# Buy-low and sell-high investment strategies

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**Abstract**

Buy-low and sell-high investment strategies are a recurrent theme in the considerations of many investors. In this paper, we consider an investor who aims at maximising the expected discounted cash-flow that can be generated by sequentially buying and selling one share of a given asset at fixed transaction costs. We model the underlying asset price by means of a general one-dimensional Itô diffusion $X$, we solve the resulting stochastic control problem in a closed analytic form, and we completely characterise the optimal strategy. In particular, we show that, if 0 is a natural boundary point of $X$, e.g., if $X$ is a geometric Brownian motion, then it is never optimal to sequentially buy and sell. On the other hand, we prove that, if 0 is an entrance point of $X$, e.g., if $X$ is a mean-reverting constant elasticity of variance (CEV) process, then it may be optimal to sequentially buy and sell, depending on the problem data.

Key Words: optimal investment strategies, optimal switching, sequential entry and exit decisions, variational inequalities.

## 1 Introduction

We consider an asset with price process $X$ that is modelled by the one-dimensional Itô diffusion

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x > 0, \tag{1}$$

where $W$ is a standard one-dimensional Brownian motion. An investor follows a strategy that consists of sequentially buying and selling one share of the asset. We use a controlled finite variation càglàd process $Y$ that takes values in $\{0, 1\}$ to model the investor's position in the market. In particular, $Y_t = 1$ (resp., $Y_t = 0$) represents the state where the investor holds (resp., does not hold) the asset, while, the jumps of $Y$ occur at the sequence of times $(\tau_n,\ n \geq 1)$

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at which the investor buys or sells. Given an initial condition $(Y_0, X_0) = (y, x) \in \{0, 1\} \times ]0, \infty[$, the investor's objective is to select a strategy $Y$ that maximises the performance criterion

$$J_{y,x}(Y) = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_x \left[ e^{-\Lambda_{\tau_j}} \left[ H_{\mathrm{s}}(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j} = -1\}} - H_{\mathrm{b}}(X_{\tau_j}) \mathbf{1}_{\{\Delta Y_{\tau_j} = 1\}} \right] \mathbf{1}_{\{\tau_j < \infty\}} \right], \tag{2}$$

where

$$H_{\mathrm{b}}(x) = x + c_{\mathrm{b}} \quad \text{and} \quad H_{\mathrm{s}}(x) = x - c_{\mathrm{s}}, \quad \text{for } x > 0, \tag{3}$$

the state-dependent discounting factor $\Lambda$ is defined by

$$\Lambda_t = \int_0^t r(X_s)\, ds, \tag{4}$$

for some function $r > 0$, and the constant $c_{\mathrm{b}} > 0$ (resp., $c_{\mathrm{s}} > 0$) represents the transaction cost of buying (resp., selling) one share of the asset. Accordingly, we define the problem's value function $v$ by

$$v(y, x) = \sup_{Y \in \mathcal{A}_{y,x}} J_{y,x}(Y), \quad \text{for } y \in \{0, 1\} \text{ and } x > 0, \tag{5}$$

where $\mathcal{A}_{y,x}$ is the set of admissible investment strategies, which is introduced by Definition 1 in Section 2.

In the presence of the general assumptions that we make on $b$, $\sigma$ and $r$, the optimisation problem defined by (1)–(5) is well-posed, in particular, the limit in (2) exists (see Theorem 3, our main result). We solve this problem in a closed analytic form and we characterise fully the optimal strategy. It turns out that, if 0 is a natural boundary point of the diffusion $X$, e.g., if $X$ is a geometric Brownian motion, then it is optimal to never enter the market, and to sell the asset as soon as its price exceeds a given level $\alpha > 0$ if the investor is long in the market at time 0 (see Remark 2 and Theorem 3). The situation is different if 0 is an entrance boundary point of the diffusion $X$. In this case, the strategy of just exiting the market appropriately may still be optimal. However, depending on the problem data, it may be optimal for the investor to sequentially buy as soon as the asset price falls below a given level $\beta > 0$ and then sell the asset as soon as its price rises above another level $\gamma > \beta$.

Despite the fundamental nature of buy-low and sell-high investment strategies, there have been few papers studying models with sequential buying and selling decision strategies. The reason for this can be attributed to the fact that, as we have briefly discussed above, the prime example of an asset price process, namely, the geometric Brownian motion, does not allow for optimal buying and selling strategies that have a sequential nature. The results presented in Shiryaev, Xu and Zhou (2008) and Dai, Jin, Zhong and Zhou (2010) support such a conclusion: assuming that a stock price follows a geometric Brownian motion, it is optimal for an investor to either sell the stock immediately or hold it until their planning horizon. Other related models involving optimal selling decisions with a geometric Brownian motion type of price model have been studied by Zhang (2001) and Guo and Zhang (2005).

In the context of one-dimensional Itô diffusions other than a standard geometric Brownian motion, sequential buying and selling investment strategies can indeed be optimal. Such a result has already been established by Zhang and Zhang (2008) and Song, Yin and Zhang (2009) who model the underlying asset price dynamics by means of a mean-reverting Ornstein-Uhlenbeck process such as the one appearing in Vasicek's interest rate model and consider proportional transaction costs. Apart from highlighting the significance of the classification of the diffusion's $X$ boundary point 0 in determining the character of the optimal strategy, our results have a substantially more general nature. In particular, they can account for a rather large family of stochastic processes that includes the mean-reverting CEV processes, which have been proposed in the empirical finance literature as better models for a range of asset prices, particularly, in the commodity markets (e.g., see Geman and Shih (2009) and the references therein). Having made this comment, we note that Dai, Zhang and Zhu (2010a) have studied the interesting case with proportional costs that arises if the underlying asset price dynamics are modelled by a geometric Brownian motion with drift switching between two possible values that are not directly observable by the investor. Further in this modelling direction, Dai, Zhang and Zhu (2010b) have studied the more realistic situation in which the investor is not restricted to buying and selling just one share of stock at a time but aims at maximising their portfolio's expected return.

The problem that we solve has the characteristics of an entry and exit decision problem. Stochastic optimal control problems involving sequential switching decisions have attracted considerable interest in the literature, particularly, in relation to the management of commodity production facilities. Following Brennan and Schwartz (1985), Dixit and Pindyck (1994), and Trigeorgis (1996), who were the first to address this type of a decision problem in the economics literature, Brekke and Øksendal (1994), Bronstein and Zervos (2006), Costeniuc, Schnetzer and Taschini (2008), Djehiche and Hamadène (2009), Djehiche, Hamadène and Popier (2009/10), Duckworth and Zervos (2001), Guo and Pham (2005), Guo and Tomecek (2008), Hamadène and Jeanblanc (2007), Hamadène and Zhang (2010), Johnson and Zervos (2010), Lumley and Zervos (2001), Ly Vath and Pham (2007), Pham (2004), Pham, Ly Vath and Zhou (2009), Tang and Yong (1993), and Zervos (2003), provide an incomplete, alphabetically ordered, list of authors who have studied a number of related models by means of rigorous mathematics. The contributions of these authors range from explicit solutions to characterisations of the associated value functions in terms of classical as well as viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations, as well as in terms of backward stochastic differential equation characterisations of the optimal strategies. Chapter 5 of Pham (2009) provides a nice overview of the area.

The paper is organised as follows. Section 2 is concerned with the setting of the problem that we study. We derive the solution to this problem in Section 3. In Section 4, we consider a couple of examples that illustrate our results.

# 2 Problem formulation and assumptions

We assume that the data of the one-dimensional Itô diffusion given by (1) in the introduction satisfy the following assumption.

**Assumption 1** The functions $b, \sigma : ]0, \infty[ \to \mathbb{R}$ are Borel-measurable,

$$\sigma^2(x) > 0 \quad \text{for all } x > 0,$$

and

$$\int_{\underline{\alpha}}^{\overline{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)} \, ds < \infty \quad \text{for all } 0 < \underline{\alpha} < \overline{\beta} < \infty.$$

□

The conditions in this assumption are sufficient for the SDE (1) to have a weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ that is unique in the sense of probability law up to a possible explosion time (e.g., see Section 5.5.C of Karatzas and Shreve (1991)). We assume that such a weak solution is fixed for each initial condition $x > 0$ throughout the paper. In particular, Assumption 1 implies that the scale function $p$, given by

$$p(x) = \int_1^x \exp\left( -2 \int_1^s \frac{b(u)}{\sigma^2(u)} \, du \right) ds, \quad \text{for } x > 0, \tag{6}$$

is well-defined, and the speed measure $m$, given by

$$m(dx) = \frac{2}{\sigma^2(x) p'(x)} \, dx, \tag{7}$$

is a Radon measure.

We also assume that the solution of (1) in non-explosive, i.e., the hitting time of the boundary $\{0, \infty\}$ of the interval $]0, \infty[$ is infinite with probability 1 (see Theorem 5.5.29 in Karatzas and Shreve (1991) for appropriate necessary and sufficient analytic conditions).

**Assumption 2** The solution of (1) is non-explosive. □

Relative to the discounting factor $\Lambda$ defined by (4), we make the following assumption.

**Assumption 3** The function $r : ]0, \infty[ \to ]0, \infty[$ is Borel-measurable and uniformly bounded away from 0, i.e., there exists a constant $r_0 > 0$ such that $r(x) \geq r_0$ for all $x > 0$. Also,

$$\int_{\bar{\alpha}}^{\bar{\beta}} \frac{r(s)}{\sigma^2(s)} \, ds < \infty \quad \text{for all } 0 < \bar{\alpha} < \bar{\beta} < \infty.$$

□

In the presence of Assumptions 1–3, there exists a pair of $C^1$ functions $\varphi, \psi : ]0, \infty[ \to ]0, \infty[$ with absolutely continuous first derivatives and such that

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{8}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0, \tag{9}$$

$$\lim_{x \downarrow 0} \varphi(x) = \lim_{x \to \infty} \psi(x) = \infty, \tag{10}$$

$$\varphi(x) = \varphi(y) \mathbb{E}_x \left[ e^{-\Lambda_{T_y}} \right] \text{ for all } y < x \quad \text{and} \quad \psi(x) = \psi(y) \mathbb{E}_x \left[ e^{-\Lambda_{T_y}} \right] \text{ for all } x < y, \tag{11}$$

where $T_y$ is the first hitting time of $\{y\}$, which is defined by $T_y = \inf\{t \geq 0 \mid \; X_t = y\}$. Also, the functions $\varphi$ and $\psi$ are classical solutions of the homogeneous ODE

$$\mathcal{L}g(x) := \frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0, \tag{12}$$

and satisfy

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp'(x) \quad \text{for all } x > 0, \tag{13}$$

where $C = \varphi(1)\psi'(1) - \varphi'(1)\psi(1)$ and $p$ is the scale function defined by (6).

Now, let $g$ be a $C^1$ function with absolutely continuous first derivative satisfying

$$\lim_{x\downarrow 0} \frac{g(x)}{\varphi(x)} = \lim_{x\to\infty} \frac{g(x)}{\psi(x)} = 0 \quad \text{and} \quad \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\mathcal{L}g(X_t)\big|\,dt\right] < \infty, \tag{14}$$

where $\mathcal{L}$ is the operator defined by (12). Such a function admits the analytic representation

$$g(x) = -\varphi(x)\int_0^x \Psi(s)\mathcal{L}g(s)\,ds - \psi(x)\int_x^\infty \Phi(s)\mathcal{L}g(s)\,ds, \tag{15}$$

where

$$\Phi(x) = \frac{2\varphi(x)}{C\sigma^2(x)p'(x)} \quad \text{and} \quad \Psi(x) = \frac{2\psi(x)}{C\sigma^2(x)p'(x)}. \tag{16}$$

Furthermore, given any $(\mathcal{F}_t)$-stopping time $\tau$, Dynkin's formula

$$\mathbb{E}_x\left[e^{-\Lambda_\tau}g(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = g(x) + \mathbb{E}_x\left[\int_0^\tau e^{-\Lambda_t}\mathcal{L}g(X_t)\,dt\right] \tag{17}$$

holds, and

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_n}}\left|g(X_{\tau_n})\right|\mathbf{1}_{\{\tau_n<\infty\}}\right] = 0, \tag{18}$$

for every sequence of $(\mathcal{F}_t)$-stopping times $(\tau_n)$ such that $\tau_n \to \infty$, $\mathbb{P}_x$-a.s.. The existence of the functions $\varphi$, $\psi$ and the various results that we have listed can be found in several references, including Chapter II of Borodin and Salminen (2002). For future reference, we also note that a straightforward calculation involving (13) and (15) implies that

$$\left(\frac{g}{\varphi}\right)'(x) = -\frac{Cp'(x)}{\varphi^2(x)}\int_x^\infty \Phi(s)\mathcal{L}g(s)\,ds \tag{19}$$

and

$$\left(\frac{g}{\psi}\right)'(x) = \frac{Cp'(x)}{\psi^2(x)}\int_0^x \Psi(s)\mathcal{L}g(s)\,ds. \tag{20}$$

We can now complete the set of our assumptions.

**Assumption 4** The problem data is such that

$$\lim_{x\to\infty} \frac{H(x)}{\psi(x)} = 0 \quad \text{and} \quad \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} \left| \mathcal{L}H(X_t) \right| dt \right] < \infty, \tag{21}$$

where $H$ is the identity function, i.e., $H(x) = x$ for all $x > 0$. Also, $c_{\mathrm{b}}, c_{\mathrm{s}} > 0$, and there exist constants $0 \le x_{\mathrm{b}} < x_{\mathrm{s}}$ such that

$$\mathcal{L}H_{\mathrm{b}}(x) \begin{cases} > 0, & \text{for } x < x_{\mathrm{b}} \text{ if } x_{\mathrm{b}} > 0, \\ < 0, & \text{for } x > x_{\mathrm{b}}, \end{cases} \quad \text{and} \quad \mathcal{L}H_{\mathrm{s}}(x) \begin{cases} > 0, & \text{for } x < x_{\mathrm{s}}, \\ < 0, & \text{for } x > x_{\mathrm{s}}. \end{cases} \tag{22}$$

□

The following definition introduces the class of all admissible investment strategies over which we maximise the performance criterion $J_{y,x}$ defined by (2).

**Definition 1** Given an initial condition $(y, x) \in \{0, 1\} \times \,]0, \infty[$ and the associated weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ of (1), an admissible investment strategy is any $(\mathcal{F}_t)$-adapted finite-variation càglàd process $Y$ with values in $\{0, 1\}$ such that $Y_0 = y$. Given such a process $Y$, we denote by $(\tau_n)$ the strictly increasing sequence of $(\mathcal{F}_t)$-stopping times at which the jumps of $Y$ occur, which can be defined recursively by

$$\tau_1 = \inf\{t > 0 \mid \; Y_t \neq y\} \quad \text{and} \quad \tau_{j+1} = \inf\{t > \tau_j \mid \; Y_t \neq Y_{\tau_j}\}, \quad \text{for } j = 1, 2, \ldots, \tag{23}$$

with the usual convention that $\inf \emptyset = \infty$. We denote by $\mathcal{A}_{y,x}$ the set of all admissible strategies. □

We conclude this section with the following remarks.

**Remark 1** In view of the definition (12) of the operator $\mathcal{L}$, the definition (3) of the functions $H_{\mathrm{b}}$, $H_{\mathrm{s}}$ and the inequality in (21), we can see that

$$\begin{aligned} &\mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} \left[ |\mathcal{L}H_{\mathrm{b}}(X_t)| + |\mathcal{L}H_{\mathrm{s}}(X_t)| \right] dt \right] \\ &\qquad \le 2\mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} |\mathcal{L}H(X_t)| \, dt \right] + (c_{\mathrm{b}} + c_{\mathrm{s}}) \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} r(X_t) \, dt \right] \\ &\qquad = 2\mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} |\mathcal{L}H(X_t)| \, dt \right] + (c_{\mathrm{b}} + c_{\mathrm{s}}) \mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} \, d\Lambda_t \right] \\ &\qquad = 2\mathbb{E}_x \left[ \int_0^\infty e^{-\Lambda_t} |\mathcal{L}H(X_t)| \, dt \right] + (c_{\mathrm{b}} + c_{\mathrm{s}}) \\ &\qquad < \infty. \end{aligned}$$

It follows that the functions $H_{\mathrm{b}}$, $H_{\mathrm{s}}$ satisfy the corresponding requirements of (14), and, therefore, they have all of the corresponding properties in (15)–(18). □

**Remark 2** In the context of Assumption 2, 0 can be a natural boundary point of the diffusion $X$, in which case, $\lim_{x\downarrow 0}\psi(x) = 0$, or an entrance boundary point, in which case, $\lim_{x\downarrow 0}\psi(x) > 0$. If 0 is a natural boundary point, then the conditions in Assumption 4 can all be true only if $x_{\rm b} = 0$. To see this claim, we argue by contradiction and we assume that 0 is a natural boundary point and Assumption 4 holds with $x_{\rm b} = 0$. In view of the strict positivity of $c_{\rm b}$, we can see that

$$\infty = \lim_{x\downarrow 0}\frac{H_{\rm b}(x)}{\psi(x)} \leq -\lim_{x\downarrow 0}\int_x^\infty \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds < -\int_{x_{\rm b}}^\infty \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds < \infty, \tag{24}$$

which is a contradiction. It follows that Assumption 4 can hold with $x_{\rm b} > 0$ only if 0 is an entrance boundary point.

For future reference, we note that, if 0 is an entrance boundary point and $x_{\rm b} > 0$, then

$$-\lim_{x\downarrow 0}\int_x^\infty \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds = \lim_{x\downarrow 0}\frac{H_{\rm b}(x)}{\psi(x)}. \tag{25}$$

To see this claim, we argue as in (24) to obtain

$$0 < \lim_{x\downarrow 0}\frac{H_{\rm b}(x)}{\psi(x)} \leq -\lim_{x\downarrow 0}\int_x^\infty \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds < \infty,$$

which implies that

$$0 < \int_0^x \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds < \infty \quad \text{for all } x \in\, ]0, x_{\rm b}[.$$

In view of the identity $\Phi = \varphi\Psi/\psi$, which follows from (16), and the fact that the function $\varphi/\psi$ is decreasing, we can see that

$$\infty > \int_0^x \Phi(s)\mathcal{L}H_{\rm b}(s)\,ds = \int_0^x \Psi(s)\frac{\varphi(s)}{\psi(s)}\mathcal{L}H_{\rm b}(s)\,ds \geq \frac{\varphi(x)}{\psi(x)}\int_0^x \Psi(s)\mathcal{L}H_{\rm b}(s)ds > 0$$

for all $x \in\, ]0, x_{\rm b}[$. It follows that

$$\lim_{x\downarrow 0}\frac{\varphi(x)}{\psi(x)}\int_0^x \Psi(s)\mathcal{L}H_{\rm b}(s)\,ds = 0,$$

which, combined with (15), implies (25). □

# 3 The solution of the control problem

The existing theory on sequential switching problems suggests that the value function $v$ of our control problem should identify with a classical solution $w$ of the HJB equation

$$\max\Big\{\frac{1}{2}\sigma^2(x)w_{xx}(y,x) + b(x)w_x(y,x) - r(x)w(y,x),$$
$$w(1-y,x) - w(y,x) + yH_{\rm s}(x) - (1-y)H_{\rm b}(x)\Big\} = 0, \quad y = 0, 1, \tag{26}$$

where $H_{\mathrm{b}}$ and $H_{\mathrm{s}}$ are defined by (3). We now solve our control problem by constructing an appropriate solution to this equation. To this end, we have to consider two qualitatively different cases. The first one arises when it is optimal for the investor to sell as soon as the asset price exceeds a given level $\alpha > 0$, if $y = 1$, and never enter the market otherwise. In this case, we look for a solution of the HJB equation (26) of the form given by

$$w(0,x) = 0 \quad \text{and} \quad w(1,x) = \begin{cases} A\psi(x), & \text{if } x < \alpha, \\ H_{\mathrm{s}}(x), & \text{if } x \geq \alpha, \end{cases} \tag{27}$$

for some constant $A$. To determine the parameter $A$ and the free-boundary point $\alpha$, we appeal to the so-called "principle of smooth fit" of sequential switching and we require that $w(1,\cdot)$ is $C^1$ along $\alpha$, which yields the system of equations

$$A\psi(\alpha) = H_{\mathrm{s}}(\alpha) \quad \text{and} \quad A\psi'(\alpha) = H_{\mathrm{s}}'(\alpha).$$

This system is equivalent to

$$A = \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)} = \frac{H_{\mathrm{s}}'(\alpha)}{\psi'(\alpha)}. \tag{28}$$

In view of (20), we can check that these identities imply that $\alpha > 0$ should satisfy the equation

$$q(\alpha) = 0, \tag{29}$$

where

$$q(x) = \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds. \tag{30}$$

The following result, which we prove in the Appendix, is concerned with the solvability of (29) as well as with a necessary and sufficient condition for the function $w$ given by (27) to satisfy the HJB equation (26).

**Lemma 1** *In the presence of Assumptions 1–4, there exists a unique $\alpha > 0$ satisfying equation (29). Furthermore, $\alpha > c_{\mathrm{s}}$ and the function $w$ defined by (27), where $A > 0$ is given by (28), satisfies the HJB equation (26) if and only if the problem data is such that either $x_{\mathrm{b}} = 0$ or*

$$x_{\mathrm{b}} > 0 \quad \textit{and} \quad c_{\mathrm{b}} \geq (\alpha - c_{\mathrm{s}}) \frac{\lim_{x \downarrow 0} \psi(x)}{\psi(\alpha)}. \tag{31}$$

The second possibility arises when it is optimal for the investor to sequentially enter and exit the market. In this case, we postulate that the value function of our control problem should identify with a solution $w$ of the HJB equation (26) that has the form given by the expressions

$$w(1,x) = \begin{cases} A\psi(x), & \text{if } x < \gamma, \\ B\varphi(x) + H_{\mathrm{s}}(x), & \text{if } x \geq \gamma, \end{cases} \tag{32}$$

$$w(0,x) = \begin{cases} A\psi(x) - H_{\mathrm{b}}(x), & \text{if } x \leq \beta, \\ B\varphi(x), & \text{if } x > \beta, \end{cases} \tag{33}$$

for some constants $A$, $B$ and free-boundary points $\beta$, $\gamma$ such that $0 < \beta < \gamma$. To determine these variables, we conjecture that the functions $w(1, \cdot)$ and $w(0, \cdot)$ should be $C^1$ at the free-boundary points $\gamma$ and $\beta$, respectively, which yields the system of equations

$$A\psi(\gamma) = B\varphi(\gamma) + H_{\mathrm{s}}(\gamma), \quad A\psi(\beta) - H_{\mathrm{b}}(\beta) = B\varphi(\beta), \tag{34}$$
$$A\psi'(\gamma) = B\varphi'(\gamma) + H_{\mathrm{s}}'(\gamma) \quad \text{and} \quad A\psi'(\beta) - H_{\mathrm{b}}'(\beta) = B\varphi'(\beta). \tag{35}$$

We can check that these equations are equivalent to

$$\begin{aligned} A =& \frac{H_{\mathrm{b}}'(\beta)\varphi(\beta) - H_{\mathrm{b}}(\beta)\varphi'(\beta)}{\varphi(\beta)\psi'(\beta) - \varphi'(\beta)\psi(\beta)} = \frac{H_{\mathrm{s}}'(\gamma)\varphi(\gamma) - H_{\mathrm{s}}(\gamma)\varphi'(\gamma)}{\varphi(\gamma)\psi'(\gamma) - \varphi'(\gamma)\psi(\gamma)}, \\ B =& \frac{H_{\mathrm{b}}'(\beta)\psi(\beta) - H_{\mathrm{b}}(\beta)\psi'(\beta)}{\varphi(\beta)\psi'(\beta) - \varphi'(\beta)\psi(\beta)} = \frac{H_{\mathrm{s}}'(\gamma)\psi(\gamma) - H_{\mathrm{s}}(\gamma)\psi'(\gamma)}{\varphi(\gamma)\psi'(\gamma) - \varphi'(\gamma)\psi(\gamma)}, \end{aligned}$$

and then appeal to Remark 1, (13) and (19)–(20) to obtain

$$A = -\int_\beta^\infty \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds = -\int_\gamma^\infty \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds, \tag{36}$$

$$B = \int_0^\beta \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds = \int_0^\gamma \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds. \tag{37}$$

It follows that the free-boundary points $0 < \beta < \gamma$ should satisfy the system of equations

$$q_\varphi(\beta, \gamma) = 0 \quad \text{and} \quad q_\psi(\beta, \gamma) = 0, \tag{38}$$

where

$$q_\varphi(x, z) = \int_x^\infty \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - \int_z^\infty \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds \tag{39}$$

and

$$q_\psi(x, z) = \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - \int_0^z \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds. \tag{40}$$

The following result is concerned with conditions, under which, there exist points $0 < \beta < \gamma$ that satisfy (38) and the corresponding function $w$ defined by (32)–(33) satisfies the HJB equation (26).

**Lemma 2** *In the presence of Assumptions 1–4, the system of equations (38) has a unique solution $0 < \beta < \gamma$ if and only if the problem data is such that*

$$x_{\mathrm{b}} > 0 \quad \text{and} \quad c_{\mathrm{b}} < (\alpha - c_{\mathrm{s}})\frac{\lim_{x\downarrow 0}\psi(x)}{\psi(\alpha)}, \tag{41}$$

*where $\alpha > c_{\mathrm{s}}$ is the unique solution of (29). In this case, $\beta < \alpha < \gamma$ and the function $w$ defined by (32)–(33), for $A > 0$ and $B > 0$ given by (36) and (37), satisfies the HJB equation (26).*

We can now establish our main result.

**Theorem 3** *The stochastic optimisation problem introduced by (1)–(5) and formulated in Section 2 is well-posed. Furthermore,*
(I) *if the problem data is such that either $x_{\mathrm{b}} = 0$ or (31) holds true, then $v = w$, where $w$ is as in Lemma 1, and the optimal strategy $Y^*$ is given by*

$$Y_t^* = y\mathbf{1}_{[0,\tau_1^*]}(t), \quad \textit{where} \quad \tau_1^* = \inf\{t \geq 0 \mid \ X_t \geq \alpha\}, \tag{42}$$

*with the convention that $\inf \emptyset = \infty$, and $\alpha > 0$ is the unique solution of (29);*
(II) *if the problem data is such that (41) holds true, then $v = w$, where $w$ is as in Lemma 2, and the optimal strategy $Y^*$ is characterised by the unique solution $0 < \beta < \gamma$ of the system of equations (38) and is given by*

$$Y_t^* = \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{\infty} \mathbf{1}_{]\tau_{2j}^*, \tau_{2j+1}^*]}(t), \tag{43}$$

*where $\tau_0^* = 0$,*

$$\tau_{2j}^* = \inf\{t \geq \tau_{2j-1}^* \mid \ X_t \geq \gamma\} \quad \textit{and} \quad \tau_{2j-1}^* = \inf\{t \geq \tau_{2j-2}^* \mid \ X_t \leq \beta\}, \quad \textit{for } j \geq 1,$$

*if $y = 1$, and by*

$$Y_t^* = \sum_{j=0}^{\infty} \mathbf{1}_{]\tau_{2j+1}^*, \tau_{2j+2}^*]}(t), \tag{44}$$

*where $\tau_0^* = 0$,*

$$\tau_{2j+1}^* = \inf\{t \geq \tau_{2j}^* \mid \ X_t \geq \gamma\} \quad \textit{and} \quad \tau_{2j}^* = \inf\{t \geq \tau_{2j-1}^* \mid \ X_t \leq \beta\}, \quad \textit{for } j \geq 1,$$

*if $y = 0$.*

**Proof.** We first note that, in view of (8)–(9), there exists a constant $K > 0$ such that $|w(y,x)| \leq K(1+x)$ for all $x > 0$ and $y = 0, 1$. This observation, the calculations

$$\mathcal{L}w(1,\cdot)(x) = \begin{cases} 0, & \text{if } x < \alpha, \\ \mathcal{L}H_{\mathrm{s}}(x), & \text{if } x > \alpha, \end{cases} \quad \text{and} \quad \mathcal{L}w(0,\cdot) = 0,$$

which hold if $w$ is as in Lemma 1, the calculations

$$\mathcal{L}w(1,\cdot)(x) = \begin{cases} 0, & \text{if } x < \gamma, \\ \mathcal{L}H_{\mathrm{s}}(x), & \text{if } x > \gamma, \end{cases} \quad \text{and} \quad \mathcal{L}w(0,\cdot)(x) = \begin{cases} -\mathcal{L}H_{\mathrm{b}}(x), & \text{if } x < \beta, \\ 0, & \text{if } x > \beta, \end{cases}$$

which hold if $w$ is as in Lemma 2, (21) in Assumption 4 and Remark 1 imply that the functions $w(y,\cdot)$, $y = 0, 1$, satisfy the corresponding requirements of (14). It follows that

$$w(0,\cdot) \text{ and } w(1,\cdot) \text{ have all of the corresponding properties in (14)–(18).} \tag{45}$$

In the rest of the analysis, we may assume that the investor is long in the market at time 0, i.e., that $y = 1$; the analysis of the case associated with $y = 0$ follows exactly the same steps. To start with, we consider any admissible investment strategy $Y \in \mathcal{A}_{1,x}$, and we recall that the jumps of $Y$ occur at the times composing the sequence $(\tau_n,\ n \geq 1)$ defined by (23) in Definition 1. For notational simplicity, we define $\tau_0 = 0$, and we note that

$$\tau_j(\omega) < \tau_{j+1}(\omega) \quad \text{for all } \omega \in \{\tau_j < \infty\} \text{ and } j \geq 1.$$

Also, we note that $\lim_{j\to\infty} \tau_j = \infty$, $\mathbb{P}_x$-a.s., because $Y$ is a finite-variation process.

Iterating Dynkin's formula (17), we obtain

$$\begin{aligned}
&\sum_{j=1}^{2n-1} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_j}} \Big[H_{\mathrm{s}}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_{\mathrm{b}}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\Big] \mathbf{1}_{\{\tau_j<\infty\}}\Big] \\
&\quad = \sum_{j=0}^{n-1} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j+1}}} H_{\mathrm{s}}(X_{\tau_{2j+1}})\mathbf{1}_{\{\tau_{2j+1}<\infty\}}\Big] - \sum_{j=1}^{n-1} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}} H_{\mathrm{b}}(X_{\tau_{2j}})\mathbf{1}_{\{\tau_{2j}<\infty\}}\Big] \\
&\quad = H_{\mathrm{s}}(x) + \sum_{j=1}^{n-1} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}} \big[H_{\mathrm{s}}(X_{\tau_{2j}}) - H_{\mathrm{b}}(X_{\tau_{2j}})\big]\mathbf{1}_{\{\tau_{2j}<\infty\}}\Big] + \mathbb{E}_x\left[\int_0^{\tau_{2n}} e^{-\Lambda_t} Y_t \mathcal{L} H_{\mathrm{s}}(X_t)\, dt\right] \\
&\quad \overset{(3)}{=} H_{\mathrm{s}}(x) - \big(c_{\mathrm{s}} + c_{\mathrm{b}}\big) \sum_{j=1}^{n-1} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}} \mathbf{1}_{\{\tau_{2j}<\infty\}}\Big] + \mathbb{E}_x\left[\int_0^{\tau_{2n}} e^{-\Lambda_t} Y_t \mathcal{L} H_{\mathrm{s}}(X_t)\, dt\right].
\end{aligned} \tag{46}$$

Also, (14) and the dominated convergence theorem imply that

$$\mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_t} Y_t \mathcal{L} H_{\mathrm{s}}(X_t)\, dt\right] = \lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{\tau_n} e^{-\Lambda_t} Y_t \mathcal{L} H_{\mathrm{s}}(X_t)\, dt\right] \in \mathbb{R}.$$

Combining these observations with the limit

$$\lim_{n\to\infty} \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2n}}} H_{\mathrm{b}}(X_{\tau_{2n}})\Big] = 0,$$

which holds thanks to (18) and the fact that $\lim_{n\to\infty} \tau_n = \infty$, we can see that

$$\begin{aligned}
J_{1,x}(Y) &= \lim_{n\to\infty} \sum_{j=1}^{n} \mathbb{E}_x\left[e^{-\Lambda_{\tau_j}} \Big[H_{\mathrm{s}}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_{\mathrm{b}}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\Big] \mathbf{1}_{\{\tau_j<\infty\}}\right] \\
&\in [-\infty, \infty[.
\end{aligned} \tag{47}$$

It follows that $J_{1,x}(Y)$ is well-defined and our optimisation problem is well-posed.

To proceed further, we recall (45) and we iterate Dynkin's formula (17) to calculate

$$\begin{aligned}
&\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2n}}}w(0,X_{\tau_{2n}})\mathbf{1}_{\{\tau_{2n}<\infty\}}\Big]\\
&\qquad = w(1,x) + \sum_{j=0}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j+1}}}\big[w(0,X_{\tau_{2j+1}}) - w(1,X_{\tau_{2j+1}})\big]\,\mathbf{1}_{\{\tau_{2j+1}<\infty\}}\Big]\\
&\qquad\quad + \sum_{j=1}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}}\big[w(1,X_{\tau_{2j}}) - w(0,X_{\tau_{2j}})\big]\mathbf{1}_{\{\tau_{2j}<\infty\}}\Big]\\
&\qquad\quad + \sum_{j=0}^{n-1}\mathbb{E}_x\left[\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\Lambda_t}\mathcal{L}w(1,X_t)\,dt\right] + \sum_{j=0}^{n-1}\mathbb{E}_x\left[\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\Lambda_t}\mathcal{L}w(0,X_t)\,dt\right].
\end{aligned}$$

Adding the term

$$\begin{aligned}
&\sum_{j=1}^{2n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_j}}\Big[H_{\rm s}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_{\rm b}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\Big]\mathbf{1}_{\{\tau_j<\infty\}}\Big]\\
&\qquad \equiv \sum_{j=0}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j+1}}}H_{\rm s}(X_{\tau_{2j+1}})\mathbf{1}_{\{\tau_{2j+1}<\infty\}}\Big] - \sum_{j=1}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}}H_{\rm b}(X_{\tau_{2j}})\mathbf{1}_{\{\tau_{2j}<\infty\}}\Big]
\end{aligned}$$

on both sides of this identity, we obtain

$$\begin{aligned}
&\sum_{j=1}^{2n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_j}}\,\Big[H_{\rm s}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_{\rm b}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\Big]\,\mathbf{1}_{\{\tau_j<\infty\}}\Big]\\
&\qquad = w(1,x) - \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2n}}}w(0,X_{\tau_{2n}})\mathbf{1}_{\{\tau_{2n}<\infty\}}\Big]\\
&\qquad\quad + \sum_{j=0}^{n-1}\mathbb{E}_x\left[\int_{\tau_{2j}}^{\tau_{2j+1}} e^{-\Lambda_t}\mathcal{L}w(1,X_t)\,dt\right] + \sum_{j=0}^{n-1}\mathbb{E}_x\left[\int_{\tau_{2j+1}}^{\tau_{2j+2}} e^{-\Lambda_t}\mathcal{L}w(0,X_t)\,dt\right]\\
&\qquad\quad + \sum_{j=0}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j+1}}}\,\big[w(0,X_{\tau_{2j+1}}) - w(1,X_{\tau_{2j+1}}) + H_{\rm s}(X_{\tau_{2j+1}})\big]\,\mathbf{1}_{\{\tau_{2j+1}<\infty\}}\Big]\\
&\qquad\quad + \sum_{j=1}^{n-1}\mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2j}}}\,\big[w(1,X_{\tau_{2j}}) - w(0,X_{\tau_{2j}}) - H_{\rm b}(X_{\tau_{2j}})\big]\,\mathbf{1}_{\{\tau_{2j}<\infty\}}\Big].
\end{aligned}$$

This calculation and the fact that $w$ satisfies the HJB equation (26), imply that

$$\begin{aligned}
&\mathbb{E}_x\left[\sum_{j=1}^{2n-1} e^{-\Lambda_{\tau_j}}\,\Big[H_{\rm s}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=-1\}} - H_{\rm b}(X_{\tau_j})\mathbf{1}_{\{\Delta Y_{\tau_j}=1\}}\Big]\,\mathbf{1}_{\{\tau_j<\infty\}}\right]\\
&\qquad\qquad \le w(1,x) - \mathbb{E}_x\Big[e^{-\Lambda_{\tau_{2n}}}w(0,X_{\tau_{2n}})\mathbf{1}_{\{\tau_{2n}<\infty\}}\Big].
\end{aligned} \tag{48}$$

In view of (47) and the fact that

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_{2n}}} w(0, X_{\tau_{2n}}) \mathbf{1}_{\{\tau_{2n}<\infty\}}\right] = 0,$$

which follows from (18) and (45), we can pass to the limit $n \to \infty$ in (48) to obtain the inequality $J_{1,x}(Y) \leq w(1,x)$. Therefore, $v(1,x) \leq w(1,x)$.

The nature of the strategy $Y^*$ given by (42) or (43)–(44), depending on the case, is such that (48) hold with equality. By passing to the limit $n \to \infty$ as above, we therefore obtain $J_{1,x}(Y^*) = w(1,x)$, which establishes the inequality $v(1,x) \geq w(1,x)$.

To complete the proof, we still have to show that the process $Y^*$ given by (43)–(44) is a finite variation process. In particular, we have to show that $\mathbb{P}_x\left(\lim_{n\to\infty} \tau_n^* < \infty\right) = 0$. To this end, we use the definition (4) of the discounting factor $\Lambda$, the strong Markov property of the process $X$ and (11) to obtain

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n+1}}}\right] &= \mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n}}} \mathbb{E}_x\left[\exp\left(-\int_0^{\tau^*_{2n+1}-\tau^*_{2n}} r(X_{\tau^*_{2n}+s})\, ds\right) \mid \mathcal{F}_{\tau^*_{2n}}\right]\right] \\
&= \mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n}}} \mathbb{E}_\gamma\left[\exp\left(-\int_0^{T_\beta} r(X_s)\, ds\right)\right]\right] \\
&= \frac{\varphi(\gamma)}{\varphi(\beta)} \mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n}}}\right].
\end{aligned}$$

Similarly, we can see that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n}}}\right] = \frac{\psi(\beta)}{\psi(\gamma)} \mathbb{E}_x\left[e^{-\Lambda_{\tau^*_{2n-1}}}\right].$$

These calculations and the dominated convergence theorem imply that

$$\mathbb{E}_x\left[\lim_{n\to\infty} e^{-\Lambda_{\tau^*_{2n+1}}}\right] = \lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau^*_1}}\right] \left(\frac{\varphi(\gamma)\psi(\beta)}{\varphi(\beta)\psi(\gamma)}\right)^n = 0,$$

the second equality following from the facts that $\varphi$ (resp., $\psi$) is strictly decreasing (resp., increasing) and $\gamma > \beta$. This conclusion contradicts the possibility that $\mathbb{P}_x\left(\lim_{n\to\infty} \tau_n^* < \infty\right) > 0$, and the proof is complete. □

# 4 Examples

## 4.1 The underlying is a geometric Brownian motion

Suppose that $X$ is a geometric Brownian motion, so that

$$dX_t = bX_t\, dt + \sigma X_t\, dW_t,$$

for some constants $b$ and $\sigma > 0$, and that $r > b \vee 0$ is a constant. In this case, Assumptions 1–3 are all satisfied,

$$\varphi(x) = x^m \quad \text{and} \quad \psi(x) = x^n,$$

where the constants $m < 0$ and $1 < n$ are given by

$$m, n = \frac{-\left(b - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(b - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2}.$$

It is straightforward to check that all of the conditions in Assumption 4 hold true with

$$x_{\mathrm{b}} = 0 \quad \text{and} \quad x_{\mathrm{s}} = \frac{r c_{\mathrm{s}}}{r - b}.$$

Since $x_{\mathrm{b}} = 0$, we are in the context of part (I) of Theorem 3. In particular, we can check that the function $q$ defined by (30) admits the expression

$$q(x) = \frac{1}{n - m} \left[ \frac{r - b}{m - 1} x^{-m+1} - \frac{r c_{\mathrm{s}}}{m} x^{-m} \right],$$

and conclude that the free-boundary point $\alpha > 0$ determining the optimal strategy is given by

$$\alpha = \frac{r c_{\mathrm{s}}(m - 1)}{(r - b)m}.$$

## 4.2 The underlying is a mean-reverting CEV process

Suppose that $X$ is the mean-reverting CEV process given by

$$dX_t = \kappa(\vartheta - X_t)\, dt + \sigma X_t^{\ell}\, dW_t.$$

for some constants $\kappa, \vartheta, \sigma > 0$ and $\ell \in [\frac{1}{2}, 1[$ such that $2\kappa\vartheta > \sigma^2$ if $\ell = \frac{1}{2}$, and that $r > 0$ is a constant. Assumptions 1–3 are all satisfied, while, the mean-reverting nature of $X$ implies that (21) in Assumption 4 also holds true. Furthermore, it is straightforward to check that the inequalities in (22) of Assumption 4 are satisfied with

$$x_{\mathrm{b}} = 0 \vee \frac{\kappa\vartheta - r c_{\mathrm{b}}}{\kappa + r} \quad \text{and} \quad x_{\mathrm{s}} = \frac{\kappa\vartheta + r c_{\mathrm{s}}}{\kappa + r}.$$

It is well-known that, if $\ell = \frac{1}{2}$, then

$$\phi(x) = \mathrm{U}\left(\frac{r}{\kappa}, \frac{2\kappa\vartheta}{\sigma^2}; \frac{2\kappa}{\sigma^2} x\right) \quad \text{and} \quad \psi(x) = {}_1\mathrm{F}_1\left(\frac{r}{\kappa}, \frac{2\kappa\vartheta}{\sigma^2}; \frac{2\kappa}{\sigma^2} x\right),$$

where U and ${}_1\mathrm{F}_1$ are confluent hypergeometric functions (see Chapter 13 of Abramowitz and Stegun (1972)). If $\ell \in ]\frac{1}{2}, 1[$, then we are not aware of any similar analytic expressions for the functions $\varphi$ and $\psi$. In either case, 0 is an entrance boundary point and $\lim_{x \downarrow 0} \psi(x) > 0$. Therefore, either of (I) or (II) in Theorem 3 can be the case, depending on the problem data.

## Appendix: proof of Lemmas 1 and 2

**Proof of Lemma 1.** In view of (22) in Assumption 4, we can see that

$$q'(x) = \Psi(x)\mathcal{L}H_{\mathrm{s}}(x) \begin{cases} > 0, & \text{if } x < x_{\mathrm{s}}, \\ < 0, & \text{if } x > x_{\mathrm{s}}. \end{cases} \tag{49}$$

This observation and the fact that $q(0) = 0$ imply that there exists a unique $\alpha > 0$ satisfying equation (29) if and only if $\lim_{x\to\infty} q(x) < 0$. To see that this inequality indeed holds, we note that the definition of $H_{\mathrm{s}}$ and (21) in Assumption 4 imply that

$$\frac{H_{\mathrm{s}}(x)}{\psi(x)} \equiv \frac{x - c_{\mathrm{s}}}{\psi(x)} > 0 \quad \text{for all } x > c_{\mathrm{s}} \quad \text{and} \quad \lim_{x\to\infty} \frac{H_{\mathrm{s}}(x)}{\psi(x)} = 0,$$

while (20) and the definition (30) of $q$ imply that

$$\left(\frac{H_{\mathrm{s}}}{\psi}\right)'(x) = \frac{Cp'(x)}{\psi^2(x)} \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds = \frac{Cp'(x)}{\psi^2(x)} q(x). \tag{50}$$

Since $q$ is strictly decreasing in $]x_{\mathrm{s}}, \infty[$, we can see that these facts can all be true only if $\lim_{x\to\infty} q(x) < 0$, as required. For future reference, we also note that this conclusion and (49) imply that the unique solution $\alpha > 0$ of the equation $q(\alpha) = 0$ is such that

$$x_{\mathrm{s}} < \alpha \quad \text{and} \quad q(x) = \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds > 0 \quad \text{for all } x < \alpha. \tag{51}$$

Since $H_{\mathrm{s}}'(x) = 1$ and $\psi'(x) > 0$ for all $x > 0$, the second expression in (28) implies that $A > 0$. This observation and the first expression in (28) imply that $H_{\mathrm{s}}(\alpha) \equiv \alpha - c_{\mathrm{s}} > 0$.

By construction, we will show that the function $w$ defined by (27) satisfies the HJB equation (26) if we show that

$$w(0,x) - w(1,x) + H_{\mathrm{s}}(x) = -A\psi(x) + H_{\mathrm{s}}(x) \le 0 \quad \text{for all } x < \alpha, \tag{52}$$

$$\mathcal{L}w(1,x) = \mathcal{L}H_{\mathrm{s}}(x) \le 0 \quad \text{for all } x > \alpha, \tag{53}$$

$$w(1,x) - w(0,x) - H_{\mathrm{b}}(x) = A\psi(x) - H_{\mathrm{b}}(x) \le 0 \quad \text{for all } x < \alpha, \tag{54}$$

and

$$w(1,x) - w(0,x) - H_{\mathrm{b}}(x) = x - c_{\mathrm{s}} - (x + c_{\mathrm{b}}) \le 0 \quad \text{for all } x > \alpha. \tag{55}$$

The last of these inequalities is plainly true, while (53) follows immediately from the first inequality in (51) and assumption (22). In view of (28), we can see that (52) is equivalent to

$$\frac{H_{\mathrm{s}}(x)}{\psi(x)} \equiv \frac{x - c_{\mathrm{s}}}{\psi(x)} \le \frac{\alpha - c_{\mathrm{s}}}{\psi(\alpha)} \equiv \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)} \quad \text{for all } x < \alpha,$$

which is indeed true, thanks to (50) and (51). Similarly, we can verify that (54) is equivalent to

$$\frac{H_{\mathrm{b}}(x)}{\psi(x)} \equiv \frac{x+c_{\mathrm{b}}}{\psi(x)} \geq \frac{\alpha-c_{\mathrm{s}}}{\psi(\alpha)} \equiv \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)} \quad \text{for all } x<\alpha. \tag{56}$$

In view of the identities

$$\left(\frac{H_{\mathrm{b}}}{\psi}\right)'(x) \overset{(20)}{=} \frac{Cp'(x)}{\psi^2(x)} \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds \overset{(30)}{=} \frac{Cp'(x)}{\psi^2(x)} \left[q(x) - \bigl(c_{\mathrm{b}}+c_{\mathrm{s}}\bigr) \int_0^x \Psi(s)r(s)\,ds\right],$$

we can see that, if $x_{\mathrm{b}}=0$, then the function $H_{\mathrm{b}}/\psi$ is strictly decreasing and (56) is true, while, if $x_{\mathrm{b}}>0$, then there exists $\tilde{x} \in \,]x_{\mathrm{b}}, \alpha[$ such that the function $H_{\mathrm{b}}/\psi$ is strictly increasing in $]0,\tilde{x}[$ and strictly decreasing in $]\tilde{x},\alpha]$, in which case (56) is true if and only if the second inequality in (31) is true. □

**Proof of Lemma 2.** We first note that (22) in Assumption 4 implies that

$$\begin{aligned} q_{\varphi}(x,z) &= \int_x^z \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds + \int_z^\infty \Phi(s)\bigl[\mathcal{L}H_{\mathrm{b}}(s) - \mathcal{L}H_{\mathrm{s}}(s)\bigr]\,ds \\ &= \int_x^z \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - (c_{\mathrm{b}}+c_{\mathrm{s}}) \int_z^\infty \Phi(s)r(s)\,ds \\ &< 0 \qquad \text{for all } x_{\mathrm{b}} \leq x < z. \end{aligned} \tag{57}$$

This observation implies that the system of equations (38) has no solution $0<\beta<\gamma$ if $x_{\mathrm{b}}=0$. Furthermore, if the system of equations (38) has a solution $0<\beta<\gamma$, then $\beta \in \,]0,x_{\mathrm{b}}[$. We therefore assume that the problem data is such that $x_{\mathrm{b}}>0$ in what follows, and we look for a solution $0<\beta<\gamma$ of (38) such that $\beta \in \,]0,x_{\mathrm{b}}[$.

In view of the equation (29) that $\alpha>0$ satisfies and the first inequality in (51), we can see that

$$\lim_{z\to\infty} q_{\psi}(x,z) = \int_0^x \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds - \int_\alpha^\infty \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds > 0 \quad \text{for all } x \leq x_{\mathrm{b}}.$$

Also, (22) in Assumption 4 implies that

$$\frac{\partial q_{\psi}}{\partial z}(x,z) = -\Psi(z)\mathcal{L}H_{\mathrm{s}}(z) \begin{cases} <0, & \text{if } z<x_{\mathrm{s}}, \\ >0, & \text{if } z>x_{\mathrm{s}}. \end{cases}$$

Combining these observations with the fact that

$$q_{\psi}(x,x) = \int_0^x \Psi(s)\mathcal{L}\bigl[H_{\mathrm{b}}(s) - H_{\mathrm{s}}(s)\bigr]\,ds = -\bigl(c_{\mathrm{b}}+c_{\mathrm{s}}\bigr) \int_0^x \Psi(s)r(s)\,ds < 0,$$

we can see that there exists a unique function $L : \,]0,x_{\mathrm{b}}[\, \to \mathbb{R}_+$ such that

$$x_{\mathrm{s}} < L(x) \quad \text{and} \quad q_{\psi}\bigl(x,L(x)\bigr) = 0. \tag{58}$$

Furthermore, we can differentiate the identity $q_\psi\big(x, L(x)\big) = 0$ with respect to $x$ to obtain

$$L'(x) = \frac{\Psi(x)\mathcal{L}H_{\mathrm{b}}(x)}{\Psi\big(L(x)\big)\mathcal{L}H_{\mathrm{s}}\big(L(x)\big)} > 0 \quad \text{for all } x < x_{\mathrm{b}}, \tag{59}$$

and combine the equation (29) that $\alpha > 0$ satisfies with the definition of $q_\psi$ to see that

$$\lim_{x \downarrow 0} L(x) = \alpha. \tag{60}$$

To proceed further, we calculate

$$\begin{aligned}
\frac{d}{dx} q_\varphi\big(x, L(x)\big) &\overset{(59)}{=} -\frac{\Phi(x)\Psi\big(L(x)\big) - \Phi\big(L(x)\big)\Psi(x)}{\Psi\big(L(x)\big)} \mathcal{L}H_{\mathrm{b}}(x) \\
&\overset{(16)}{=} -\frac{2\psi(x)}{C\sigma^2(x)p'(x)} \left[\frac{\varphi(x)}{\psi(x)} - \frac{\varphi\big(L(x)\big)}{\psi\big(L(x)\big)}\right] \mathcal{L}H_{\mathrm{b}}(x) \\
&< 0 \qquad \text{for all } x < x_{\mathrm{b}},
\end{aligned}$$

the inequality following because the function $\varphi/\psi$ is strictly decreasing. This result and the observation that $\lim_{x \uparrow x_{\mathrm{b}}} q_\varphi\big(x, L(x)\big) < 0$, which follows from (57), imply that there exists a unique point $\beta > 0$ such that $q_\varphi\big(\beta, L(\beta)\big) = 0$ if and only if

$$\lim_{x \downarrow 0} q_\varphi\big(x, L(x)\big) > 0. \tag{61}$$

In light of the calculations

$$\begin{aligned}
\lim_{x \downarrow 0} q_\varphi\big(x, L(x)\big) &\overset{(60)}{=} \lim_{x \downarrow 0} \int_x^\infty \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\, ds - \int_\alpha^\infty \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds \\
&\overset{(25),(29)}{=} -\lim_{x \downarrow 0} \frac{H_{\mathrm{b}}(x)}{\psi(x)} - \frac{\varphi(\alpha)}{\psi(\alpha)} \int_0^\alpha \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds - \int_\alpha^\infty \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\, ds \\
&\overset{(15)}{=} -\lim_{x \downarrow 0} \frac{H_{\mathrm{b}}(x)}{\psi(x)} + \frac{H_{\mathrm{s}}(\alpha)}{\psi(\alpha)},
\end{aligned} \tag{62}$$

and the definition (3) of the functions $H_{\mathrm{b}}$ and $H_{\mathrm{s}}$, we can see that (61) is equivalent to the second inequality in (41). We conclude this part of the analysis by observing that the system of equations (38) has a unique solution if and only if (41) is true. In particular, when this solution exists,

$$0 < \beta < x_{\mathrm{b}} \quad \text{and} \quad \gamma = L(\beta) > x_{\mathrm{s}}. \tag{63}$$

Also, the second expression in (36) and the inequality $\gamma > x_{\mathrm{s}}$ imply that $A > 0$, while the first expression in (37) and the inequality $\beta < x_{\mathrm{b}}$ imply that $B > 0$.

By construction, we will show that the function $w$ defined by (32)–(33) satisfies the HJB equation (26) if we show that

$$g_{\mathrm{s}}(x) := w(0, x) - w(1, x) + H_{\mathrm{s}}(x) \le 0 \quad \text{for all } x < \gamma, \tag{64}$$

$$\mathcal{L}w(1, x) = \mathcal{L}H_{\mathrm{s}}(x) \le 0 \quad \text{for all } x > \gamma, \tag{65}$$

$$\mathcal{L}w(0, x) = -\mathcal{L}H_{\mathrm{b}}(x) \le 0 \quad \text{for all } x < \beta, \tag{66}$$

and

$$g_{\mathrm{b}}(x) := w(1,x) - w(0,x) - H_{\mathrm{b}}(x) \le 0 \quad \text{for all } x > \beta. \tag{67}$$

The inequalities (65) and (66) follow immediately from (22) in Assumption 4 and (63). Also, (64) for $x \le \beta$, as well as (67) for $x > \gamma$, is equivalent to $-(c_{\mathrm{b}} + c_{\mathrm{s}}) \le 0$, which is true by assumption. To establish (64) and (67) for $x \in [\beta, \gamma]$, we use (15) and (36)–(37) to calculate

$$\begin{aligned} g_{\mathrm{s}}(x) &= -A\psi(x) + B\varphi(x) + H_{\mathrm{s}}(x) \\ &= -\psi(x) \int_x^\gamma \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds + \varphi(x) \int_x^\gamma \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds, \\ g_{\mathrm{b}}(x) &= A\psi(x) - B\varphi(x) - H_{\mathrm{b}}(x) \\ &= -\psi(x) \int_\beta^x \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds + \varphi(x) \int_\beta^x \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds. \end{aligned}$$

Furthermore, we use the definition (16) of the functions $\Phi$, $\Psi$ to obtain

$$g_{\mathrm{s}}'(x) = -\psi'(x) \int_x^\gamma \Phi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds + \varphi'(x) \int_x^\gamma \Psi(s)\mathcal{L}H_{\mathrm{s}}(s)\,ds, \tag{68}$$

$$g_{\mathrm{b}}'(x) = -\psi'(x) \int_\beta^x \Phi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds + \varphi'(x) \int_\beta^x \Psi(s)\mathcal{L}H_{\mathrm{b}}(s)\,ds. \tag{69}$$

In view of (22) in Assumption 4 and the inequalities $\varphi' < 0 < \psi'$, we can see that these identities imply that

$$g_{\mathrm{s}}'(x) > 0 \text{ for all } x \in [x_{\mathrm{s}}, \gamma[ \quad \text{and} \quad g_{\mathrm{b}}'(x) < 0 \text{ for all } x \in ]\beta, x_{\mathrm{b}}[.$$

These inequalities and the fact that $g_{\mathrm{s}}(\gamma) = g_{\mathrm{b}}(\beta) = 0$ imply that

$$g_{\mathrm{s}}(x) < 0 \text{ for all } x \in [x_{\mathrm{s}}, \gamma[ \quad \text{and} \quad g_{\mathrm{b}}(x) < 0 \text{ for all } x \in ]\beta, x_{\mathrm{b}}[.$$

Combining these inequalities with the identities

$$g_{\mathrm{s}}(\beta) = H_{\mathrm{s}}(\beta) - H_{\mathrm{b}}(\beta) = -(c_{\mathrm{s}} + c_{\mathrm{b}}) \quad \text{and} \quad g_{\mathrm{b}}(\gamma) = H_{\mathrm{s}}(\gamma) - H_{\mathrm{b}}(\gamma) = -(c_{\mathrm{s}} + c_{\mathrm{b}}),$$

which follow from (34), we can see that $g_{\mathrm{s}}(x), g_{\mathrm{b}}(x) \le 0$ for all $x \in [\beta, \gamma]$, as required, provided that $g_{\mathrm{s}}$ (resp., $g_{\mathrm{b}}$) does not have a strictly positive local maximum in $]\beta, x_{\mathrm{s}}[$ (resp., $]x_{\mathrm{b}}, \gamma[$). We can see that this is indeed the case by noting that

$$\mathcal{L}g_{\mathrm{s}}(x) = \mathcal{L}H_{\mathrm{s}}(x) > 0 \text{ for all } x \in ]\beta, x_{\mathrm{s}}], \quad \mathcal{L}g_{\mathrm{b}}(x) = -\mathcal{L}H_{\mathrm{b}}(x) > 0 \text{ for all } x \in [x_{\mathrm{b}}, \gamma[,$$

and then appealing to the maximum principle. □

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