# A Problem of Sequential Entry and Exit Decisions Combined with Discretionary Stopping

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November 4, 2010

**Abstract**

We consider a stochastic control problem that has emerged in the economics literature as an investment model under uncertainty. This problem combines features of both stochastic impulse control and optimal stopping. The aim is to discover the form of the optimal strategy. It turns out that this has a priori rather unexpected features. The results that we establish are of an explicit nature. We also construct an example whose value function does not possess $C^1$ regularity.

**Keywords and phrases:** Stochastic Impulse Control, Optimal Switching, Optimal Stopping, Real Options.

## 1 Introduction

Problems that combine features of both stochastic optimal control and optimal stopping have attracted the interest of several researchers. Models of absolutely continuous control of the drift and discretionary stopping have been studied by Krylov [K], Beneš [B], Karatzas and Sudderth [KS], Karatzas and Wang [KW], and Karatzas and Ocone [KO]. Models of combined singular stochastic control where the control effort takes the form of a finite variation process and discretionary stopping have been studied by Davis and Zervos [DZ], and Karatzas, Ocone, Wang and Zervos [KOWZ]. These two families of problems have been motivated by applications in target tracking where the controller has to steer a system close to a target and then decide on an engagement time, as well as by applications in finance.

The latter ones include the classical consumption/investment problem for a small investor who can decide on the time of their "exit" from the market (see Karatzas and Wang [KW]), as well as the pricing of American contingent claims under constraints or with transaction costs.

In this paper, we consider a problem of stochastic impulse control combined with optimal stopping with a view to discovering the form of the optimal strategy. Note that the impulse control component of the control strategy is not of the standard form because the sizes of the jumps associated with each intervention strategy are not discretionary, but are constrained to follow the pattern $\ldots, 1, -1, 1, -1, \ldots$. This simplification makes the problem easier to analyse. However, it is offset by the extra complexity that is introduced by the additional control variable which is the discretionary stopping.

Problems of this type arise in the context of various applications in which the system dynamics involve discrete actions. For instance, in manufacturing, one needs to choose a machine setup mode over time so as to switch optimally among a finite number of different product types (see Sethi and Zhang [SZ]). The actual motivation of this paper arises from the area of "real options" that has emerged in the economics literature over the past two decades. This area is concerned with the development of new stochastic models that can lead to more accurate pricing of investments in real assets by taking into account the value of managerial flexibility; the interested reader can consult the books by Dixit and Pindyck [DP], and Trigeorgis [T].

To fix ideas, consider an economic activity that is centred on a project that can operate in two modes, an "open" one and a "closed" one. Whenever the project is in its "open" operating mode, it yields a stream of profits or losses which is a functional of the uncertain prices of input and output commodities. Whenever the project is in its "closed" operating mode, it yields neither profits nor losses. The transition of the project from one of its operating modes to the other one forms a sequence of managerial decisions and is associated with certain fixed costs. The problem is to determine the switching strategy that maximises the expected present value of all profits and losses resulting from the project. Variants of this problem have been developed in the economics literature as models for the valuation of investments in real assets by Brennan and Schwartz [BS], Dixit [D], and Dixit and Pindyck [DP]. Such a problem has the features of stochastic impulse control, and explicit solutions have been obtained in the mathematics literature by Brekke and Øksendal [BØ1, BØ2], Lumley and Zervos [LZ], and Duckworth and Zervos [DuZ1].

Suppose now that the option of totally abandoning the project at a discretionary time and at a certain fixed cost is added in the set of available managerial decisions. The resulting problem then combines stochastic impulse control with discretionary stopping. In fact, such a model is extensively discussed in Dixit and Pindyck [DP, Section 7.2], and is a special case of the one developed by Brennan and Schwartz [BS]. However, these authors make very little progress in actually solving the problem. The purpose of this paper is to solve completely the resulting optimisation problem under the assumption that the rate at which the project yields profits or losses is a standard Brownian motion. Such an assumption is probably crude as long as real life applications are concerned. However, it leads to explicit, non-trivial results that unveil the qualitative nature of the optimal strategy.

The results of our analysis take qualitatively different forms, depending on parameter values, and can be summarised informally as follows. Suppose that the switching costs are fixed. If the abandonment cost is very large (see case I in Theorem 6 and Figure 1), then it is optimal to perpetuate the project by switching it to its "closed" mode as soon as its output cash flow falls below a certain level, and by switching it to its "open" mode as soon as its potential output cash flow rises above a certain higher level. If the abandonment cost is very small (see case III of Theorem 6 and Figure 4), then abandonment is optimal, sooner or later. If the project is in its "closed" mode at time 0, then it is switched to its "open" mode as soon as its potential output cash flow exceeds a certain level. Once in it, the project should be kept in its "open" operating mode for as long as its output cash flow is above a given level, and should be abandoned as soon as its output cash flow falls below this level. For intermediate values of the abandonment cost, we have an a priori rather unexpected combination of the two cases above (see case II of Theorem 6 and Figure 3). If the project starts from its "closed" mode, then it is never abandoned, and the situation resembles the case where the abandonment cost is very large. A similar scenario pertains to the case when the project is originally "open" and its output cash flow assumes sufficiently high levels. However, if the project is originally "open" and its output cash flow assumes very low values, then it is optimal to abandon the project immediately. The most interesting possibility arises when the project is originally "open" and its output cash flow assumes moderately low values. In this case, it is optimal to keep the project live and keep on accumulating losses until its output cash flow either falls below a certain level, on which event the project is totally abandoned, or rises above another certain level, on which event its operation enters the perpetual life-cycle pertaining to the case of a large abandonment cost. As a result, the abandonment time of the project is either finite or infinite, and each of the two possibilities has positive probability.

The paper is organised as follows. Section 2 is concerned with the formulation of the stochastic optimisation problem that we address. In Section 3, we prove a verification theorem that will play a crucial role in our subsequent analysis. The assumptions of the theorem allow for the possibility that the value function is not $C^1$, and the proof is developed using Itô-Tanaka's formula, and relies on the properties of local times. The explicit solution of the non-trivial case discussed above is developed in Section 4. Finally, an example whose value function is not $C^1$ is presented in Section 5.

# 2 Problem formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)$ satisfying the usual conditions of right continuity and augmentation by $P$-negligible sets, and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion $W$. We denote by $\mathcal{Z}$ the family of all adapted, finite variation, càglàd processes $Z$ with values in $\{0, 1\}$, and by $\mathcal{S}$ the set of all $(\mathcal{F}_t)$-stopping times.

We consider a stochastic system that can operate in two modes, an "open" one and a "closed" one. The system's mode of operation can be changed at a sequence of $(\mathcal{F}_t)$-stopping times. These transition times constitute a decision strategy that we model by a process $Z \in \mathcal{Z}$. Specifically, given any time $t$, $Z_t = 1$ if the system is "open" at time $t$, whereas $Z_t = 0$ if the system is "closed" at time $t$. The stopping times at which the jumps of $Z$ occur are the intervention times at which the system's operating mode is changed. We denote by $z \in \{0, 1\}$ the system's mode at time 0. We also assume that the operation of this system can be permanently abandoned at an $(\mathcal{F}_t)$-stopping time $T$, which is an additional decision variable. We define the set of all admissible strategies to be

$$\Pi_z = \{(Z, T) \mid \ Z \in \mathcal{Z}, \ Z_0 = z, \ T \in \mathcal{S}\}.$$

We assume that the rate at which the system yields payoff, the switching costs associated with the transition of the system from its "closed" mode to its "open" one, and vice versa, as well as the permanent abandonment cost are all functions of a state process $X$ which satisfies the one-dimensional SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \qquad X_0 = x \in \mathcal{I}, \tag{1}$$

where $\mathcal{I}$ is a given interval. We assume that the functions $b, \sigma : \mathcal{I} \to \mathbb{R}$ satisfy assumptions such that this SDE has a unique strong solution with values in $\mathcal{I}$, for all $t \geq 0$, $P$-a.s.. In the problem that we solve in Section 4, $\mathcal{I} = \mathbb{R}$. However, if, following several of the references mentioned in the introduction, we use $X$ to model commodity prices, we must have $\mathcal{I} = ]0, \infty[$.

With each admissible strategy $(Z, T) \in \Pi_z$ we associate the expected payoff

$$\begin{aligned} J_{z,x}(Z, T) = &E \left[ \int_0^T R_s \big[H_1(X_s) Z_s + H_0(X_s)(1 - Z_s)\big]\, ds \right. \\ &\left. - \sum_{0 \leq s \leq T} 1_{\{s<\infty\}} R_s \left[G_1(X_s)\, (\Delta Z_s)^+ + G_0(X_s)\, (\Delta Z_s)^-\right] - 1_{\{T<\infty\}} R_T F(X_T) \right], \end{aligned} \tag{2}$$

where $\Delta Z_t = Z_{t+} - Z_t$, $(\Delta Z_t)^\pm = \max\{\pm \Delta Z_t, 0\}$, and the discounting process $R$ is given by

$$R_t = \exp\left( - \int_0^t r(X_s)\, ds \right), \tag{3}$$

for some positive function $r : \mathcal{I} \to \mathbb{R}$. Here, $H_1(X_t)$ (resp. $H_0(X_t)$) is the rate at which the system yields payoff assuming that, at time $t$, it is in its "open" (resp. "closed") operating mode. Also, $G_1(X_t)$, $G_0(X_t)$ are the costs associated with switching the investment from its "closed" to its "open" mode, and vice versa, respectively, at time $t$, whereas $F(X_t)$ is the cost faced if the system is completely abandoned at time $t$.

The objective is to maximise $J_{z,x}(Z,T)$ over $\Pi_z$. Accordingly we define the value function

$$v(z,x) = \sup_{(Z,T)\in\Pi_z} J_{z,x}(Z,T).$$

We assume that the problem is well posed in the sense that all of the integrals in (2) are well defined, for every admissible strategy, and non-trivial in the sense that $v(z,x) < \infty$ for for every initial condition $(z,x)$. For the problem to be well posed, we also need to assume that no strategy associated with a finite payoff involves an infinite number of switchings prior to abandonment on a set of positive probability, so that every switching strategy can be modelled by a process in $\mathcal{Z}$. A sufficient condition for this assumption to hold is $G_1(x) + G_0(x) > \epsilon > 0$, for all $x \in \mathcal{I}$. From an economics perspective, this assumption is a natural one because it rules out the unrealistic situation where arbitrarily high profits can be made by rapidly changing the system's operating mode.

All of the assumptions discussed above are of an implicit nature. Further assumptions will appear in the statement of Theorem 1, again in an implicit way. On the other hand, the results of Sections 4 and 5 will assume that the problem's data have specific forms.

At this point, it would be of interest to make a comment on a possible generalisation of the model considered here. The dynamics of the state process $X$ can be modified to include an additional, regime switching process, so that (1) becomes

$$dX_t = b(\theta_t, X_t)\,dt + \sigma(\theta_t, X_t)\,dW_t, \qquad X_0 = x \in \mathcal{I}.$$

The process $\theta$ can be taken to be a finite-state Markov chain representing a number of different economic outlooks (e.g., a state of economic growth and a state of recession). Models involving regime switchings have been considered in the literature, and include Guo [G] who solves the problem of pricing a Russian option in such a context. A generalisation of the model studied here in this direction would multiply the complexity of the problem by the number of states that the process $\theta$ can assume, and we leave it as an interesting open problem.

# 3 A verification theorem

The problem considered in the previous section combines features of both stochastic impulse control and optimal stopping. Therefore, we expect that the value function $v$ should satisfy the Hamilton-Jacobi-Bellman (HJB) equation

$$\max\bigl\{\mathcal{L}v(z,x) + zH_1(x) + (1-z)H_0(x),\quad v(1-z,x) - v(z,x) - zG_0(x) - (1-z)G_1(x),\\ - v(z,x) - F(x)\bigr\} = 0, \quad z = 1, 0 \quad x \in \mathcal{I}, \tag{4}$$

where the second order elliptic operator $\mathcal{L}$ is defined by

$$\mathcal{L}v(z,x) = \tfrac{1}{2}\sigma^2(x)v_{xx}(z,x) + b(x)v_x(z,x) - r(x)v(z,x).$$

The ideas behind the origins of this equation are the following. Suppose that, at time 0, the system is in its "open" operating mode, i.e. $z = 1$. The controller's immediate decision consists of choosing between three actions. The first action is to totally terminate the system's operation at the cost of $-F(x)$. Such a possibility gives rise to the inequality

$$v(1,x) \geq -F(x). \tag{5}$$

The second option is to pay the cost of $G_0(x)$ to switch the system to its "closed" operating mode, and then continue optimally. This possibility yields the inequality

$$v(1,x) \geq -G_0(x) + v(0,x). \tag{6}$$

The third action is to leave the system in its "open" operating mode for a short time $\Delta t$, and then continue optimally. This action is associated with the inequality

$$v(1,x) \geq E\left[\int_0^{\Delta t} R_s H_1(X_s)\, ds + R_{\Delta t} v(1, X_{\Delta t})\right].$$

Under the assumption that $v(1,\cdot)$ is sufficiently smooth, we may apply Itô's formula to the last term, and then divide by $\Delta t$ before letting $\Delta t \downarrow 0$, to obtain

$$\mathcal{L}v(1,x) + H_1(x) \equiv \tfrac{1}{2}\sigma^2(x)v_{xx}(1,x) + b(x)v_x(1,x) - r(x)v(1,x) + H_1(x) \leq 0. \tag{7}$$

Now, each of (5)–(7) can hold with strict inequality because the corresponding action may not be optimal. However, we expect that the three actions considered above form a complete repertoire of optimal tactics. Therefore, given any $x \in \mathcal{I}$, we expect that one of (5)–(7) should hold with equality. Combining all of these relationships, we can conclude that the value function $v(1,\cdot)$ should satisfy

$$\max\bigl\{\mathcal{L}v(1,x) + H_1(x),\ v(0,x) - v(1,x) - G_0(x),\ -v(1,x) - F(x)\bigr\} = 0. \tag{8}$$

Using a similar reasoning, we can also conclude that the value function $v(0,\cdot)$ associated with the system in its "closed" operating mode (i.e. when $z = 0$) should satisfy

$$\max\bigl\{\mathcal{L}v(0,x) + H_0(x), v(1,x) - v(0,x) - G_1(x),\ -v(0,x) - F(x)\bigr\} = 0. \tag{9}$$

Now, combining (8) and (9), we conclude that the value function $v$ should satisfy (4). Without any further conditions, this equation has, in general, uncountably many solutions.

**Example 1** Suppose that $\mathcal{I} = \mathbb{R}$, and, for all $x \in \mathbb{R}$, $b(x) = 0$, $\sigma(x) = \sqrt{2}$, $r(x) = 4$, $H_1(x) = 3e^x + 4$, $H_0(x) = 0$, $G_1(x) = G_0(x) = 1$ and $F(x) = c$, for some constant $c > 0$. It is straightforward to verify that each of the functions defined by

$$w(z,x) = Ae^{2x} + Be^{-2x} + e^x + z, \qquad A, B \geq 0,$$

satisfies (4).

It turns out that the functions $v(1,\cdot)$ and $v(0,\cdot)$ composing the value function of the special case of the control problem that we explicitly solve in Section 4 are both $C^1$ but not $C^2$. However, it is clear that, as long as the general problem is concerned, we cannot expect such regularity of the value function unless we impose appropriate assumptions on the problem's data. For instance, we cannot in general expect $C^1$ regularity unless the abandonment cost function $F$ is $C^1$. An explicitly solvable example illustrating this issue is presented in Section 5.

In the next theorem, we consider candidates for the value functions $v(1,\cdot)$ and $v(0,\cdot)$ which are differences of convex functions; for a survey of the results needed here, see Revuz and Yor [RY, Appendix 3]. In particular, we consider solutions of (4) in the following sense.

**Definition 1** A function $w : \{0,1\} \times \mathcal{I} \to \mathbb{R}$ satisfies (4) if each of $w(1,\cdot)$, $w(0,\cdot)$ is a difference of two convex functions and (4) is true Lebesgue-a.e., with $\hat{\mathcal{L}}$ in place of $\mathcal{L}$, where the operator $\hat{\mathcal{L}}$ is defined by

$$\hat{\mathcal{L}}w(z,x) = \tfrac{1}{2}\sigma^2(x) w_{xx}^{\mathrm{ac}}(z,x) + b(x) w_x^-(z,x) - r(x) w(z,x).$$

Here, $w_x^-(z,\cdot)$ is the left hand derivative of $w(z,\cdot)$. Also,

$$w_{xx}(z,dx) = w_{xx}^{\mathrm{ac}}(z,x)\,dx + w_{xx}^{\mathrm{s}}(z,dx) \tag{10}$$

is the Lebesgue decomposition of the second distributional derivative $w_{xx}(z,dx)$ of $w(z,\cdot)$ to the measure $w_{xx}^{\mathrm{ac}}(z,x)\,dx$ which is absolutely continuous with respect to the Lebesgue measure and the measure $w_{xx}^{\mathrm{s}}(z,dx)$ which is mutually singular with the Lebesgue measure.

We can now prove conditions which are sufficient for optimality in our problem.

**Theorem 1** *Consider the control problem described in Section 2. Suppose that $G_1$, $G_0$, $F$ are continuous functions, $\sigma^2(x) > 0$, for all $x \in \mathcal{I}$, and, for every admissible strategy $(Z,T) \in \Pi_z$, there exists a sequence of times $t_m \to \infty$ such that*

$$\lim_{m\to\infty} J_{z,x}(Z, T \wedge t_m) = J_{z,x}(Z,T). \tag{11}$$

*Suppose that there exist functions $w(1,\cdot), w(0,\cdot) : \mathcal{I} \to \mathbb{R}$ which are differences of convex functions such that*

$$-w_{xx}^{\mathrm{s}}(1,dx) \text{ and } -w_{xx}^{\mathrm{s}}(0,dx) \text{ are positive measures,} \tag{12}$$

*and which satisfy the HJB equation (4) in the sense of Definition 1. Also, suppose that the process $M$ defined by*

$$M_t = \int_0^t R_s \sigma(X_s) w_x^-(Z_s, X_s)\, dW_s \tag{13}$$

*is a martingale for every switching strategy $Z \in \mathcal{Z}$. Then, given any initial condition $(z, x) \in \{0, 1\} \times \mathcal{I}$,*
*(a) $v(z, x) \leq w(z, x)$, and*
*(b) if*

$$\operatorname{supp} w_{xx}^{\mathrm{s}}(z, dx) \subseteq \mathcal{I} \setminus \operatorname{int} \overline{\left\{ x \in \mathcal{I} \mid \ \hat{\mathcal{L}} w(z, x) + z H_1(x) + (1 - z) H_0(x) = 0 \right\}} =: \mathcal{O}_z, \tag{14}$$

*and there exists $Z^* \in \mathcal{Z}$ such that*

$$\hat{\mathcal{L}} w(Z_t^*, X_t) + Z_t^* H_1(X_t) + (1 - Z_t^*) H_0(X_t) = 0, \tag{15}$$

*for Lebesgue almost all $t \leq T^*$, $P$-a.s., and*

$$[w(1, X_t) - w(0, X_t) - G_1(X_t)]\, (\Delta Z_t^*)^+ = 0, \tag{16}$$
$$[w(0, X_t) - w(1, X_t) - G_0(X_t)]\, (\Delta Z_t^*)^- = 0, \tag{17}$$

*for all $t \leq T^*$, $P$-a.s., where*

$$T^* = \inf \left\{ t \geq 0 \mid \ w(Z_t^*, X_t) = -F(X_t) \right\}, \tag{18}$$

*as well as a sequence of times $t_m \to \infty$ satisfying (11) as well as*

$$\lim_{m \to \infty} E\big[R_{t_m} \left| w(Z_{t_m}^*, X_{t_m}) \right|\big] = 0 \quad \textit{and} \quad \lim_{m \to \infty} E\big[R_{t_m} \left| F(X_{t_m}) \right|\big] = 0, \tag{19}$$

*then $v(z, x) = w(z, x)$, and the optimal strategy is $(Z^*, T^*)$.*

**Proof.** Fix any $z = 0, 1$. Using Itô-Tanaka's formula (see Revuz and Yor [RY, Theorem VI.1.5]), we obtain

$$\begin{aligned} w(z, X_t) = w(z, x) &+ \int_0^t b(X_s) w_x^-(z, X_s)\, ds + \int_0^t \sigma(X_s) w_x^-(z, X_s)\, dW_s \\ &+ \tfrac{1}{2} \int_{\mathcal{I}} L_t^a\, w_{xx}(z, da), \end{aligned} \tag{20}$$

where $L^a$ is the local time of the process $X$ at level $a$. We assume that

$$\text{the mapping } (t, a) \to L_t^a \text{ is continuous in } t \text{ and càdlàg in } a, \tag{21}$$

$P$-a.s. (see Revuz and Yor [RY, Theorem VI.1.7]). With reference to (10) and the occupation times formula (see Revuz and Yor [RY, Corollary VI.1.6])

$$\int_{\mathcal{I}} L_t^a w_{xx}^{\mathrm{ac}}(z, a)\, da = \int_0^t \sigma^2(X_s) w_{xx}^{\mathrm{ac}}(z, X_s)\, ds,$$

so (20) implies

$$\begin{aligned} w(z, X_t) = w(z, x) &+ \int_0^t \left[\tfrac{1}{2}\sigma^2(X_s) w_{xx}^{\mathrm{ac}}(z, X_s) + b(X_s) w_x^-(z, X_s)\right] ds \\ &+ \int_0^t \sigma(X_s) w_x^-(z, X_s)\, dW_s + A_t^z \end{aligned}$$

where

$$A_t^z = \tfrac{1}{2} \int_{\mathcal{I}} L_t^a\, w_{xx}^{\mathrm{s}}(z, da). \tag{22}$$

For future reference, observe that (12) implies

$$-A^z \text{ is a continuous, increasing process,} \tag{23}$$

because such a statement is true for local times. Now, using the integration by parts formula for semimartingales, we obtain

$$R_t w(z, X_t) = w(z, x) + \int_0^t R_s \hat{\mathcal{L}} w(z, X_s)\, ds + \int_0^t R_s \sigma(X_s) w_x^-(z, X_s)\, dW_s + \int_0^t R_s\, dA_s^z. \tag{24}$$

We can now prove the two statements of the theorem.

(a) Fix any admissible strategy $(Z, T) \in \Pi_z$, and suppose that the abandonment time $T$ is bounded by a constant. Define the increasing sequence of $(\mathcal{F}_t)$-stopping times $(T_n)$ by

$$T_1 = \inf\{t \geq 0 \mid \ Z_t \neq z\} \quad \text{and} \quad T_{n+1} = \inf\{t > T_n \mid \ Z_t \neq Z_{T_n+}\}, \tag{25}$$

with the usual convention that $\inf \emptyset = \infty$. Note that the assumption that $Z$ is a finite variation process implies that its discontinuities cannot accumulate within any compact subset of $\mathbb{R}_+$, so $T_n \to \infty$, $P$-a.s.. Therefore,

$$\begin{aligned} R_T w(Z_T, X_T) = R_T w(Z_T, X_T) 1_{\{T \leq T_1\}} &+ \sum_{n=1}^{\infty} \Bigg[ R_T w(Z_T, X_T) - R_{T_n} w(Z_{T_n+}, X_{T_n}) \\ &+ \sum_{j=1}^{n-1} \left[ R_{T_{j+1}} w(Z_{T_{j+1}}, X_{T_{j+1}}) - R_{T_j} w(Z_{T_j+}, X_{T_j}) \right] + R_{T_1} w(Z_{T_1}, X_{T_1}) \\ &+ \sum_{j=1}^{n} R_{T_j} \left[ w(Z_{T_j+}, X_{T_j}) - w(Z_{T_j}, X_{T_j}) \right] \Bigg] 1_{\{T_n < T \leq T_{n+1}\}}. \end{aligned} \tag{26}$$

Now, since $Z$ is constant on the stochastic interval $]T_j, T_{j+1}]$ and $T$ is bounded, (24) implies

$$\big[R_{T_{j+1}} w(Z_{T_{j+1}}, X_{T_{j+1}}) - R_{T_j} w(Z_{T_j+}, X_{T_j})\big] 1_{\{T_{j+1}<T\}} = \Bigg[\int_{T_j}^{T_{j+1}} R_s \hat{\mathcal{L}} w(Z_s, X_s)\, ds$$
$$+ M_{T_{j+1}} - M_{T_j} + \int_{T_j}^{T_{j+1}} R_s Z_s \, dA_s^1 + \int_{T_j}^{T_{j+1}} R_s (1 - Z_s)\, dA_s^0\Bigg] 1_{\{T_{j+1}<T\}},$$

where $M$ is defined as in (13). Since the terms

$$\big[R_T w(Z_T, X_T) - w(z,x)\big] 1_{\{T \le T_1\}},$$
$$\big[R_{T_1} w(Z_{T_1}, X_{T_1}) - w(z,x)\big] 1_{\{T_1 \le T\}},$$
$$\big[R_T w(Z_T, X_T) - R_{T_n} w(Z_{T_n+}, X_{T_n})\big] 1_{\{T_n < T \le T_{n+1}\}}$$

admit similar expressions, (26) implies

$$R_T w(Z_T, X_T) = w(z,x) + \int_0^T R_s \hat{\mathcal{L}} w(Z_s, X_s)\, ds + M_T$$
$$+ \sum_{0 \le s < T} R_s \big[w(Z_{s+}, X_s) - w(Z_s, X_s)\big] + \int_0^T R_s Z_s \, dA_s^1 + \int_0^T R_s (1 - Z_s)\, dA_s^0.$$

It follows that

$$\begin{aligned}
&\int_0^T R_s \big[H_1(X_s) Z_s + H_0(X_s)(1 - Z_s)\big]\, ds \\
&\qquad - \sum_{0 \le s \le T} R_s \left[G_1(X_s)\,(\Delta Z_s)^+ + G_0(X_s)\,(\Delta Z_s)^-\right] - R_T F(X_T) \\
&= w(z,x) - R_T \big[w(Z_{T+}, X_T) + F(X_T)\big] + M_T \\
&\quad + \int_0^T R_s \big[\hat{\mathcal{L}} w(Z_s, X_s) + H_1(X_s) Z_s + H_0(X_s)(1 - Z_s)\big]\, ds \\
&\quad + \sum_{0 \le s \le T} R_s \big[w(1, X_s) - w(0, X_s) - G_1(X_s)\big] (\Delta Z_s)^+ + \int_0^T R_s Z_s \, dA_s^1 \\
&\quad + \sum_{0 \le s \le T} R_s \big[w(0, X_s) - w(1, X_s) - G_0(X_s)\big] (\Delta Z_s)^- + \int_0^T R_s (1 - Z_s)\, dA_s^0.
\end{aligned} \tag{27}$$

In view of (23) and the fact that $w$ satisfies (4) in the sense of Definition 1, this implies

$$\begin{aligned}
&\int_0^T R_s \big[H_1(X_s) Z_s + H_0(X_s)(1 - Z_s)\big]\, ds \\
&\qquad - \sum_{0 \le s \le T} R_s \left[G_1(X_s)\,(\Delta Z_s)^+ + G_0(X_s)\,(\Delta Z_s)^-\right] - R_T F(X_T) \\
&\quad \le w(z,x) + M_T.
\end{aligned}$$

Taking expectations and noting that the stochastic integral has expectation 0, we obtain $J_{z,x}(Z,T) \leq w(z,x)$.

Now, consider the general case where the abandonment time $T$ is not necessarily bounded by a constant, and let $(t_m)$ be a sequence satisfying (11). From our analysis above, it follows that $J_{z,x}(Z, T \wedge t_m) \leq w(z,x)$, for all $m$. However, this and (11) imply $J_{z,x}(Z,T) \leq w(z,x)$, which establishes this part of the theorem.

(b) Suppose that there exists a strategy $(Z^*, T^*)$ satisfying (15)–(18), let $(T_n^*)$ be the associated sequence of stopping times defined as in (25), and let $(t_m)$ be a sequence satisfying (11) as well as (19). Fix any of the stochastic intervals $]T_n^* \wedge T^* \wedge t_m, T_{n+1}^* \wedge T^* \wedge t_m]$, and observe that $Z^*$ is constant on this interval, i.e. $Z_t^* = z$, for some $z \in \{0,1\}$, for all $t \in ]T_n^* \wedge T^* \wedge t_m, T_{n+1}^* \wedge T^* \wedge t_m]$, $P$-a.s.. Since the measure $dL_t^a$ is carried by the set $\{t \geq 0 \mid \ X_t = a\}$, $P$-a.s., (14) and (15) imply

$$L^a_{T_n^* \wedge T^* \wedge t_m} = L^a_{T_{n+1}^* \wedge T^* \wedge t_m}, \quad P\text{-a.s.}, \quad \forall a \in \mathcal{O}_z.$$

Therefore,

$$L^a_{T_n^* \wedge T^* \wedge t_m} = L^a_{T_{n+1}^* \wedge T^* \wedge t_m}, \quad \forall a \in \mathcal{O}_z^d, \tag{28}$$

$P$-a.s., where $\mathcal{O}_z^d$ is any countable subset of $\mathcal{O}_z$ which is dense in $\mathcal{O}_z$. Now, let $A \in \mathcal{F}$ be such that $P(A) = 1$ and (21), (28) are true for all $\omega \in A$. Given any $a \in \mathcal{O}_z \setminus \mathcal{O}_z^d$ and any sequence $(a_m)$ in $\mathcal{O}_z^d$ such that $a_m \downarrow a$ (21) implies

$$L_t^a(\omega) = \lim_{k \to \infty} L_t^{a_k}(\omega), \quad \forall t \geq 0, \quad \forall \omega \in A.$$

However, this and (28) imply

$$L^a_{T_n^* \wedge T^* \wedge t_m}(\omega) = L^a_{T_{n+1}^* \wedge T^* \wedge t_m}(\omega), \quad \forall a \in \mathcal{O}_z, \quad \forall \omega \in A.$$

Combining this with (22), we can see that

$$A^z_{T_n^* \wedge T^* \wedge t_m} = A^z_{T_{n+1}^* \wedge T^* \wedge t_m}, \qquad \text{if } Z_t^* = z, \text{ for } t \in ]T_n^* \wedge T^* \wedge t_m, T_{n+1}^* \wedge T^* \wedge t_m].$$

It follows that

$$\int_0^{T^* \wedge t_m} R_s Z_s^* \, dA_s^1 = \int_0^{T^* \wedge t_m} R_s (1 - Z_s^*) \, dA_s^0 = 0.$$

Therefore, in view of (15)–(18), (27) implies

$$\begin{aligned}
\int_0^{T^* \wedge t_m} & R_s \big[ H_1(X_s) Z_s^* + H_0(X_s)(1 - Z_s^*) \big] \, ds \\
& - \sum_{0 \leq s \leq T^* \wedge t_m} R_s \left[ G_1(X_s) \, (\Delta Z_s^*)^+ + G_0(X_s) \, (\Delta Z_s^*)^- \right] - R_{T^* \wedge t_m} F(X_{T^* \wedge t_m}) \\
&= w(z,x) - 1_{\{T^* > t_m\}} R_{t_m} \left[ w(Z^*_{t_m+}, X_{t_m}) + F(X_{t_m}) \right] + M_{T^* \wedge t_m}.
\end{aligned}$$

Taking expectations and letting $m \to \infty$, we obtain $J_{z,x}(Z^*, T^*) = w(z,x)$, by virtue of (11) and (19), and the proof is complete. □

**Remark 1** To obtain some further insight into the assumptions of the theorem above, suppose that, given a finite number of points $a_1^1 < a_2^1 < \cdots < a_{N^1}^1$ (resp. $a_1^0 < a_2^0 < \cdots < a_{N^0}^0$), $w(1, \cdot)$ (resp. $w(0, \cdot)$) is twice continuously differentiable at every point $x \in \mathcal{I} \setminus \{a_1^1, \ldots, a_{N^1}^1\}$ (resp. $x \in \mathcal{I} \setminus \{a_1^0, \ldots, a_{N^0}^0\}$). Also, suppose that each of the functions $w_x^-(z, \cdot)$ is locally bounded. In this case, assumptions (12), (14) are equivalent to

$$w_x^-(z, a_{i^z}^z) \geq w_x^-(z, a_{i^z}^z+) \equiv \lim_{x \downarrow a_{i^z}^z} w_x^-(z, x), \quad \forall i^z = 1, 2, \ldots, N^z, \tag{29}$$

$$a_1^z, \ldots, a_{N^z}^z \subseteq \mathcal{I} \setminus \operatorname{int} \overline{\Big\{x \in \mathcal{I} \mid \ \hat{\mathcal{L}} w(z, x) + zH_1(x) + (1-z)H_0(x) = 0\Big\}}, \quad z = 0, 1, \tag{30}$$

respectively. For future reference, we should stress that we cannot dispense with either of these two assumptions. Also, it is worth observing the *asymmetry* presented by (12) or (29): had the optimisation problem been a minimisation one, we would have to replace (12) by the assumption that $w_{xx}^{\mathrm{s}}(1, dx)$ and $w_{xx}^{\mathrm{s}}(0, dx)$ are positive measures, and we would have to consider the reverse inequalities in (29). With regard to (30), we can conclude that the points where $C^1$ regularity fails should not belong to the interior of the "continuation" region, but can be allowed in the closure of the "switching" or "stopping" regions.

**Remark 2** The result proved above can be trivially extended to the case where the system's operating modes are not just two, namely "open" and "closed", but are any finite positive integer. On the other hand, the proof cannot be trivially modified to account for the case where the process $X$ assumes values in a higher dimensional state space because it relies heavily on the use of local times and Itô-Tanaka's formula.

To analyse the problem arising if the state process $X$ is a $n$-dimensional diffusion under similarly general assumptions, one would have to resort to the use of viscosity solutions of the associated HJB equation (see Fleming and Soner [FS] and Yong and Zhou [YZ]). This project would aim at proving that the value function identifies with the unique viscosity solution of the HJB equation. Furthermore, characterising the optimal strategy would require a viscosity solution version of the verification Theorem 1 in the spirit of Theorem 5.5.3 in Yong and Zhou [YZ]. Such an analysis lies beyond the scope of this article, and we leave it as an interesting open problem.

# 4 The explicit solution of a special case

We now solve completely the special case of the general control problem formulated in Section 2 that arises if we impose the following assumption.

**Assumption 1** $\mathcal{I} = \mathbb{R}$, and $b(x) = 1$, $\sigma(x) = 1$, $r(x) = r$, $H_1(x) = x$, $H_0(x) = 0$, $G_1(x) = K_1$, $G_0(x) = K_0$ and $F(x) = K$, for some constants $r, K_1, K_0, K > 0$, for all $x \in \mathbb{R}$.

In this case, the HJB equation (4) reduces to the following pair of coupled quasi-variational inequalities:

$$\max\left\{\tfrac{1}{2}v_1''(x) - rv_1(x) + x,\ v_0(x) - v_1(x) - K_0,\ -v_1(x) - K\right\} = 0, \tag{31}$$

$$\max\left\{\tfrac{1}{2}v_0''(x) - rv_0(x),\ v_1(x) - v_0(x) - K_1,\ -v_0(x) - K\right\} = 0. \tag{32}$$

Here, we write $v_1$ and $v_0$ in place of $v(1,\cdot)$ and $v(0,\cdot)$, respectively, to simplify the notation.

To make some headway, we first make some qualitative observations. Since the system yields 0 payoff whenever it operates in its "closed" mode and the abandonment cost $K$ is positive, it follows that abandonment cannot be optimal when the system is in its "closed" mode. As a consequence, abandonment can be part of the optimal strategy only if the system is in its "open" operating mode. Moreover, the system should be in its "open" operating mode if the state process $X$ assumes sufficiently large values and should be in its "closed" operating mode or should be abandoned if the state process $X$ takes sufficiently low values.

Now, a first possibility arises if abandonment is not part of the optimal scenario. In such a case, we should switch the system from its "closed" to its "open" mode whenever the state process $X$ exceeds a level specified by a constant $\alpha$, and we should switch the system from its "open" to its "closed" mode whenever the state process $X$ falls below a level given by a constant $\beta$. Clearly, such a strategy is well defined only if $\beta < \alpha$. It can be depicted by Figure 1.

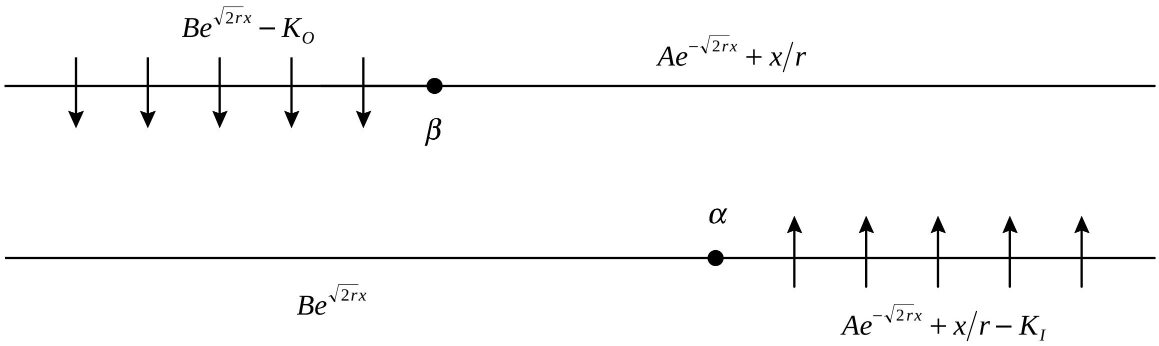


Figure 1: The "no-abandonment case".

If such a strategy is indeed optimal, the value function should be given by a solution $w_1$, $w_0$ of the HJB equations (31)–(32) described as follows. For $x > \beta$, $w_1$ should satisfy $\frac{1}{2}w_1''(x) - rw_1(x) + x = 0$, namely $w_1(x) = Ae^{-\sqrt{2r}x} + C_1e^{\sqrt{2r}x} + x/r$, for some constants $A, C_1 \in \mathbb{R}$, whereas, for $x \leq \beta$, $w_1$ should be given by $w_1(x) = w_0(x) - K_0$. On the other hand, if $x < \alpha$, $w_0$ should satisfy $\frac{1}{2}w_0''(x) - rw_0(x) = 0$, namely $w_0(x) = C_2e^{-\sqrt{2r}x} + Be^{\sqrt{2r}x}$, for some constants $C_2, B \in \mathbb{R}$, whereas, for $x \geq \alpha$, $w_0$ should be given by $w_0(x) = w_1(x) - K_1$. Now, we must have $C_1 = C_2 = 0$, because, otherwise, the assumptions of Theorem 1 cannot be satisfied. In view of these conditions, $w_1$, $w_0$ should be given by

$$w_1(x) = \begin{cases} Be^{\sqrt{2r}x} - K_0 & \text{if } x \leq \beta, \\ Ae^{-\sqrt{2r}x} + x/r & \text{if } x > \beta, \end{cases} \qquad w_0(x) = \begin{cases} Be^{\sqrt{2r}x} & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_1 & \text{if } x \geq \alpha, \end{cases} \tag{33}$$

respectively. To specify the parameters $A$, $B$, $\alpha$, $\beta$, we postulate that $w_1$, $w_0$ are $C^1$ at the free boundary points $\beta$, $\alpha$, respectively. This requirement gives rise to the system of equations

$$Be^{\sqrt{2r}\alpha} - Ae^{-\sqrt{2r}\alpha} = \frac{\alpha - rK_1}{r}, \tag{34}$$

$$Be^{\sqrt{2r}\alpha} + Ae^{-\sqrt{2r}\alpha} = \frac{1}{r\sqrt{2r}}, \tag{35}$$

$$Be^{\sqrt{2r}\beta} - Ae^{-\sqrt{2r}\beta} = \frac{\beta + rK_0}{r}, \tag{36}$$

$$Be^{\sqrt{2r}\beta} + Ae^{-\sqrt{2r}\beta} = \frac{1}{r\sqrt{2r}.} \tag{37}$$

It is straightforward to verify that these are equivalent to

$$A = -\frac{\beta + rK_0 - 1/\sqrt{2r}}{2r} e^{\sqrt{2r}\beta}, \tag{38}$$

$$B = \frac{\beta + rK_0 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\beta}, \tag{39}$$

$$\left(\alpha - rK_1 - 1/\sqrt{2r}\right) e^{\sqrt{2r}\alpha} = \left(\beta + rK_0 - 1/\sqrt{2r}\right) e^{\sqrt{2r}\beta}, \tag{40}$$

$$\left(\alpha - rK_1 + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\alpha} = \left(\beta + rK_0 + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\beta}. \tag{41}$$

The next lemma is concerned with the solvability of (40)–(41) and with necessary and sufficient conditions under which the functions $w_1$, $w_0$ given above satisfy the HJB equations (31)–(32). To derive the results of Lemma 4 below, we assume here that the constants $K_1$, $K_0$ can take negative as well as positive values, subject to the condition that $K_1 + K_0 > 0$.

**Lemma 2** *Suppose that $r > 0$ and $K_1, K_0 \in \mathbb{R}$ satisfy $K_1 + K_0 > 0$. There exists a unique pair of points $\alpha = \alpha(r, K_1, K_0)$ and $\beta = \beta(r, K_1, K_0)$ which satisfies (40)–(41). Point $\beta$ is the unique solution of*

$$H(\beta) := \frac{\beta + rK_0 + 1/\sqrt{2r}}{\beta + rK_0 - 1/\sqrt{2r}} \exp\left(-\sqrt{2r}(2\beta + rK_0 - rK_1)\right) = -1, \tag{42}$$

*and satisfies*

$$-rK_0 - \frac{1}{\sqrt{2r}} < \beta < -rK_0, \tag{43}$$

$$-2e^{-2} > \sqrt{2r}\left(\beta + rK_0 - 1/\sqrt{2r}\right)\exp\left(\sqrt{2r}(\beta + rK_0 - 1/\sqrt{2r})\right) > -e^{-1}, \tag{44}$$

*whereas*

$$\alpha = -\beta - rK_0 + rK_1 > \beta. \tag{45}$$

*The functions $w_1$, $w_0$ defined by (33), where $\alpha$ and $\beta$ are as above and $A, B > 0$ are given by (38), (39), respectively, are convex, non-decreasing, $C^1$ for all $x \in \mathbb{R}$ and $C^2$ for all $x \in \mathbb{R} \setminus \{\beta\}$, $x \in \mathbb{R} \setminus \{\alpha\}$, respectively, and satisfy*

$$\max\left\{\tfrac{1}{2}w_1''(x) - rw_1(x) + x,\ w_0(x) - K_0 - w_1(x)\right\} = 0, \quad \forall x \in \mathbb{R} \setminus \{\beta\}, \tag{46}$$

$$\max\left\{\tfrac{1}{2}w_0''(x) - rw_0(x),\ w_1(x) - K_1 - w_0(x),\ -K - w_0(x)\right\} = 0, \quad \forall x \in \mathbb{R} \setminus \{\alpha\}. \tag{47}$$

*Moreover, $w_1(x) \geq -K$ if and only if $K \geq K_0$.*

We collect in the Appendix the proofs of those results that are not developed in the text.

If the condition $K \geq K_0$ is not satisfied, we expect that abandonment becomes part of the optimal scenario. Now, assuming that the optimal strategy has a continuous qualitative character, we should expect that, as $K_0$ rises above $K$, abandonment should become optimal if the system is "open" and the state process $X$ assumes sufficiently small values. The obvious modification of the strategy studied above, can be depicted by Figure 2. Such a possibility involves 5 parameters and 3 free boundary points, so we cannot impose a $C^1$ fit at all of the free boundary points.

By an obvious symmetry argument, we can conclude that the value function is $C^1$ at the points $\alpha$, $\beta$ and $C^0$ at the point $\gamma$. However, by elementary considerations, we can see that the value function is non-decreasing in $x$. Therefore, if the optimal strategy identifies with the one depicted by Figure 2, we must have $w_1(\gamma-) = 0 < w_1(\gamma+)$, which is unacceptable in the light of Remark 1. Alternatively, we can postulate that the value function is $C^1$ at $\gamma$ and $\beta$ (resp. $\alpha$), and $C^0$ at $\alpha$ (resp. $\beta$). However, such a possibility would impose a discontinuity of the first derivative of the candidate value functions inside the interior of the "continuation" region, which is again contradicting the conclusions of Remark 1. It turns out that a strategy having the form depicted by Figure 2 cannot be optimal. However, the idea that the optimal strategy should possess a character which depends continuously on the problem's data leads us to the conclusion that we should look for a further modification of this strategy. Such a modification can be obtained by inserting a "do-not-abandon-or-switch-off" region around $\gamma$, so that the interface of the "abandonment" and the "switch off" regions is not just a point but an interval. This strategy can be depicted by Figure 3.

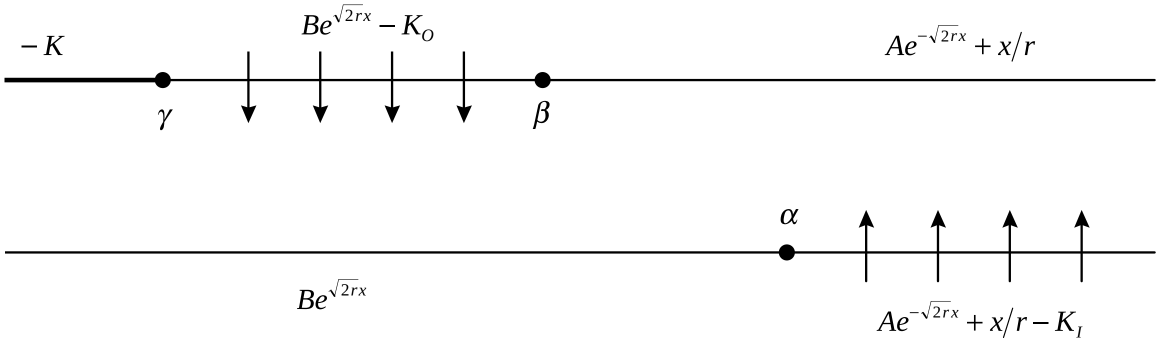


Figure 2: An obvious modification of the “no-abandonment case”.

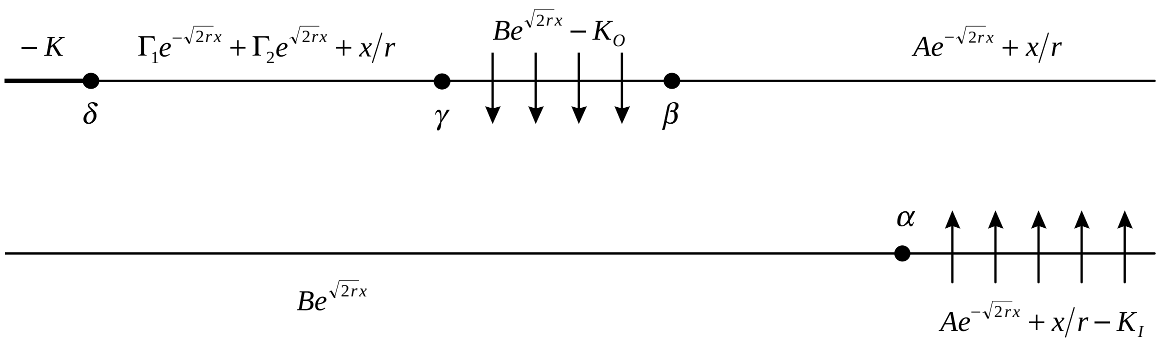


Figure 3: The case where abandonment becomes part of the optimal tactics.

If this case is indeed optimal, the value function of the control problem should identify with a solution $w_1$, $w_0$ of the HJB equations (31)–(32) described by

$$w_1(x) = \begin{cases} -K & \text{if } x \leq \delta, \\ \Gamma_1 e^{-\sqrt{2r}x} + \Gamma_2 e^{\sqrt{2r}x} + x/r & \text{if } \delta < x < \gamma, \\ Be^{\sqrt{2r}x} - K_0 & \text{if } \gamma \leq x \leq \beta, \\ Ae^{-\sqrt{2r}x} + x/r & \text{if } x > \beta, \end{cases} \tag{48}$$

$$w_0(x) = \begin{cases} Be^{\sqrt{2r}x} & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_1 & \text{if } x \geq \alpha. \end{cases} \tag{49}$$

The parameters $A$, $B$, $\Gamma_1$, $\Gamma_2$, $\alpha$, $\beta$, $\gamma$, $\delta$ can then be specified by the requirement that $w_1$, $w_0$ are $C^1$ at the free boundary points $\alpha$, $\beta$, $\gamma$, $\delta$. Now, it is a straightforward calculation to verify that this requirement implies that $\alpha$, $\beta$, $A$, $B$, should satisfy (38)–(41),

$$\Gamma_1 = -\frac{\gamma + rK_0 - 1/\sqrt{2r}}{2r} e^{\sqrt{2r}\gamma}, \tag{50}$$

$$\Gamma_2 = B - \frac{\gamma + rK_0 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\gamma}, \tag{51}$$

and $\gamma$, $\delta$ should satisfy the system of equations

$$F_1(\gamma, \delta) := \left(\delta + rK - 1/\sqrt{2r}\right) e^{\sqrt{2r}\delta} - \left(\gamma + rK_0 - 1/\sqrt{2r}\right) e^{\sqrt{2r}\gamma} = 0, \tag{52}$$

$$F_2(\gamma, \delta) := \left(\delta + rK + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\delta} - \left(\gamma + rK_0 + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\gamma} + 2rB = 0. \tag{53}$$

The next lemma is concerned with the solvability of (52)–(53) as well as with necessary and sufficient conditions under which the functions $w_1$, $w_0$ considered above satisfy the HJB equations (31)–(32).

**Lemma 3** *Let $\alpha = \alpha(r, K_1, K_0)$, $\beta = \beta(r, K_1, K_0)$, $A$, $B$ be as in Lemma 2. The system of equations (52)–(53) has a unique solution $\gamma = \gamma(r, K_1, K_0, K)$, $\delta = \delta(r, K_1, K_0, K)$ such that $\delta < \gamma < \beta$ if and only if*

$$K_* \vee 0 < K < K_0, \tag{54}$$

*where $K_* = K_*(r, K_1, K_0) < K_0$ is defined by*

$$K_* = -\frac{1}{r\sqrt{2r}} \ln \left( -\frac{\sqrt{2r}}{2} \left(\beta + rK_0 - 1/\sqrt{2r}\right) \exp\left(\sqrt{2r}(\beta + 1/\sqrt{2r})\right) \right). \tag{55}$$

*If $K_* > 0$ and $K = K_*$, then $\gamma = \beta$, $\delta = -rK - 1/\sqrt{2r}$, $\Gamma_1 = A$, and $\Gamma_2 = 0$. If (54) is true, then the functions $w_1$, $w_0$ defined by (48), (49), respectively, where $\Gamma_1, \Gamma_2 > 0$ are given by (50)–(51), are convex, non-decreasing, $C^1$ for all $x \in \mathbb{R}$ and $C^2$ for all $x \in \mathbb{R} \setminus \{\delta, \gamma, \beta\}$, $x \in \mathbb{R} \setminus \{\alpha\}$, respectively, and satisfy the HJB equations (31)–(32).*

The optimality of the case considered in the previous lemma depends crucially on the parameter $K_*$. If $K_* \leq 0$ for every admissible choice of the problem's data, then our solution is complete. However, it turns out that this is not in general the case.

**Lemma 4** *Given any values of the parameters $r, K_1 > 0$, the function $K_*(r, K_1, \cdot)$ is well defined on $]-K_1, \infty[$, is strictly increasing and at least $C^1$ on this interval, and satisfies $\lim_{K_0 \to \infty} K_*(r, K, K_0) = \infty$ and $K_*(r, K, 0) < 0$.*

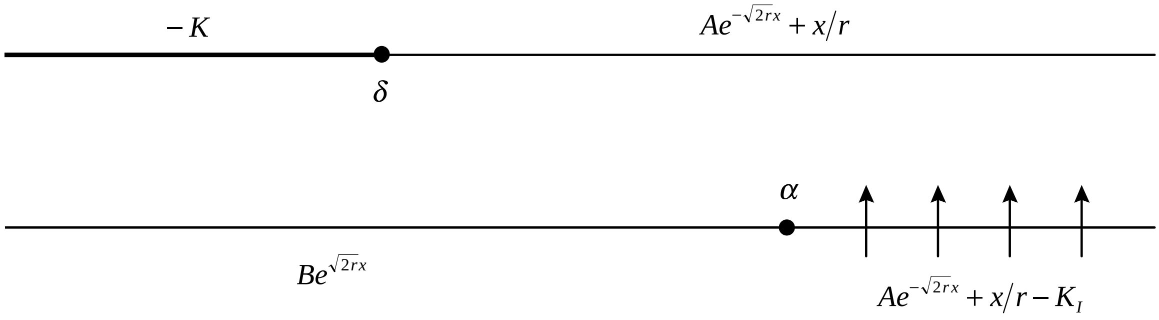


Figure 4: The case where switching the system to its "closed" mode is never optimal.

In Lemma 3, we proved that if $K_* > 0$ and $K = K_*$, then $\gamma = \beta$, so the "switch-from-open-to-closed" region disappears, and the optimal strategy can be depicted by Figure 4. For $K < K_*$, we can expect that it is not optimal to switch the system from its "open" to its "closed" operating mode at any time, so that the optimal strategy can again be depicted by Figure 4.

If this strategy is indeed optimal, the value function should be given in terms of the functions

$$w_1(x) = \begin{cases} -K & \text{if } x \le \delta, \\ Ae^{-\sqrt{2r}x} + x/r & \text{if } x > \delta, \end{cases} \qquad w_0(x) = \begin{cases} Be^{\sqrt{2r}x} & \text{if } x < \alpha, \\ Ae^{-\sqrt{2r}x} + x/r - K_1 & \text{if } x \ge \alpha. \end{cases} \tag{56}$$

Again, we require that $w_1$, $w_0$ are $C^1$ at the free boundary points $\delta$, $\alpha$, respectively. Straightforward calculations show that $C^1$ fit at $\delta$ yields

$$A = \frac{1}{r\sqrt{2r}} \exp\left(-\sqrt{2r}(rK + 1/\sqrt{2r})\right), \tag{57}$$

$$\delta = -rK - \frac{1}{\sqrt{2r}}, \tag{58}$$

whereas $C^1$ fit at $\alpha$ yields the system of equations

$$Be^{\sqrt{2r}\alpha} - Ae^{-\sqrt{2r}\alpha} = \frac{\alpha - rK_1}{r}, \tag{59}$$

$$Be^{\sqrt{2r}\alpha} + Ae^{-\sqrt{2r}\alpha} = \frac{1}{r\sqrt{2r}}, \tag{60}$$

which is equivalent to

$$B = \frac{\alpha - rK_1 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\alpha}, \tag{61}$$

$$G(\alpha) := \frac{\sqrt{2r}}{2} \left(\alpha - rK_1 - 1/\sqrt{2r}\right) \exp\left(\sqrt{2r}(\alpha + rK + 1/\sqrt{2r})\right) = -1. \tag{62}$$

The next lemma is concerned with the solvability of (62) and with necessary and sufficient conditions under which this case is optimal.

**Lemma 5** *Equation (62) has a unique solution* $\alpha = \alpha(r, K_1, K)$ *such that* $\alpha > -rK - 1/\sqrt{2r}$*. For this value of* $\alpha$*, and for* $A, B > 0$ *and* $\delta = \delta(r, K)$ *given by (57), (61) and (58), respectively, the functions* $w_1$*,* $w_0$ *defined by (56) are convex, non-decreasing,* $C^1$ *for all* $x \in \mathbb{R}$ *and* $C^2$ *for all* $x \in \mathbb{R} \setminus \{\delta\}$*,* $\mathbb{R} \setminus \{\alpha\}$*, respectively. Moreover, assuming that* $K_* > 0$*, they satisfy the HJB equations (31)–(32) if and only if* $0 < K \leq K_*$*.*

We can now prove the main result of the section.

**Theorem 6** *Consider the stochastic optimisation problem defined in Section 2, and suppose that its data are as in Assumption 1. The value function* $v$ *is* $C^1$*, convex and non-decreasing in* $x$*, and is given by* $v(1, \cdot) = w_1$ *and* $v(0, \cdot) = w_0$*, where:*
(I) *If* $K_0 \leq K$*,* $w_1$*,* $w_0$ *are given by Lemma 2 (see Figure 1).*
(II) *If* $K_* < K < K_0$*, where* $K_* < K_0$ *is given by (55),* $w_1$*,* $w_0$ *are given by Lemma 3 (see Figure 3).*
(III) *If* $K_* > 0$ *and* $K \leq K_*$*,* $w_1$*,* $w_0$ *are given by Lemma 5 (see Figure 4).*
*In each of the three cases, the optimal strategy can be constructed as in the proof below.*

**Proof.** First, observe that, given any sequence $t_m \to \infty$, (11) and the second limit in (19) are true for all $(Z, T) \in \Pi_z$. Also, in each of the three cases, the functions $w_1$ and $w_0$ are convex and non-decreasing.

Now, consider any of the three cases. Since $w_1 \equiv w(1, \cdot)$ and $w_0 \equiv w(0, \cdot)$ are $C^1$ for all $x$, and $C^2$ for all $x$ outside a finite set, their second distributional derivatives are measures that are absolutely continuous with respect to the Lebesgue measure. With regard to the notation of Definition 1, this implies that $w^{\mathrm{s}}_{xx}(1, dx) \equiv 0$ and $w^{\mathrm{s}}_{xx}(0, dx) \equiv 0$, and therefore, (12) as well as (14) are true. Moreover, $w_1 \equiv w(1, \cdot)$ and $w_0 \equiv w(0, \cdot)$ satisfy the HJB equations (31)–(32) in the classical sense, by construction, and therefore, in the sense of Definition 1.

Since $w(1, \cdot)$ and $w(0, \cdot)$ have bounded first derivatives the process $M$ defined as in (13) is a square integrable martingale, for all $Z \in \mathcal{Z}$. Furthermore, there exist constants $C_1$ and $C_2$ such that

$$w(z, x) \leq C_1 + C_2|x|, \quad \text{for all } z = 1, 0 \text{ and } x \in \mathbb{R}.$$

It follows that, given any $Z \in \mathcal{Z}$,

$$\lim_{t\to\infty} E\left[e^{-rt}\left|w(Z_t, X_t)\right|\right] \leq \lim_{t\to\infty} E\left[e^{-rt}\left(C_1 + C_2\left|x + W_t\right|\right)\right] = 0.$$

However, this shows that (18) is satisfied for all $Z \in \mathcal{Z}$, and therefore, for the optimal switching process.

The above arguments prove that, in any of the three cases, $w_1 \equiv w(1, \cdot)$ and $w_0 \equiv w(0, \cdot)$ satisfy all of the assumptions related to part (a) of Theorem 1, as well as (14) and (19). As a consequence, to complete the proof, we have to construct a strategy $(Z^*, T^*)$ satisfying (15)–(18).

Now, in Case I, if $z = 1$, then we can see that the strategy $(Z^*, T^*)$, where $T^* = \infty$ and the process $Z^* \in \mathcal{Z}$ is defined by

$$Z_t^* = 1_{\{t=0\}} + \sum_{j=0}^{\infty} 1_{\{T_{2j}^* < t \leq T_{2j+1}\}^*}, \tag{63}$$

where $T_0^* = 0$ and the stopping times $T_n^*$, $n \in \mathbb{N}^*$, are defined recursively by

$$T_{2n+1}^* = \inf\{t \geq T_{2n}^* \mid X_t \leq \beta\}, \quad n = 0, 1, 2, \ldots, \tag{64}$$
$$T_{2n}^* = \inf\{t \geq T_{2n-1}^* \mid X_t \geq \alpha\}, \quad n = 1, 2, \ldots, \tag{65}$$

satisfies (15)–(18). If $z = 0$, we again have $T^* = \infty$ and the optimal switching process $Z^*$ can be constructed in a similar fashion.

In Case II, if $z = 1$ and $x \geq \gamma$, or if $z = 0$, the optimal strategy is the same as in Case I. If $z = 1$ and $x \leq \delta$, then the optimal strategy is characterised by $T^* = 0$. If $z = 1$ and $\delta < x < \gamma$, then define

$$T_\delta = \inf\{t \geq 0 \mid X_t \leq \delta\} \quad \text{and} \quad T_\gamma = \inf\{t \geq 0 \mid X_t \geq \gamma\} \tag{66}$$

and let

$$T^* = T_\delta 1_{\{T_\delta < T_\gamma\}} + \infty 1_{\{T_\delta > T_\gamma\}} \in \mathcal{S}.$$

Also let $T_0^* = 0$, $T_1^* = T_\gamma$ and define $T_n^*$, $n \geq 2$, as in (65). Then we can see that the strategy $(Z^*, T^*)$ where $Z^*$ is defined as in (63), satisfies (15)–(18).

In Case III, if $z = 1$, then $T^* = T^\delta$, where $T_\delta$ is defined as in (66), and $Z^*$ defined by $Z_t^* = 1$, for all $t \geq 0$, provide the optimal strategy. Finally, if $z = 0$, then $T^* = \inf\{t \geq T_\alpha \mid X_t \leq \delta\}$, where $T_\alpha = \inf\{t \geq 0 \mid X_t \geq \alpha\}$, and $Z^*$ defined by $Z_t^* = 1_{\{T_\alpha < t\}}$, $t \geq 0$, are the optimal strategy, and the proof is complete. □

**Remark 3** The rather unexpected qualitative nature of Case II is intimately related to optimal stopping. To understand this claim, consider the case as a perturbation of Case I, where stopping is not part of the optimal strategy. With regard to the heuristic discussion at the beginning of the section, abandonment can be optimal only if the system is "open" and the state process $X$ assumes sufficiently low values. As a result, the optimal strategy should possess the same qualitative nature as in Case I if the system is "closed" or if the system is "open" and the state process $X$ assumes sufficiently large values. Now, as the abandonment cost $K$ falls marginally below the critical value $K_0$ and abandonment comes into the picture, a continuity argument dictates that the switching boundary points $\alpha$ and $\beta$ as given by Lemma 2 should be "close" to the optimal ones. However, these points determine completely the function $w_0$. As a consequence, for sufficiently low values of $X$, the function $w_1$ should be "close" to the value function of the purely optimal stopping problem which seeks to maximise

$$E\left[\int_0^\tau e^{-rs} X_s\, ds - e^{-r\tau}\big[K \vee (K_0 - w_0(X_\tau))\big]\right],$$

over all stopping times $\tau \in \mathcal{S}$. In fact, we have proved that $w_1$ identifies with the value function of this purely optimal stopping problem for appropriate parameter values. Note that the terminal payoff function $-K \vee (K_0 - w_0(\cdot))$ of this problem is not $C^1$. From these observations, we can conclude that the existence of a "continuation" region such as the interval $]\delta, \gamma[$ in Case II should characterise the optimal strategy in purely optimal stopping problems where the first derivative of the terminal payoff function has appropriate discontinuities.

# 5 An example where $C^1$ regularity of the value function fails

Based on the results established in the previous sections, we can easily construct an example whose value function is not composed by $C^1$ functions. To this end, consider the problem formulated in Section 2, and assume that $\mathcal{I} = \mathbb{R}$, and

$$b(x) = 0, \quad \sigma(x) = 1, \quad r(x) = \tfrac{1}{2}, \quad H_1(x) = x, \quad H_0(x) = 0,$$
$$G_1(x) = 2, \quad G_0(x) = 10, \quad F(x) = \begin{cases} 5 - e^{x+10} & \text{if } x < -10, \\ 4 & \text{if } x \geq -10, \end{cases}$$

for all $x \in \mathbb{R}$.

With regard to (42) and (55), we calculate

$$\beta\left(\tfrac{1}{2}, 2, 10\right) = -5.999328399 \quad \text{and} \quad K_*\left(\tfrac{1}{2}, 2, 10\right) = 9.999328511.$$

Since $K_* > F(x)$, for all $x \in \mathbb{R}$, this example is akin to Case III of Theorem 6. The values of the associated parameters are

$$\delta\left(\tfrac{1}{2}, 4\right) = -3, \quad \alpha\left(\tfrac{1}{2}, 2, 4\right) = 1.986338745,$$
$$A = 0.099574137 \quad \text{and} \quad B = 0.272519358.$$

The value function of the example under consideration is given by

$$v(1,x) = \begin{cases} e^{x+10} - 5 & \text{if } x < -10, \\ -4 & \text{if } -10 \le x < -3, \\ 0.099574137e^{-x} + 2x & \text{if } -3 \le x, \end{cases}$$

$$v(0,x) = \begin{cases} 0.272519358e^{x} & \text{if } x < 1.9886338745, \\ 0.099574137e^{-x} + 2x - 2 & \text{if } x \ge 1.9886338745. \end{cases}$$

To see this, observe first that the only point where $C^1$ regularity fails is given by $z = 1$ and $x = -10$. Clearly, (12) and (14) are satisfied (see also Remark 1). Now, with reference to the proofs of Lemma 5 and Theorem 6, all of the assumptions of Theorem 1 will follow if we verify that

$$\begin{aligned} \tfrac{1}{2}w_1''(x) - \tfrac{1}{2}w_1(x) + x \le 0, &\quad \text{for } x < -3, \\ w_0(x) - 10 - w_1(x) \le 0, &\quad \text{for } x < -3, \\ w_1(x) - 2 - w_0(x) \le 0, &\quad \text{for } x < -3. \end{aligned}$$

However, this is a trivial exercise.

# Acknowledgements

I am grateful to Kate Duckworth for several discussions on the problem solved in this paper. A preliminary exposition of part of the results obtained here has been presented in Duckworth and Zervos [DuZ2].

# Appendix: proofs of results in Section 4

**Proof of Lemma 2.** Multiplying (40) and (41) side by side, and solving for $\alpha$, we obtain

$$\alpha = \beta + r(K_1 + K_0) \quad \text{or} \quad \alpha = -\beta - rK_0 + rK_1.$$

Substituting $\beta + r(K_1 + K_0)$ for $\alpha$ in (40) and (41), we obtain

$$\begin{aligned} \left(\beta + rK_0 - 1/\sqrt{2r}\right) e^{r\sqrt{2r}(K_1+K_0)} &= \beta + rK_0 - 1/\sqrt{2r}, \\ \left(\beta + rK_0 + 1/\sqrt{2r}\right) e^{-r\sqrt{2r}(K_1+K_0)} &= \beta + rK_0 + 1/\sqrt{2r}, \end{aligned}$$

respectively. Since $K_1 + K_0 > 0$, there is no $\beta$ satisfying both of these equations. Therefore, $\alpha$ must be as in (45). Now, (45) and either (40) or (41) yield (42).

Since $H(\beta) > 0$, for all $\beta < -rK_0 - 1/\sqrt{2r}$ and all $\beta > -rK_0 + 1/\sqrt{2r}$, if (42) has a solution, then this has to satisfy $-rK_0 - 1/\sqrt{2r} \le \beta \le -rK_0 + 1/\sqrt{2r}$. Now,

$$H'(\beta) = -2\sqrt{2r}\frac{(\beta + rK_0)^2}{(\beta + rK_0 - 1/\sqrt{2r})^2} \exp\left(-\sqrt{2r}(2\beta + rK_0 - rK_1)\right),$$

which implies that $H$ is strictly decreasing in $\mathbb{R} \setminus \{-rK_0, -rK_0 + 1/\sqrt{2r}\}$. Combining this with the facts that $H(-rK_0 - 1/\sqrt{2r}) = 0$ and $H(-rK_0) = -\exp(r\sqrt{2r}(K_1 + K_0)) < -1$, we conclude that (42) has a unique solution which satisfies (43). Observe that the inequality $\beta < -rK_0$ and the expression $\alpha = -\beta - rK_0 + rK_1$ imply trivially that $\beta < \alpha$. Also, (43) implies (44) because the function $x \to xe^x$ is strictly decreasing in $]-\infty, -1[$. Furthermore, (43) along with (38) and (39) imply that $A, B > 0$.

Since $A, B > 0$, the functions $w_1$, $w_0$ are convex and non-decreasing. As a consequence, $-K \le w_1$ if and only if $K \ge K_0$, and $-K \le w_0$. Now, to verify that $w_1$, $w_0$ satisfy (46) and (47), we have to prove that

$$\tfrac{1}{2}w_1''(x) - rw_1(x) + x \le 0, \quad \text{for } x < \beta, \tag{67}$$
$$w_1(x) - K_1 - w_0(x) \le 0, \quad \text{for } x \le \beta, \tag{68}$$
$$w_0(x) - K_0 - w_1(x) \le 0, \quad \text{for } \beta \le x \le \alpha, \tag{69}$$
$$w_1(x) - K_1 - w_0(x) \le 0, \quad \text{for } \beta \le x \le \alpha, \tag{70}$$
$$w_0(x) - K_0 - w_1(x) \le 0, \quad \text{for } x \ge \alpha, \tag{71}$$
$$\tfrac{1}{2}w_0''(x) - rw_0(x) \le 0, \quad \text{for } x > \alpha. \tag{72}$$

Each of (68) and (71) is equivalent to $-K_1 - K_0 \le 0$, which is true. Inequality (67) is trivially implied by $rK_0 + \beta < 0$ (see (43)), whereas (72) is trivially implied by $-\alpha + rK_1 = \beta + rK_0 < 0$.

Now, consider the function $g$ defined by

$$g(x) := Be^{\sqrt{2r}x} - Ae^{-\sqrt{2r}x} - \frac{x}{r} - K_0.$$

Since $g(x) = w_0(x) - K_0 - w_1(x)$, for $x \in [\beta, \alpha]$, (69) and (70) will follow if we prove that

$$-K_1 - K_0 \le g(x) \le 0, \quad \text{for all } x \in [\beta, \alpha]. \tag{73}$$

The function $g'$ is strictly convex because

$$g'''(x) = 2r\sqrt{2r}\left(Be^{\sqrt{2r}x} + Ae^{-\sqrt{2r}x}\right) > 0,$$

the inequality being true because $A, B > 0$. As a consequence, $g'(x) < 0$, for all $x \in ]\beta, \alpha[$, because $g'(\beta) = g'(\alpha) = 0$, by construction. However, combining this observation with the fact that $g(\beta) = 0$ and $g(\alpha) = -K_0 - K_1$, we conclude that (73) is true. $\square$

**Proof of Lemma 3.** Fix any $\gamma < -rK_0$, and consider the equation

$$f(\delta) := F_1(\gamma, \delta) = 0. \tag{74}$$

From the calculations

$$\lim_{\delta \to -\infty} f(\delta) = -\left(\gamma + rK_0 - 1/\sqrt{2r}\right) e^{\sqrt{2r}\gamma} > 0, \tag{75}$$

$$f'(\delta) = \sqrt{2r}(\delta + rK)e^{\sqrt{2r}\delta}, \tag{76}$$

$$f(\gamma) = r(K - K_0)e^{\sqrt{2r}\gamma}, \tag{77}$$

we can see that (74) has a unique solution $\delta < \gamma$ if $K < K_0$. These calculations also imply that (74) does not have a solution $\delta < \gamma$ if $K > K_0$ and $\gamma < -rK$. If $K > K_0$ and $-rK < \gamma$, (74) will have a solution only if $f$ is negative at $\delta = -rK$ where its minimum over $\delta \in ]-\infty, \gamma]$ occurs. However,

$$f(-rK) = -(\gamma + rK_0)e^{\sqrt{2r}\gamma} + \frac{1}{\sqrt{2r}}\left[e^{\sqrt{2r}\gamma} - e^{\sqrt{2r}(-rK)}\right] > 0,$$

the inequality following because $-rK < \gamma < -rK_0$. From these considerations, we conclude that, given any $\gamma < -rK_0$, (74) has a unique solution $\delta < \gamma$ if and only if $K < K_0$. For the rest of this proof, we assume that this condition is satisfied.

From the above, we can see that, as $\gamma$ varies, (74) defines uniquely a function $\delta = \delta(\gamma)$ on $]-\infty, -rK_0[$ such that $\delta(\gamma) < \gamma$, for all $\gamma < -rK_0$. Also, by implicit differentiation of (74), we obtain

$$\delta'(\gamma) = \frac{\gamma + rK_0}{\delta(\gamma) + rK} e^{\sqrt{2r}(\gamma - \delta(\gamma))}. \tag{78}$$

Now consider the equation

$$g(\gamma) := F_2(\gamma, \delta(\gamma)) = 0, \quad \text{for } \gamma < \beta. \tag{79}$$

Since

$$g(\gamma) = \sqrt{2r}\int_{\delta(\gamma)}^{\gamma} e^{-\sqrt{2r}s}(s + rK)\, ds + r(K - K_0)e^{-\sqrt{2r}\gamma} + 2rB,$$

and $K < K_0$, it follows that

$$\lim_{\gamma \to -\infty} g(\gamma) = -\infty.$$

Also, using (78), we can calculate

$$g'(\gamma) = -2\sqrt{2r}(\gamma + rK_0)e^{-\sqrt{2r}\delta(\gamma)} \sinh\left[\sqrt{2r}(\gamma - \delta(\gamma))\right] > 0,$$

whereas, in view of (39),

$$g(\beta) = \left(\delta(\beta) + rK + 1/\sqrt{2r}\right) e^{-\sqrt{2r}\delta(\beta)}.$$

From (80)–(80), we can see that (79) has a unique solution $\gamma < \beta$ if and only if $\delta(\beta) > -rK - \frac{1}{\sqrt{2r}}$. With regard to the analysis relating to (74), this will be true if and only if $F_1(\beta, -rK - 1/\sqrt{2r}) > 0$, i.e. if and only if

$$-2e^{-2} > \sqrt{2r}\left(\beta + rK_0 - 1/\sqrt{2r}\right) \exp\left(\sqrt{2r}(\beta + rK - 1/\sqrt{2r})\right). \tag{80}$$

With reference to (44), this is true for $K = K_0$. Furthermore, the right hand side of this inequality is increasing as $K$ decreases. As a consequence, (80) is true for all $K \in ]K_* \vee 0, K_0[$, where $K_* < K_0$ is given by (55).

With regard to the arguments above, if $K_* > 0$ and $K = K_*$, then (80) holds with equality, $\gamma = \beta$ and $\delta = -rK - 1/\sqrt{2r}$. From (38), (50) and (39), (51), it then follows that $\Gamma_1 = A$ and $\Gamma_2 = 0$, respectively.

Since $\gamma < \beta < -rK_0$, (50) implies that $\Gamma_1 > 0$. Furthermore, since

$$\frac{d}{dy}\left[\frac{y + rK_0 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}y}\right] = -\frac{1}{\sqrt{2r}}(y + rK_0)e^{-\sqrt{2r}y} > 0, \quad \forall y < -rK_0,$$

it follows that

$$\frac{\beta + rK_0 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\beta} > \frac{\gamma + rK_0 + 1/\sqrt{2r}}{2r} e^{-\sqrt{2r}\gamma}.$$

Therefore, (39) and (51) imply $\Gamma_2 > 0$.

Since $A, B, \Gamma_1, \Gamma_2 > 0$, the functions $w_1$, $w_0$ are convex and non-decreasing, so $w_1, w_0 \geq -K$. Also, all of the inequalities associated with the HJB equations (31)–(32) for $x \geq \gamma$ follow from Lemma 2. Therefore, to verify that $w_1$, $w_0$ satisfy the HJB equations (31)–(32), it remains to show that

$$\tfrac{1}{2}w_1''(x) - rw_1(x) + x \leq 0, \quad \text{for } x < \delta, \tag{81}$$
$$w_0(x) - K_0 - w_1(x) \leq 0, \quad \text{for } x \leq \delta, \tag{82}$$
$$w_1(x) - K_1 - w_0(x) \leq 0, \quad \text{for } x \leq \delta, \tag{83}$$
$$w_0(x) - K_0 - w_1(x) \leq 0, \quad \text{for } \delta \leq x \leq \gamma, \tag{84}$$
$$w_1(x) - K_1 - w_0(x) \leq 0, \quad \text{for } \delta \leq x \leq \gamma. \tag{85}$$

The inequalities $\delta + rK < \beta + rK_0 < 0$ imply trivially (81). Also, since $B > 0$, (83) is straightforward, whereas (82) is implied by (84) and the continuity of $w_1$, $w_0$.

Now, consider the function $g$ defined by

$$g(x) := -\Gamma_1 e^{-\sqrt{2r}x} + (B - \Gamma_2)e^{\sqrt{2r}x} - \frac{x + rK_0}{r}.$$

Since $g(x) = w_0(x) - K_0 - w_1(x)$, if $x \in [\delta, \gamma]$, (84) and (85) will follow if we show that

$$-K_1 - K_0 \le g(x) \le 0, \quad \text{for all } x \in [\delta, \gamma]. \tag{86}$$

By construction,

$$g(\gamma) = g'(\gamma) = 0 \quad \text{and} \quad g''(\gamma) = 2(\gamma + rK_0) < 0. \tag{87}$$

If $B - \Gamma_2 < 0$, then

$$g''(x) = 2r\left[-\Gamma_1 e^{-\sqrt{2r}x} + (B - \Gamma_2)e^{\sqrt{2r}x}\right] < 0,$$

so $g'$ is strictly decreasing, which, combined with $g'(\gamma) = 0$, implies $g'(x) > 0$, for all $x < \gamma[$. On the other hand, if $B - \Gamma_2 > 0$, then

$$g'''(x) = 2r\sqrt{2r}\left[\Gamma_1 e^{-\sqrt{2r}x} + (B - \Gamma_2)e^{\sqrt{2r}x}\right] > 0,$$

which proves that $g'$ is strictly convex. However, this observation and (87) imply that $g'(x) > 0$, for all $x < \gamma$. Finally, since $g$ is increasing in $[\delta, \gamma]$ and $g(\gamma) = 0$, (86) follows from the observation that

$$g(\delta) = Be^{\sqrt{2r}\delta} - K_0 + K > -K_1 - K_0.$$

□

**Proof of Lemma 4.** Suppose that the values of $r, K_1 > 0$ are fixed, and consider the unique solution $\beta = \beta(r, K_1, K_0)$ of (42) as a function of $K_0$ on $]-K_1, \infty[$. By implicit differentiation of (42), we obtain

$$\frac{\partial \beta}{\partial K_0} + r = \frac{r(\beta + rK_0 + 1/\sqrt{2r})(\beta + rK_0 - 1/\sqrt{2r})}{2(\beta + rK_0)^2}. \tag{88}$$

Now, differentiating (55) with respect to $K_0$, we obtain

$$\frac{\partial K_*}{\partial K_0} = -\frac{1}{r(\beta + rK_0 - 1/\sqrt{2r})}\left[(\beta + rK_0)\left(\frac{\partial \beta}{\partial K_0} + r\right) - r(\beta + rK_0 - 1/\sqrt{2r})\right].$$

Substituting for $\partial\beta/\partial K_0 + r$ from (88), we obtain

$$\frac{\partial K_*}{\partial K_0} = \frac{\beta + rK_0 - 1/\sqrt{2r}}{2(\beta + rK_0)} > 0,$$

the inequality following because $\beta < -rK_0$. As a consequence, $K_*(r, K_1, \cdot)$ is strictly increasing in $]-K_1, \infty[$

Now, (44) and (55) imply

$$K_0 > K_* > -\frac{1-\ln 2}{r\sqrt{2r}} + K_0,$$

However, these inequalities imply $\lim_{K_0\to\infty} K_*(r, K_1, K_0) = \infty$ and $K_*(r, K_1, 0) < 0$, and the proof is complete. $\square$

**Proof of Lemma 5.** The fact that (62) has a unique solution $\alpha > -rK - 1/\sqrt{2r}$ follows from the calculations

$$G(-rK - 1/\sqrt{2r}) = -\frac{\sqrt{2r}}{2} r(K + K_1) - 1 < -1,$$
$$G'(\alpha) = r(\alpha - rK_1)\exp\left(\sqrt{2r}(\alpha + rK + 1/\sqrt{2r})\right),$$
$$\lim_{\alpha\to\infty} G(\alpha) = \infty.$$

Also, this solution satisfies

$$rK_1 < \alpha < rK_1 + 1/\sqrt{2r}, \tag{89}$$

the second inequality holding because $G(rK_1 + 1/\sqrt{2r}) = 0 > -1$.

Now, (61) and (89) imply $B > 0$, whereas $A > 0$ is obvious from (57). Since $A, B > 0$, $w_1$, $w_0$ are convex and non-decreasing, so $w_1, w_0 \geq -K$. To verify that they satisfy the HJB equations (31)–(32), we have to establish conditions under which

$$\tfrac{1}{2}w_1''(x) - rw_1(x) + x \leq 0, \quad \text{for } x \leq \delta, \tag{90}$$
$$w_0(x) - K_0 - w_1(x) \leq 0, \quad \text{for } x \leq \delta, \tag{91}$$
$$w_1(x) - K_1 - w_0(x) \leq 0, \quad \text{for } x \leq \delta, \tag{92}$$
$$w_0(x) - K_0 - w_1(x) \leq 0, \quad \text{for } \delta \leq x \leq \alpha, \tag{93}$$
$$w_1(x) - K_1 - w_0(x) \leq 0, \quad \text{for } \delta \leq x \leq \alpha, \tag{94}$$
$$w_0(x) - K_0 - w_1(x) \leq 0, \quad \text{for } x \geq \alpha, \tag{95}$$
$$\tfrac{1}{2}w_0''(x) - rw_0(x) \leq 0, \quad \text{for } x \geq \alpha. \tag{96}$$

Inequalities, (90) and (96) are implied trivially by the fact that $\delta = -rK - 1/\sqrt{2r}$ and the first inequality in (89), respectively. Also, (95) is equivalent to $-K_1 - K_0$ which is true, whereas (92) follows immediately because $B > 0$. In view of the continuity of $w_1$, $w_0$ and the fact that $B > 0$, we can also see that (91) is implied by (93).

To study (93), (94), define the function $g$ by

$$g(x) := Be^{\sqrt{2r}x} - Ae^{-\sqrt{2r}x} - \frac{x + rK_0}{r},$$

so that $g(x) = w_0(x) - K_0 - w_1(x)$, if $x \in [\delta, \alpha]$. By construction,

$$g(\alpha) = -K_1 - K_0, \quad g'(\alpha) = 0, \quad g''(\alpha) = 2(\alpha - rK_1) > 0, \tag{97}$$

the inequality following by virtue of (89). Now, since $A, B > 0$,

$$g'''(x) = 2r\left[g'(x) + \frac{1}{r}\right] = 2r\sqrt{2r}\left[Be^{\sqrt{2r}x} + Ae^{-\sqrt{2r}x}\right] > 0 \tag{98}$$

imply that $g'$ is strictly convex and $\lim_{x\to-\infty} g'(x) = \infty$. However, these observations and (97) imply that there exists a unique $\hat{x} < \alpha$ such that $g'(\hat{x}) = 0$. Furthermore, $\delta > \hat{x}$ because $g'(\delta) = \sqrt{2r}Be^{\sqrt{2r}\delta} > 0$. From these considerations, we conclude that

$$g'(x) > 0, \quad \forall x \in [\delta, \hat{x}[ \qquad \text{and} \qquad g'(x) < 0, \quad \forall x \in ]\hat{x}, \alpha[. \tag{99}$$

Now, (94) follows from the fact that

$$-K_1 - K_0 \leq g(x), \quad \forall x \in [\delta, \alpha],$$

which is true in view of (92) and the continuity of $w_1$, $w_0$, (97) and (99). On the other hand, (93) will follow if we show that $g(x) \leq 0$, for all $x \in [\delta, \alpha]$. In view of (99), this will be true if and only if $g(\hat{x}) \leq 0$, i.e. if and only if

$$Be^{\sqrt{2r}\hat{x}} - Ae^{-\sqrt{2r}\hat{x}} \leq \frac{\hat{x} + rK_0}{r}. \tag{100}$$

All of the results proved above are true for any positive values of the problem's data $r$, $K_1$, $K_0$, $K$. Therefore, given any positive value of these parameters, there exists a unique $\Delta \in \mathbb{R}$ such that

$$Be^{\sqrt{2r}\hat{x}} - Ae^{-\sqrt{2r}\hat{x}} = \frac{\hat{x} + r(K_0 + \Delta)}{r}, \tag{101}$$

Clearly, (100) will be true if and only if $\Delta \leq 0$. Now, recall that $\hat{x} \in ]\delta, \alpha[$ satisfies $g'(\hat{x}) = 0$, i.e.

$$Be^{\sqrt{2r}\hat{x}} + Ae^{-\sqrt{2r}\hat{x}} = \frac{1}{r\sqrt{2r}}. \tag{102}$$

With regard to these two equations, we can eliminate $B$, substitute for $A$ from (57), and solve for $K$ to obtain

$$K = -\frac{1}{r\sqrt{2r}} \ln\left(-\frac{\sqrt{2r}}{2}\left(\hat{x} + r(K_0 + \Delta) - 1/\sqrt{2r}\right)\exp\left(\sqrt{2r}(\hat{x} + 1/\sqrt{2r})\right)\right). \tag{103}$$

Furthermore, by comparing (59), (60), (101), (102) with (34), (35), (36), (37), respectively, we can see that $\hat{x} = \beta(r, K_1, K_0 + \Delta)$, where $\beta$ is given by Lemma 2. Therefore, (55) and (103) imply $K = K_*(r, K_1, K_0 + \Delta)$. Since $K_*(r, K_1, \cdot)$ is strictly increasing (see Lemma 4), $\Delta \leq 0$ if and only if $K \leq K_*(r, K_1, K_0)$. However, these arguments establish that (100) is true if and only if $K \leq K_*(r, K_1, K_0)$, and the proof is complete.

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