105) and the fact that $m < 0 < n$. Also, differentiating the equation $f(\bar{F}(y), L(\bar{F}(y), y), y) = 0$ with respect to y, and using (132) and (139), we calculate

$$\begin{aligned}\bar{G}'(y) = &-\bar{F}^{-n}(y)\bar{G}^{m+1}\left[\bar{G}^{-(n-m)}(y) - \bar{F}^{-(n-m)}(y)\right]^{-1}\left[H(\bar{G}(y), y) - rK^+\right]^{-1} \\ &\times \int_{\bar{F}(y)}^{\bar{G}(y)} \left[\left(\frac{\bar{F}(y)}{s}\right)^n - \left(\frac{\bar{F}(y)}{s}\right)^m\right]\frac{1}{s}H_y(s, y)\, ds > 0,\end{aligned}$$

the inequality following thanks to (105) and (21). However, these calculations show that that $\bar{F}$ and $\bar{G}$ both are C^1 and strictly increasing.

Step 5. Finally, suppose that (25) in Assumption 2 is also true. With reference to the equation $f(\bar{F}(y), \bar{G}(y), y) = 0$, we calculate

$$\begin{aligned}0 = &-\int_{\bar{F}(y)}^{\bar{G}(y)} s^{-m-1}\left[H(s, y) - rK^+\right] ds - \frac{1}{m}r\left(K^+ + K^-\right)\bar{F}^{-m}(y) \\ \geq &-\left[\frac{\beta C}{\alpha - m}\bar{G}^{\alpha-m}(y)y^{-(1-\beta)} + \frac{r\vartheta}{m}\bar{G}^{-m}(y)\right] \\ &+\left[\frac{\beta C}{\alpha - m}\bar{F}^{\alpha-m}(y)y^{-(1-\beta)} - \frac{1}{m}r\left(K^+ + K^- - \vartheta\right)\bar{F}^{-m}(y)\right].\end{aligned}$$

Since $\vartheta < K^+ + K^-$ by assumption, the second term on the right hand side of this expression is strictly positive. Therefore, we must have

$$\frac{\beta C}{\alpha - m}\bar{G}^{\alpha-m}(y)y^{-(1-\beta)} + \frac{r\vartheta}{m}\bar{G}^{-m}(y) > 0.$$

This inequality can be true only if $\bar{G}(y)$ is strictly greater than the unique strictly positive solution of the equation

$$\frac{\beta C}{\alpha - m}z^{\alpha-m}y^{-(1-\beta)} + \frac{r\vartheta}{m}z^{-m} = 0,$$

which yields

$$\bar{G}(y) \geq \left(- \frac{r\vartheta(\alpha - m)}{\beta C m} \right)^{\frac{1}{\alpha}} y^{\frac{1-\beta}{\alpha}}, \quad \text{for all } y > \bar{y}^*.$$

However, this implies (106). □

Proof of lemma 8. We develop the proof along a series of steps.

Step 1. We first prove (114). Consider (109), and note that the upper bound in (25) in Assumption 2 implies

$$0 < A(y) \leq \frac{\beta C}{\sigma^2 (n - m)(n - \alpha)} \int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du. \tag{140}$$

Recalling the inequalities $\alpha < \frac{\alpha}{1-\beta} < n$, we fix any $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < n - \frac{\alpha}{1-\beta} < n - \alpha.$$

Using the fact that G is increasing and the estimate provided by (91) and (106), we calculate

$$\begin{aligned} \int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du &\leq G^{-\varepsilon_0}(y) \int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha-\varepsilon_0)}(u)\, du \\ &\leq \frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0) - \alpha} G^{-\varepsilon_0}(y) y^{1 - \frac{(1-\beta)(n-\varepsilon_0)}{\alpha}}, \end{aligned}$$

which implies

$$\int_y^\infty u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du \leq \frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0) - \alpha} G^{-\varepsilon_0}(y), \quad \text{for all } y \geq 1. \tag{141}$$

Also, the fact that G is increasing implies that

$$\begin{aligned} G^n(y) \int_y^1 u^{-(1-\beta)} G^{-(n-\alpha)}(u)\, du &\leq G^\alpha(y) \int_y^1 u^{-(1-\beta)}\, du \\ &\leq \frac{1}{\beta} G^\alpha(1), \quad \text{for all } y < 1. \end{aligned} \tag{142}$$

However, (140)–(142) imply

$$\begin{aligned} A(y)x^n &\leq A(y)G^n(y) \\ &\leq \frac{\beta C}{\sigma^2(n-m)(n-\alpha)} \Bigg[\frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{n-\varepsilon_0}(y) \mathbf{1}_{\{y \geq 1\}} \\ &\quad + \left(\frac{\alpha C_4^{(1-\beta)(n-\alpha-\varepsilon_0)/\alpha}}{(1-\beta)(n-\varepsilon_0)-\alpha} G^{n-\varepsilon_0}(1) + \frac{1}{\beta} G^\alpha(1) \right) \mathbf{1}_{\{y<1\}} \Bigg] \\ &= C_{51} \left(1 + G^{n-\varepsilon_0}(y)\right), \quad \text{for all } y \geq 0 \text{ and } x \leq G(y), \end{aligned} \tag{143}$$

where $C_{51} > 0$ is a constant.

If $y^* < \infty$, then (110), the assumption that $K^+ + K^- > 0$, the lower bound in (25) in Assumption 2 and the fact that F is increasing imply that, given any $y > y^*$,

$$\begin{aligned} B(y) &\leq -\frac{C + rK^+}{\sigma^2 m(n-m)} \int_{y^*}^{y} F^{-m}(u)\, du \\ &\leq -\frac{C + rK^+}{\sigma^2 m(n-m)} y F^{-m}(y). \end{aligned}$$

In light of this calculation and the fact that $m < 0$, we can see that

$$\sup_{x \in [F(y), G(y)]} B(y) x^m \leq B(y) F^m(y) \leq C_{52} y, \quad \text{for all } y > y^*, \tag{144}$$

where $C_{52} > 0$ is a constant. Since R is increasing in x (see (26) in Assumption 2 and (16)), the upper bound in Lemma 2 implies

$$\begin{aligned} \sup_{x \leq G(y)} R(x, y) &\leq R(G(y), y) \\ &\leq C_1 \left(1 + y + G^{n-\vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for all } y \geq 0. \end{aligned}$$

However, combining this estimate with (143) and (144), we can see that w satisfies

$$w(x, y) \leq C_{53} \left(1 + y + G^{n-\varepsilon_0 \wedge \vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for all } (x, y) \in \overline{\mathcal{W}}, \tag{145}$$

for some constant $C_{53} > 0$. With regard to the structure of w provided by (112)–(113), this inequality and the estimates provided by (91) and (106) imply

$$\begin{aligned} w(x, y) &\leq w(x, G^{[-1]}(x)) + K^+ y \\ &\leq C_{53} \left(1 + G^{[-1]}(x) + x^{n-\varepsilon_0 \wedge \vartheta} + x^{\alpha} \left[G^{[-1]}(x)\right]^{\beta}\right) + K^+ y \\ &\leq C_{54} \left(1 + y + x^{n-\varepsilon_0 \wedge \vartheta} + x^{\alpha/(1-\beta)}\right), \quad \text{for } (x, y) \in \mathcal{I}, \end{aligned} \tag{146}$$

for some constant $C_{54} > 0$. Also, since $\Phi(x) \leq y$, for all $(x, y) \in \mathcal{D}$, and G is increasing,

$$\begin{aligned} w(x, y) &\leq w(x, \Phi(x)) + |K^-| y \\ &\leq C_{53} \left(1 + \Phi(x) + G^{n-\varepsilon_0 \wedge \vartheta}(\Phi(x)) + G^{\alpha}(\Phi(x)) \Phi^{\beta}(x)\right) + |K^-| y \\ &\leq C_{55} \left(1 + y + G^{n-\varepsilon_0 \wedge \vartheta}(y) + G^{\alpha}(y) y^{\beta}\right), \quad \text{for } (x, y) \in \mathcal{D}, \end{aligned} \tag{147}$$

where $C_{55} > 0$ is a constant. However, in view of the assumption $\frac{\alpha}{1-\beta} < n$, if we choose any

$$\varepsilon_4 \in \left]0, \varepsilon_0 \wedge \vartheta \wedge \left(n - \frac{\alpha}{1-\beta}\right)\right[\quad \text{and} \quad C_5 \geq C_{53} \vee C_{54} \vee C_{55},$$

then we can see that (145)–(147) imply (114).

Step 2. To show that w satisfies (54), we first observe that the positivity of A, B and the lower bound in Lemma 2 imply that

$$w(x,y) \geq -C_1(1+y), \quad \text{for all } (x,y) \in \overline{\mathcal{W}}. \tag{148}$$

This estimate and the definition of w in $\mathcal{I}$, provided by (112)–(113), imply

$$\begin{aligned} w(x,y) &\geq -(C_1+K^+)G^{[-1]}(x) - C_1 \\ &\geq -(C_1+K^+)C_4 x^{\alpha/(1-\beta)} - C_1, \quad \text{for all } (x,y) \in \mathcal{I}, \end{aligned} \tag{149}$$

the second inequality following thanks to (91) and (106). Also, if $y^* < \infty$, then (148) and the definition of w in $\mathcal{D}$, given by (113), imply

$$\begin{aligned} w(x,y) &\geq -C_1(1+\Phi(x)) - |K^-| \max\{y, \Phi(x)\} \\ &\geq -(C_1+|K^-|)y - C_1. \end{aligned} \tag{150}$$

However, (148)–(150) establish (54).

Step 3. With reference to the construction of w, we will show that w is C^2 if we prove that w_x, w_{xx} and w_{yy} are continuous along the free boundaries F and G. To this end, we calculate

$$\begin{aligned} w_x(x,y) &= w_x\left(x, G^{[-1]}(x)\right) + \left[w_y\left(x, G^{[-1]}(x)\right) - K^+\right] \frac{dG^{[-1]}(x)}{dx} \\ &= w_x\left(x, G^{[-1]}(x)\right), \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{151}$$

and

$$\begin{aligned} w_{xx}(x,y) &= w_{xx}\left(x, G^{[-1]}(x)\right) + w_{xy}\left(x, G^{[-1]}(x)\right) \frac{dG^{[-1]}(x)}{dx} \\ &= w_{xx}\left(x, G^{[-1]}(x)\right), \quad \text{for } (x,y) \in \mathcal{I}, \end{aligned} \tag{152}$$

the second equalities following thanks to (80) that have been among the requirements leading to the equations specifying the function G. However, these calculations and the structure of w provided by (112)–(113) show that w_x and w_{xx} are continuous along G.

Now, if $y^* > 0$ and $y \in [0, y^*] \cap \mathbb{R}$, we can use (79) and (88) to calculate

$$\begin{aligned} \lim_{x \uparrow G(y)} w_{yy}(x,y) &= A''(y)G^n(y) + R_{yy}(G(y),y) \\ &= \frac{G^{-1}(y)}{\sigma^2(n-m)} \left[G'(y)\left[H(G(y),y) - rK^+\right] + G^{m+1}(y) \int_0^{G(y)} s^{-m-1} H_y(s,y)\, ds\right] \\ &= 0, \end{aligned} \tag{153}$$

the last equality following thanks to (125). Also, if $y^* < \infty$ and $y > y^*$, we can use (79), (95) and (97) to calculate

$$\begin{aligned}\lim_{x \uparrow G(y)} w_{yy}(x,y) &= A''(y)G^n(y) + B''(y)G^m(y) + R_{yy}(G(y),y) \\ &= 0. \end{aligned} \tag{154}$$

However, combining (153) and (154) with the fact that $w_{yy}(x,y) = 0$, for $(x,y) \in \mathcal{I}$, we conclude that w_{yy} is continuous along G.

Showing that w_x, w_{xx} and w_{yy} are continuous along F involves similar arguments.

Step 4. By construction, we will prove that w satisfies the HJB equation (49)–(50) if we show that

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \le 0, \quad \text{for } (x,y) \in \mathcal{I}, \tag{155}$$

$$w_y(x,y) + K^- \ge 0, \quad \text{for } (x,y) \in \mathcal{I},\ y > 0, \tag{156}$$

$$w_y(x,y) - K^+ \le 0, \quad \text{for } (x,y) \in \mathcal{W}, \tag{157}$$

$$w_y(x,y) + K^- \ge 0, \quad \text{for } (x,y) \in \mathcal{W},\ y > 0, \tag{158}$$

and, if $\mathcal{D} \neq \emptyset$,

$$\sigma^2 x^2 w_{xx}(x,y) + bxw_x(x,y) - rw(x,y) + h(x,y) \le 0, \quad \text{for } (x,y) \in \mathcal{D}, \tag{159}$$

$$w_y(x,y) - K^+ \le 0, \quad \text{for } (x,y) \in \mathcal{D}. \tag{160}$$

It is straightforward to see that either of (156) or (160) is equivalent to $K^+ + K^- \ge 0$, which is true by assumption. Recalling that $H \equiv h_y$, we can easily verify that, since $y \le G^{[-1]}(x)$, for all $(x,y) \in \mathcal{I}$, (151) and (152) imply that (155) is equivalent to

$$\int_y^{G^{[-1]}(x)} \left[H(x,u) - rK^+\right] du \ge 0, \quad \text{for } (x,y) \in \mathcal{I}.$$

However, this inequality follows immediately from the assumption that $H(x,\cdot)$ is strictly decreasing, for all x, and (90) together with the second inequality in (105). Similarly, we can show that, if $\mathcal{D} \neq \emptyset$, then (159) is equivalent to

$$\int_{\Phi(x)}^{y} \left[H(x,u) + rK^-\right] du \le 0, \quad \text{for } (x,y) \in \mathcal{D},$$

where Φ is defined by (111). however, we can see that this inequality is true once we combine the first inequality in (105) with the assumption that $H(x,\cdot)$ is strictly decreasing, for all x, and the assumption that $H(\cdot,0)$ is strictly increasing.

Now, suppose that $y^* < \infty$, and fix any $y > y^*$. Since $w_y(F(y),y) = -K^-$ and $w_y(G(y),y) = K^+$, we will prove that both of (157) and (158) are satisfied if we show that

$$w_{yx}(x,y) \ge 0, \quad \text{for all } x \in]F(y), G(y)[. \tag{161}$$

To this end, we consider the transformation of the independent variable $x > 0$ provided by $z = \ln x$, and we write $w(x, y) = u(\ln x, y)$ for some function $u = u(z, y)$. It follows that (161) is true if and only if

$$u_{yz}(z, y) \geq 0, \quad \text{for all } z \in \,]\ln F(y), \ln G(y)[. \tag{162}$$

Now, since $w = w(x, y)$ satisfies (77) for $x \in]F(y), G(y)[$, u_y satisfies

$$\sigma^2 u_{yzz}(z, y) + \left(b - \sigma^2\right) u_{yz}(z, y) - r u_y(z, y) + H(e^z, y) = 0, \quad \text{for } z \in \,]\ln F(y), \ln G(y)[.$$

Recalling that H_x is continuous and $H_x(\cdot, y) \geq 0$ (see Assumption 1), we can differentiate this equation with respect to z to obtain

$$\begin{aligned}\sigma^2 (u_{yz})_{zz}(z, y) + \left(b - \sigma^2\right) (u_{yz})_z(z, y) - r u_{yz}(z, y) &= -e^z H_x(e^z, y), \\ &\leq 0, \quad \text{for } z \in \,]\ln F(y), \ln G(y)[.\end{aligned}$$

This inequality and the maximum principle imply that $u_{yz}(\cdot, y)$ does not have a negative minimum in the interval $]\ln F(y), \ln G(y)[$, so

$$\begin{aligned}\inf_{z \in]\ln F(y), \ln G(y)[} u_{yz}(z, y) &\geq \min_{z = \ln F(y), \ln G(y)} 0 \wedge u_{yz}(z, y) \\ &= \min_{z = F(y), G(y)} 0 \wedge w_{yx}(x, y) \\ &= 0.\end{aligned}$$

However, this calculation implies (162).

To proceed further, fix any $y \in [0, y^*] \cap \mathbb{R}$. Using the definition of R in (79), the expression for $A'(y)$ provided by (88) and the fact that $G(y)$ satisfies (86), we can see that, if we define $\overline{u}(x, y) = w_y(x, y) - K^+$, then

$$\begin{aligned}\overline{u}_x(x, y) = \frac{1}{\sigma^2(n - m)} \Bigg[&-m x^{m-1} \int_x^{G(y)} s^{-m-1} \left[H(s, y) - rK^+\right] ds \\ &+ n x^{n-1} \int_x^{G(y)} s^{-n-1} \left[H(s, y) - rK^+\right] ds \Bigg], \quad \text{for } x \in]0, G(y)[.\end{aligned}$$

This calculation and the assumption that $H(\cdot, y)$ is strictly increasing imply that $\overline{u}_x(x, y) = w_{yx}(x, y) > 0$, for all $x \in [x^\dagger(y), G(y)[$, where $x^\dagger(y) \in]0, G(y)[$ is the unique point such that $H\left(x^\dagger(y), y\right) - rK^+ = 0$ (see Lemma 6). This observation and the boundary condition $w_y\left(G(y), y\right) = K^+$ imply

$$w_y(x, y) - K^+ < 0, \quad \text{for all } x \in [x^\dagger(y), G(y)[. \tag{163}$$

Furthermore, since

$$\sigma^2 x^2 \overline{u}_{xx}(x, y) + b x \overline{u}_x(x, y) - r \overline{u}(x, y) = -\left[H(x, y) - rK^+\right] \geq 0, \quad \text{for } x \in]0, x^\dagger(y)[,$$

the maximum principle implies that the function $x \mapsto \overline{u}(x,y) = w_y(x,y) - K^+$ has no positive maximum in the interval $]0, x^\dagger(y)[$, so

$$\sup_{x \in\,]0,x^\dagger(y)[} \left[w_y(x,y) - K^+ \right] \leq \max_{x=0,x^\dagger(y)} 0 \vee \left[w_y(x,y) - K^+ \right] = 0, \tag{164}$$

the equality following thanks to (163) and the fact that

$$\lim_{x \downarrow 0} w_y(x,y) = \lim_{x \downarrow 0} R_y(x,y) = \lim_{x \downarrow 0} \frac{H(x,y)}{r} \in [-K^-, K^+[. \tag{165}$$

The second equality here holds true because of (17), while the inclusion follows from the context (see Lemmas 6 and 7). However, (163) and (164) establish (157). Finally, if we define $\underline{u}(x,y) = w_y(x,y) + K^-$, then (165) and the assumption that $H(\cdot,y)$ is increasing imply

$$\sigma^2 x^2 \underline{u}_{xx}(x,y) + bx\underline{u}_x(x,y) - r\underline{u}(x,y) = -\left[H(x,y) + rK^- \right] \leq 0, \quad \text{for all } x \in]0, G(y)[.$$

This calculation and the maximum principle imply that the function $x \mapsto \underline{u}(x,y) = w_y(x,y) + K^-$ has no negative minimum inside $]0, G(y)[$, so

$$\inf_{x \in\,]0,G(y)[} \left[w_y(x,y) + K^- \right] = \min_{x=0,G(y)} 0 \wedge \left[w_y(x,y) + K^- \right],$$

which, combined with (165) and the boundary condition $w_y(G(y),y) + K^- = K^+ + K^- > 0$, proves (158), and the proof is complete. □

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References

[AE96] A. B. Abel and J. C. Eberly (1996), Optimal investment with costly reversibility, *Review of Economic Studies*, **63**, pp. 581–593.

[B05] P. Bank (2005), Optimal control under a dynamic fuel constraint, *SIAM Journal on Control and Optimization*, **44**, pp. 1529–1541.

[BB90] S. Bentolila and G. Bertola (1990), Firing costs and labour demand: how bad is eurosclerosis?, *The Review of Economic Studies*, **57**, pp. 381-402.

[CH05] M. B. Chiarolla and U. G. Haussmann (2005), Explicit solution of a stochastic, irreversible investment problem and its moving threshold, *Mathematics of Operations Research*, vol. **30**, pp. 91–108.

[D93] M. H. A. Davis (1993), *Markov Models and Optimization*, Chapman & Hall.

[DDSV87] M. H. A. Davis, M. A. H. Dempster, S. P. Sethi and D. Vermes (1987), Optimal capacity expansion under uncertainty, *Advances in Applied Probability*, vol. **19**, pp. 156–176.

[GP05] X. Guo and H. Pham (2005), Optimal partially reversible investment with entry decisions and general production function, *Stochastic Processes and their Applications*, vol. **115**, pp. 705–736.

[KS88] I. Karatzas and S. E. Shreve (1988), *Brownian Motion and Stochastic Calculus*, Springer-Verlag.

[KMZ98] T. S. Knudsen, B. Meister and M. Zervos (1998), Valuation of investments in real assets with implications for the stock prices, *SIAM Journal on Control and Optimization*, **36**, pp. 2082–2102.

[K93] T. Ø. Kobila (1993), A class of solvable stochastic investment problems involving singular controls, *Stochastics and Stochastics Reports*, vol. **43**, pp. 29–63.

[Ø00] A. Øksendal (2000), Irreversible investment problems, *Finance and Stochastics*, vol. **4**, pp. 223–250.

[W03] H. Wang (2003), Capacity expansion with exponential jump diffusion processes, *Stochastics and Stochastics Reports*, vol. **75**, pp. 259–274.

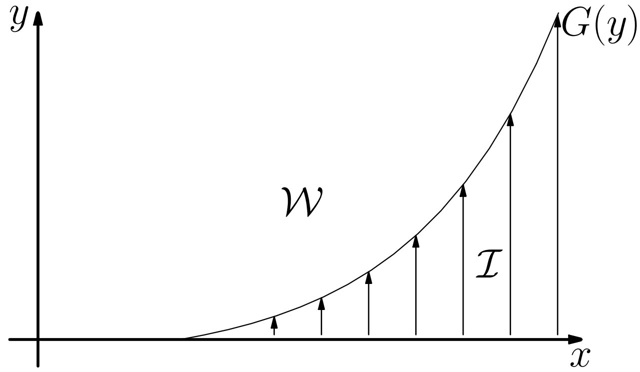

Figure 1: A possible optimal capacity control strategy. In this case, it is never optimal to decrease the project's capacity.

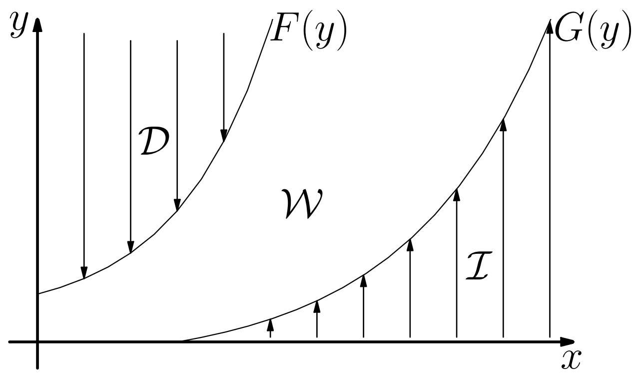

Figure 2: A possible optimal capacity control strategy. In this case, increasing the project's capacity, waiting and decreasing the project's capacity are all parts of the optimal strategy. Also, the point y^* defined by (74) is strictly positive.

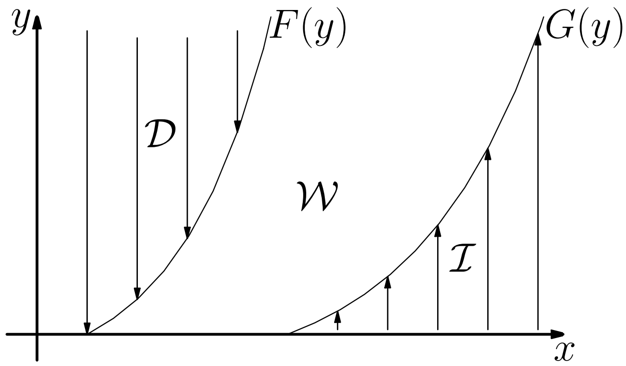

Figure 3: A possible optimal capacity control strategy. In this case, increasing the project's capacity, waiting and decreasing the project's capacity all belong to the set of optimal tactics. Also, $y^* = 0$, where y^* is defined by (74), $F(0) > 0$, and $\{(x, 0) : \ x \leq F(0)\}$ is a subset of the "wait" region $\mathcal{W}$.

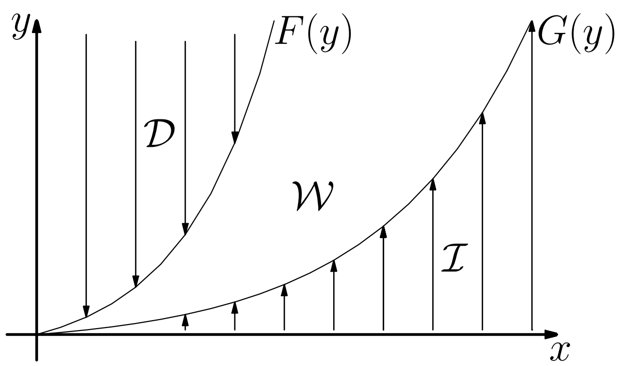

Figure 4: A possible optimal capacity control strategy. This case arises when the running payoff function h identifies with the Cobb-Douglas production function and $K^- < 0$.

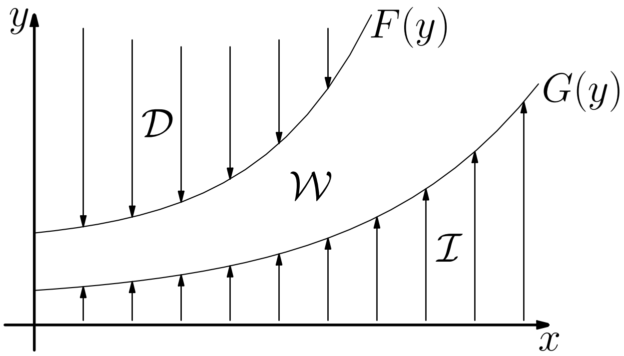

Figure 5: A possible optimal capacity control strategy. This case *cannot* arise under our assumptions.