The explicit solution to a sequential switching problem with non-smooth data*

Timothy C. Johnson[†] and Mihail Zervos[‡]

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Abstract

We consider the problem faced by a decision maker who can switch between two random payoff flows. Each of these payoff flows is an additive functional of a general one-dimensional Itô diffusion. There are no bounds on the number or on the frequency of the times at which the decision maker can switch, but each switching incurs a cost, which may depend on the underlying diffusion. The objective of the decision maker is to select a sequence of switching times that maximises the associated expected discounted payoff flow. In this context, we develop and study a model in the presence of assumptions that involve minimal smoothness requirements from the running payoff and switching cost functions, but which guarantee that the optimal strategies have relatively simple forms. In particular, we derive a complete and explicit characterisation of the decision maker's optimal tactics, which can take qualitatively different forms, depending on the problem data.

Keywords: optimal switching, sequential entry and exit decisions, stochastic impulse control, system of variational inequalities

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1 Introduction

The origins of the problem that we study are located in economics. Indeed, consider a manager who lives in an economy that is driven by a one-dimensional Itô diffusion. This manager can switch, at a cost, between two investment modes that are associated with different payoff flows and are dependent on the state of the economy. One of these investment modes is preferable

when the economic environment is poor, while the other one is preferable when the economic environment is positive. The manager has an infinite time horizon and wishes to maximise their expected discounted payoff flow by switching between the two investment modes. For instance, the manager may be switching between an asset with stochastic price dynamics and a bank account, or may be the operator of a production facility that can be shut down when it is not sufficiently profitable.

To fix ideas, we assume that the economy is driven by the one-dimensional Itô diffusion

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t, \quad X_0 = x \in \mathcal{I}, \tag{1}$$

where W is a standard one-dimensional Brownian motion, and $\mathcal{I} =]\alpha, \beta[$ is a given interval. In particular, we consider a stochastic system that can be operated in two modes, say "open" and "closed". We use the controlled finite variation process Z that takes values in $\{0, 1\}$ to keep track of the system's operating mode over time. In particular, if $Z_t = 1$ (resp., $Z_t = 0$), then the system is in its open (resp., closed) operating mode at time t, while, the jumps of Z occur at the sequence of times (T_n) when the decision maker switches the system between its two operating modes. Assuming that the system is initially in operating mode $z \in \{0, 1\}$, the decision maker's objective is to select a switching strategy $\mathbb{Z}_{z,x}$ that maximises the performance criterion

$$\tilde{J}(\mathbb{Z}_{z,x}) = \liminf_{n \to \infty} \mathbb{E}_x\left[\int_0^{T_n} e^{-\Lambda_t} Z_t\, dA_t^{h_o} + \int_0^{T_n} e^{-\Lambda_t}(1 - Z_t)\, dA_t^{h_c} \right.$$
$$\left. - \sum_{j=1}^{n-1} e^{-\Lambda_{T_j}} \left[g_o(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=1\}} + g_c(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right]. \tag{2}$$

The additive functionals A^{h_o} and A^{h_c} model the running payoff flows that the system yields while it is operated in its open and in its closed operating modes, respectively, the functions g_o and g_c provide the costs of switching the system from its closed to its open operating mode and vice versa, while the state-dependent discounting factor Λ is defined by

$$\Lambda_t = \int_0^t r(X_s)\, ds, \tag{3}$$

for some function $r > 0$. The precise definition of the additive functionals A^{h_o} and A^{h_c}, which are parametrised by the measures h_o and h_c, is given by (14) below.

It is worth noting at this point that our assumptions on the problem data imply that, given any admissible switching strategy, the "lim inf" in (2) can be replaced by "lim" (see (35) in Remark 1 and (41) in Section 3). Also, there exist functions R_{h_c} and R_h such that

$$\tilde{J}(\mathbb{Z}_{z,x}) = R_{h_c}(x) + zR_h(x) + \lim_{n \to \infty} \mathbb{E}_x\left[\sum_{j=1}^n e^{-\Lambda_{T_j}} \left[(R_h - g_o)(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=1\}} \right. \right.$$
$$\left. \left. - (R_h + g_c)(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right] \tag{4}$$

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(see (35) in Remark 1 and (42) in Section 3). In this expression, the function R_h can be interpreted as a measure for the accrual payoff differential resulting from having the system in its open rather than its closed operating mode. Furthermore, if the measures h_o and h_c are absolutely continuous with Radon-Nikodym derivatives with respect to the Lebesgue measure denoted by $\dot{h}_\mathrm{o}$ and $\dot{h}_\mathrm{c}$, respectively, then the performance index $\tilde{J}$ defined by (2) admits the expression

$$
\begin{aligned}
\tilde{J}(\mathbb{Z}_{z,x}) = \lim_{n \to \infty} \mathbb{E}_x \Bigg[&\int_0^{T_n} e^{-\Lambda_t} Z_t \dot{h}_\mathrm{o}(X_t)\, dt + \int_0^{T_n} e^{-\Lambda_t}(1 - Z_t)\dot{h}_\mathrm{c}(X_t)\, dt \\
&- \sum_{j=1}^{n-1} e^{-\Lambda_{T_j}} \left[g_\mathrm{o}(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=1\}} + g_\mathrm{c}(X_{T_j})\mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j<\infty\}} \Bigg],
\end{aligned}
\tag{5}
$$

which is more familiar in the stochastic control literature (see Remark 1).

Problems involving sequential entry and exit decisions have attracted considerable interest in the literature, particularly, in relation to the management of commodity production facilities. Following Brennan and Schwartz [BS85], Dixit and Pindyck [DP94], and Trigeorgis [T96], who were the first to address this type of a decision problem in the economics literature, Brekke and Øksendal [BØ94], Bronstein and Zervos [BZ06], Carmona and Ludkovski [CL05], Costeniuc, Schnetzer and Taschini [CST08], Djehiche and Hamadène [DH08], Djehiche, Hamadène and Popier [DHP08], Duckworth and Zervos [DZ01], Guo and Pham [GP05], Guo and Tomecek [GT08], Hamadène and Jeanblanc [HJ07], Lumley and Zervos [LZ01], Ly Vath and Pham [LVP01], Pham [P04], Porchet, Warin and Touzi [PTW06], and Zervos [Z03], provide an incomplete, alphabetically ordered, list of authors who have studied a number of related models by means of rigorous mathematics. The contributions of these authors range from explicit solutions to characterisations of the associated value functions in terms of classical as well as viscosity solutions of the corresponding Hamilton-Jacobi-Bellman (HJB) equations, as well as in terms of backward stochastic differential equation characterisations of the optimal strategies.

The paper is organised as follows. Section 2, which is composed by four parts, is mostly concerned with the problem formulation. In Section 2.1, we discuss some of the notation that we use throughout the paper, in Section 2.2, we develop our assumptions on the data of the underlying Itô diffusion defined by (1), in Section 2.3, we review a number of results regarding the solvability of a second order linear ODE on which our analysis relies, while, in Section 2.4, we complete the formulation of the control problem that we solve. Section 3 is concerned with the well-posedness of our optimisation problem as well as with establishing claims made above such as expression (4). In Section 4, we study a number of implications stemming from our Assumption 5 in Section 2.4. Indeed, Assumption 5 plays a central role in our analysis in the sense that it is this assumption that implies a relatively simple structure of the optimal strategies. In Section 5, we prove a verification theorem, which does not rely on Assumption 5, and, in Section 6, we develop the explicit solution of our control problem. Finally, in Section 7, we consider a couple of examples that provide some illustration of our results.

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2 Problem formulation, assumptions and preliminary results

2.1 Notation

We denote by $\mathcal{I}$ a given open interval with left endpoint $\alpha \geq -\infty$ and right endpoint $\beta \leq \infty$, and by $\mathcal{B}(\mathcal{I})$ the Borel σ-algebra on $\mathcal{I}$. Given a point $c \in \mathcal{I}$, we adopt the convention $]c, c[\, = \,]c, c] = [c, c[\, = \emptyset$. Also, when we consider sets $A \subseteq \mathcal{I}$, we adopt the conventions $\inf A = \beta$ and $\sup A = \alpha$ if $A = \emptyset$.

Throughout the paper, we consider signed measures of σ-finite total variation, and we refer to them as just "measures". Given a measure μ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ we denote by μ^+ and μ^- the unique positive measures on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ resulting from the Radon decomposition of μ, so that $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$, where $|\mu|$ is the total variation measure of μ. We denote by $\text{supp}\,\mu$ the support of μ. Also, we say that a measure μ on a measurable space $(\bar{\mathcal{I}}, \mathcal{B}(\bar{\mathcal{I}}))$, where $\bar{\mathcal{I}} \subseteq \mathcal{I}$ and $\mathcal{B}(\bar{\mathcal{I}})$ is the Borel σ-algebra on $\bar{\mathcal{I}}$, has *full-support* if $\text{supp}\,\mu = \bar{\mathcal{I}}$, and that it is *non-atomic* if $\mu(\{c\}) = 0$, for all $c \in \bar{\mathcal{I}}$.

Recalling that a function $f : \mathcal{I} \to \mathbb{R}$ is the difference of two convex functions if and only if its second distributional derivative is a measure, we denote by f'_- and by f'_+ the left-hand side and the right-hand side first derivatives of f, respectively, which both are functions of finite variation, and by f'' the measure on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ corresponding to the second distributional derivative of f.

2.2 The underlying Itô diffusion

We assume that the data of the one-dimensional Itô diffusion given by (1) in the introduction satisfy the following assumption.

Assumption 1 The functions $b, \sigma : \mathcal{I} \to \mathbb{R}$ are $\mathcal{B}(\mathcal{I})$-measurable,

$$\sigma^2(x) > 0, \quad \text{for all } x \in \mathcal{I},$$

and

$$\int_{\underline{\alpha}}^{\overline{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)}\, ds < \infty \quad \text{and} \quad \sup_{s \in [\underline{\alpha}, \overline{\beta}]} \sigma^2(s) < \infty, \quad \text{for all } \alpha < \underline{\alpha} < \overline{\beta} < \beta.$$

$\square$

With reference to Karatzas and Shreve [KS91, Section 5.5.C], the conditions appearing in this assumption are sufficient for the SDE (1) to have a weak solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ that is unique in the sense of probability law up to a possible explosion time, for all initial conditions $x \in \mathcal{I}$. In particular, given $c \in \mathcal{I}$, the scale function p_c and the speed measure m,

given by

$$p_c(x) = \int_c^x \exp\left(-2\int_c^s \frac{b(u)}{\sigma^2(u)}\,du\right) ds, \quad \text{for } x \in \mathcal{I},$$

$$m(dx) = \frac{2}{\sigma^2(x)p'(x)}\,dx,$$

which characterise one-dimensional diffusions, are well-defined. We also assume that the solution of (1) in non-explosive, i.e., the hitting time of the boundary $\{\alpha, \beta\}$ of the interval $\mathcal{I}$ is infinite with probability 1 (see Karatzas and Shreve [KS91, Theorem 5.5.29] for appropriate necessary and sufficient analytic conditions).

Assumption 2 The solution of (1) is non-explosive. □

Relative to the discounting factor Λ defined by (3), we make the following assumption.

Assumption 3 The function $r : \mathcal{I} \to]0, \infty[$ is $\mathcal{B}(\mathcal{I})$-measurable and locally bounded. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$, for all $x \in \mathcal{I}$. □

2.3 The solution of an associated ODE

In the presence of Assumptions 1, 2 and 3, the general solution of the second-order linear homogeneous ODE

$$\frac{1}{2}\sigma^2(x)f''(x) + b(x)f'(x) - r(x)f(x) = 0, \quad x \in \mathcal{I},$$

exists in the classical sense and is given by

$$f(x) = A\phi(x) + B\psi(x),$$

for some constants $A, B \in \mathbb{R}$. The functions ϕ and ψ are C^1, their first derivatives are absolutely continuous functions,

$$0 < \phi(x) \quad \text{and} \quad \phi'(x) < 0, \quad \text{for all } x \in \mathcal{I}, \tag{6}$$
$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0, \quad \text{for all } x \in \mathcal{I}, \tag{7}$$

and

$$\lim_{x \downarrow \alpha} \phi(x) = \lim_{x \uparrow \beta} \psi(x) = \infty. \tag{8}$$

In this context, ϕ and ψ are unique, modulo multiplicative constants, and the scale function p_c admits the expression

$$p_c'(x) = \frac{\phi(x)\psi'(x) - \phi'(x)\psi(x)}{\phi(c)\psi'(c) - \phi'(c)\psi(c)} = \frac{\mathcal{W}(x)}{\mathcal{W}(c)}, \quad \text{for all } x, c \in \mathcal{I}, \tag{9}$$

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where $\mathcal{W} > 0$ is the Wronskian of the functions ϕ and ψ. Also, given any points $x_1 < x_2$ in $\mathcal{I}$ and weak solutions $\mathbb{S}_{x_1}$, $\mathbb{S}_{x_2}$ of the SDE (1), the functions ϕ and ψ satisfy

$$\phi(x_2) = \phi(x_1)\mathbb{E}_{x_2}\left[e^{-\Lambda_{\tau_{x_1}}}\right] \quad \text{and} \quad \psi(x_1) = \psi(x_2)\mathbb{E}_{x_1}\left[e^{-\Lambda_{\tau_{x_2}}}\right]. \tag{10}$$

Here, as well as in the rest of the paper, we denote by τ_γ, where γ is any point in $\mathcal{I}$, the first hitting time of $\{\gamma\}$, which is defined by

$$\tau_\gamma = \{t \geq 0 \mid X_t = \gamma\}.$$

All of these claims are standard, and can be found in various forms in several references, including Feller [F52], Breiman [B68], Itô and McKean [IM74], Karlin and Taylor [KT81], Rogers and Williams [RW00], and Borodin and Salminen [BS02].

To proceed further, we consider the solvability of the non-homogeneous ODE

$$\mathcal{L}R_\mu + \mu = 0, \tag{11}$$

where μ is a measure on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ and the measure-valued operator $\mathcal{L}$ is defined by

$$\mathcal{L}f(dx) = \frac{1}{2}\sigma^2(x)f''(dx) + b(x)f_-'(x)\,dx - r(x)f(x)\,dx \tag{12}$$

on the space of all functions $f : \mathcal{I} \to \mathbb{R}$ that are differences of two convex functions (see also Section 2.1 above). Also, we recall Definition 2.5 from Johnson and Zervos [JZ07].

Definition 1 The space $\mathcal{I}_{\phi,\psi}$ is defined to be the set of all measures μ on $(\mathcal{I}, \mathcal{B}(\mathcal{I}))$ such that

$$\int_{]\alpha,\gamma[} \Psi(s)\,|\mu|(ds) + \int_{[\gamma,\beta[} \Phi(s)\,|\mu|(ds) < \infty, \quad \text{for all } \gamma \in \mathcal{I},$$

where the functions Φ and Ψ are defined by

$$\Phi(x) = \frac{\phi(x)}{\sigma^2(x)\mathcal{W}(x)} \quad \text{and} \quad \Psi(x) = \frac{\psi(x)}{\sigma^2(x)\mathcal{W}(x)}. \tag{13}$$

$\mathcal{I}_{\phi,\psi}$ is called the space of (ϕ, ψ)-integrable measures. $\qquad\qquad\qquad\square$

For the rest of this section, we fix a weak solution $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ of (1), and we consider the linear functional $\mu \mapsto A^\mu$ mapping the family of all σ-finite measures μ into the set of finite variation, continuous processes A^μ defined by

$$A_t^\mu = \int_\alpha^\beta \frac{L_t^y}{\sigma^2(y)}\,\mu(dy), \tag{14}$$

where L^y is the local-time process of X at $y \in \mathcal{I}$. It is straightforward to see that the total variation process $|A^\mu|$ of A^μ is given by $|A^\mu| = A^{|\mu|}$, and that

$$\text{if } \mu \text{ is a positive measure, then } A^\mu \text{ is an increasing process,} \tag{15}$$

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because L^y is an increasing process, for all $y \in \mathcal{I}$. Also,

$$\int_0^\infty \mathbf{1}_\Gamma(t)\,dA_t^{|\mu|} = 0, \quad \text{for all countable sets } \Gamma \subset \mathcal{I}, \tag{16}$$

because A^μ has continuous sample paths.

The following results, which we will need, have been established by Johnson and Zervos [JZ07]. A measure μ belongs to $\mathcal{I}_{\phi,\psi}$ if and only if

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\,dA_t^{|\mu|}\right] < \infty, \quad \text{for all } x \in \mathcal{I}. \tag{17}$$

Given any $\mu \in \mathcal{I}_{\phi,\psi}$, the function R_μ defined by

$$R_\mu(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\,dA_t^\mu\right] \tag{18}$$

admits the analytic representations

$$\begin{aligned}
R_\mu(x) &= \phi(x)\int_{]\alpha,x[} \Psi(s)\,\mu(ds) + \psi(x)\int_{[x,\beta[} \Phi(s)\,\mu(ds) \\
&= \phi(x)\int_{]\alpha,x]} \Psi(s)\,\mu(ds) + \psi(x)\int_{]x,\beta[} \Phi(s)\,\mu(ds),
\end{aligned} \tag{19}$$

and satisfies the ODE (11) as well as

$$\lim_{x\downarrow\alpha} \frac{|R_\mu(x)|}{\phi(x)} = \lim_{x\uparrow\beta} \frac{|R_\mu(x)|}{\psi(x)} = 0. \tag{20}$$

Noting that $-\mathcal{L}R_\mu = \mu$, we can see that, if $R_{-\mathcal{L}R_\mu}$ is defined as in (18)–(19) with $-\mathcal{L}R_\mu$ in place of μ, then

$$R_{-\mathcal{L}R_\mu} = R_\mu. \tag{21}$$

Given any $(\mathcal{F}_t)$-stopping time ρ,

$$\mathbb{E}_x\left[e^{-\Lambda_\rho}|R_\mu(X_\rho)|\mathbf{1}_{\{\rho<\infty\}}\right] < \infty, \tag{22}$$

and R_μ satisfies Dynkin's formula, i.e., given any $(\mathcal{F}_t)$-stopping times $\rho_1 < \rho_2$,

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\Lambda_{\rho_2}}R_\mu(X_{\rho_2})\mathbf{1}_{\{\rho_2<\infty\}}\right] &= \mathbb{E}_x\left[e^{-\Lambda_{\rho_1}}R_\mu(X_{\rho_1})\mathbf{1}_{\{\rho_1<\infty\}}\right] + \mathbb{E}_x\left[\int_{\rho_1}^{\rho_2} e^{-\Lambda_t}\,dA_t^{-\mu}\right] \\
&= \mathbb{E}_x\left[e^{-\Lambda_{\rho_1}}R_\mu(X_{\rho_1})\mathbf{1}_{\{\rho_1<\infty\}}\right] - \mathbb{E}_x\left[\int_{\rho_1}^{\rho_2} e^{-\Lambda_t}\,dA_t^\mu\right],
\end{aligned} \tag{23}$$

as well as the strong transversality condition, i.e., given any sequence of $(\mathcal{F}_t)$-stopping times (ρ_n) such that $\lim_{n\to\infty}\rho_n = \infty$,

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\rho_n}}|R_\mu(X_{\rho_n})|\mathbf{1}_{\{\rho_n<\infty\}}\right] = 0, \tag{24}$$

Furthermore,

$$\phi(x)(R_\mu)'_+(x) - \phi'(x)R_\mu(x) = -\mathcal{W}(x)\int_{]x,\beta[}\Phi(s)\,\mathcal{L}R_\mu(ds), \qquad (25)$$

$$\phi(x)(R_\mu)'_-(x) - \phi'(x)R_\mu(x) = -\mathcal{W}(x)\int_{[x,\beta[}\Phi(s)\,\mathcal{L}R_\mu(ds), \qquad (26)$$

$$\psi(x)(R_\mu)'_+(x) - \psi'(x)R_\mu(x) = \mathcal{W}(x)\int_{]\alpha,x]}\Psi(s)\,\mathcal{L}R_\mu(ds), \qquad (27)$$

$$\psi(x)(R_\mu)'_-(x) - \psi'(x)R_\mu(x) = \mathcal{W}(x)\int_{]\alpha,x[}\Psi(s)\,\mathcal{L}R_\mu(ds). \qquad (28)$$

At this point, we should note that, if μ is absolutely continuous with Radon-Nikodym derivative with respect to the Lebesgue measure denoted by $\dot{\mu}$, then, given any $(\mathcal{F}_t)$-stopping times $\rho_1 \leq \rho_2$,

$$\mathbb{E}_x\left[\int_{\rho_1}^{\rho_2} e^{-\Lambda_t}\,dA_t^\mu\right] = \mathbb{E}_x\left[\int_{\rho_1}^{\rho_2} e^{-\Lambda_t}\dot{\mu}(X_t)\,dt\right], \qquad (29)$$

which is essentially a consequence of the so-called occupation times formula. Furthermore, if α (resp., β) is a natural boundary point, i.e., if

$$\lim_{x\downarrow\alpha}\psi(x) = 0 \qquad \left(\text{resp., } \lim_{x\uparrow\beta}\phi(x) = 0\right),$$

then

$$\lim_{x\downarrow\alpha}R_h(x) = \lim_{x\downarrow\alpha}\frac{\dot{\mu}(x)}{r(x)} \qquad \left(\text{resp., } \lim_{x\uparrow\beta}R_h(x) = \lim_{x\uparrow\beta}\frac{\dot{\mu}(x)}{r(x)}\right). \qquad (30)$$

These limits are not necessarily true if, e.g., α is an entrance boundary point (an example illustrating this is given by the function R_h in Section 7.2).

2.4 The objective of the optimisation problem

We adopt a weak formulation of the optimal control problem that we solve.

Definition 2 Given an initial condition $(z,x) \in \{0,1\} \times \mathcal{I}$, an admissible *switching* strategy, is any collection $\mathbb{Z}_{z,x} = (\mathbb{S}_x, Z, T_n)$ such that
(I) $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ is a weak solution of the SDE (1),
(II) Z is an $(\mathcal{F}_t)$-adapted, finite variation, càglàd process with values in $\{0,1\}$, and such that $Z_0 = z$, and
(III) (T_n) is the strictly increasing sequence of $(\mathcal{F}_t)$-stopping times at which the jumps of Z occur, which can be defined recursively by

$$T_1 = \inf\{t > 0 \mid Z_t \neq z\} \quad \text{and} \quad T_{j+1} = \inf\{t > T_j \mid Z_t \neq Z_{T_j}\}, \quad \text{for } j = 1, 2, \ldots, \qquad (31)$$

with the usual convention that $\inf\emptyset = \infty$.
We denote by $\mathcal{A}_{z,x}$ the set of all admissible strategies. $\qquad\square$

With each admissible switching strategy, we associate the performance criterion

$$J(\mathbb{Z}_{z,x}) = \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} Z_t \, dA_t^h \right]$$

$$- \sum_{n=1}^\infty \mathbb{E}_x \left[e^{-\Lambda_{T_n}} \left[g_{\mathrm{o}}(X_{T_n}) \mathbf{1}_{\{\Delta Z_{T_n}=1\}} + g_{\mathrm{c}}(X_{T_n}) \mathbf{1}_{\{\Delta Z_{T_n}=-1\}} \right] \mathbf{1}_{\{T_n<\infty\}} \right]. \qquad (32)$$

The objective of the control problem is to maximise $J(\mathbb{Z}_{z,x})$ over all admissible $\mathbb{Z}_{z,x}$. Accordingly, we define the value function v by

$$v(z,x) = \sup_{\mathbb{Z}_{z,x} \in \mathcal{A}_{z,x}} J(\mathbb{Z}_{z,x}), \quad \text{for } z \in \{0,1\} \text{ and } x \in \mathcal{I}.$$

To ensure that our optimisation problem is well-posed, we make the following assumption. It is worth observing that among the other conditions, (33) has a simple economic interpretation because it excludes the possibility of generating arbitrarily high profits by rapidly switching between the system's two operating modes.

Assumption 4 Each of the functions $g_{\mathrm{c}}, g_{\mathrm{o}} : \mathcal{I} \to \mathbb{R}$ is the difference of two convex functions, and

$$g_{\mathrm{c}}(x) + g_{\mathrm{o}}(x) > 0, \quad \text{for all } x \in \mathcal{I}. \qquad (33)$$

The measures $\mathcal{L}g_{\mathrm{c}}, \mathcal{L}g_{\mathrm{o}}$ and h are (ϕ, ψ)-integrable,

$$g_{\mathrm{c}} = R_{-\mathcal{L}g_{\mathrm{c}}} \quad \text{and} \quad g_{\mathrm{o}} = R_{-\mathcal{L}g_{\mathrm{o}}}, \qquad (34)$$

where $R_{-\mathcal{L}g_{\mathrm{c}}}$ and $R_{-\mathcal{L}g_{\mathrm{o}}}$ are defined as in (18)–(19). $\qquad \square$

Remark 1 The structure of the performance criterion defined by (32) involves a running payoff flow only when the system is in its open operating mode. We have chosen this setting instead of the apparently more general one involving the performance index $\tilde{J}$ defined by (2) in the introduction, only with a view to simplifying the presentation of our results. Indeed, assuming that both of h_{o} and h_{c} are (ϕ, ψ)-integrable, the linearity of the mapping $\mu \mapsto A^\mu$ and (18) implies that

$$\tilde{J}(\mathbb{Z}_{z,x}) = \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \, dA_t^{h_{\mathrm{c}}} \right] + J(\mathbb{Z}_{z,x}) = R_{h_{\mathrm{c}}}(x) + J(\mathbb{Z}_{z,x}), \quad \text{for all } \mathbb{Z}_{z,x} \in \mathcal{A}_{z,x}, \qquad (35)$$

if we let $h = h_{\mathrm{o}} - h_{\mathrm{c}}$, which reveals that the two optimisation problems are equivalent. Furthermore, it is worth noting at this point that the expression (5) of $\tilde{J}$ that arises when h_{o} and h_{c} absolutely continuous with respect to the Lebesgue measure follows immediately from (29). $\qquad \square$

The next assumption, which involves the functions $-(R_h + g_{\mathrm{c}})$ and $R_h - g_{\mathrm{o}}$ appearing in the expression of the performance criterion given by (4) in the introduction, ensures that the optimal strategies of the optimisation problem that we study admit an explicit characterisation.

Assumption 5 The measure $\mathcal{L}(R_h + g_c)$ satisfies one of the following mutually exclusive conditions:

A1. $\big|\mathcal{L}(R_h + g_c)\big|(\mathcal{I}) = 0$;

A2. $\big|\mathcal{L}(R_h + g_c)\big|(\mathcal{I}) > 0$, and $-\mathcal{L}(R_h + g_c)$ is a positive measure;

A3. $\big|\mathcal{L}(R_h + g_c)\big|(\mathcal{I}) > 0$, and $\mathcal{L}(R_h + g_c)$ is a positive measure;

A4. $\big|\mathcal{L}(R_h + g_c)\big|(\mathcal{I}) > 0$, $\operatorname{supp}[\mathcal{L}(R_h + g_c)]^+ \neq \emptyset$, $\operatorname{supp}[\mathcal{L}(R_h + g_c)]^- \neq \emptyset$, and there exists a point $\tilde{a} \in \mathcal{I}$ such that

$$\operatorname{supp}[\mathcal{L}(R_h + g_c)]^+ \subseteq\,]\alpha, \tilde{a}[\quad \text{and} \quad \operatorname{supp}[\mathcal{L}(R_h + g_c)]^- \subseteq\, [\tilde{a}, \beta[. \tag{36}$$

Similarly, the measure $\mathcal{L}(R_h - g_o)$ satisfies one of the mutually exclusive conditions:

B1. $\big|\mathcal{L}(R_h - g_o)\big|(\mathcal{I}) = 0$;

B2. $\big|\mathcal{L}(R_h - g_o)\big|(\mathcal{I}) > 0$, and $\mathcal{L}(R_h - g_o)$ is a positive measure;

B3. $\big|\mathcal{L}(R_h - g_o)\big|(\mathcal{I}) > 0$, and $-\mathcal{L}(R_h - g_o)$ is a positive measure;

B4. $\big|\mathcal{L}(R_h - g_o)\big|(\mathcal{I}) > 0$, $\operatorname{supp}[\mathcal{L}(R_h - g_o)]^+ \neq \emptyset$, $\operatorname{supp}[\mathcal{L}(R_h - g_o)]^- \neq \emptyset$, and there exists a point $\tilde{b} \in \mathcal{I}$ such that

$$\operatorname{supp}[\mathcal{L}(R_h - g_o)]^+ \subseteq\,]\alpha, \tilde{b}] \quad \text{and} \quad \operatorname{supp}[\mathcal{L}(R_h - g_o)]^- \subseteq\,]\tilde{b}, \beta[. \tag{37}$$

Furthermore, if the conditions **A4** and **B4** hold simultaneously, then

$$\tilde{a} \leq \tilde{b}, \tag{38}$$

$$\int_{]\alpha,u]} \Psi(s)\,\mathcal{L}(g_c + g_o)(ds) < 0, \quad \text{for all } u \in\,]\alpha, \tilde{a}], \tag{39}$$

and

$$\int_{[u,\beta[} \Phi(s)\,\mathcal{L}(g_c + g_o)(ds) < 0, \quad \text{for all } u \in\, [\tilde{b}, \beta[. \tag{40}$$

$\square$

The previous assumptions are sufficient for the existence of an optimal strategy, which is not in general unique. To address uniqueness issues, we have to make additional assumptions, which are captured by the following conditions.

Assumption 6 Cases **A1** and **B1** cannot occur. If case **A3** (resp., case **B3**) occurs, then $\mathcal{L}(R_h + g_c)(]x, \beta[) > 0$ (resp., $\mathcal{L}(R_h - g_o)(]\alpha, x[) < 0$), for all $x \in \mathcal{I}$. Also, in case **A4** of Assumption 5, the point $\tilde{a}$ can be chosen so that the restriction of the measure $\mathcal{L}(R_h + g_c)$ in $(]\alpha, \tilde{a}[, \mathcal{B}(]\alpha, \tilde{a}[))$ has full support, while, in case **B4** of Assumption 5, the point $\tilde{b}$ can be chosen so that the restriction of the measure $\mathcal{L}(R_h - g_o)$ in $(]\tilde{b}, \beta[, \mathcal{B}(]\tilde{b}, \beta[))$ has full support. □

Assumption 7 In cases **A4** and **B4** of Assumption 5, the restriction of the measure $\mathcal{L}(R_h+g_c)$ in $(]\alpha, \tilde{a}[, \mathcal{B}(]\alpha, \tilde{a}[))$ and the restriction of the measure $\mathcal{L}(R_h - g_o)$ in $(]\tilde{b}, \beta[, \mathcal{B}(]\tilde{b}, \beta[))$ both are non-atomic. □

3   Well-posedness of the optimisation problem

The following result is mainly concerned with establishing that the optimisation problem that we study is non-trivial in the sense that there are no switching strategies with infinite payoff.

Lemma 1 *Consider the stochastic control problem formulated in Section 2, and suppose that Assumptions 1–4 hold true. Given any initial condition $(z, x) \in \{0, 1\} \times \mathcal{I}$ and any admissible switching strategy $\mathbb{Z}_{z,x} \in \mathcal{A}_{z,x}$, $J(\mathbb{Z}_{z,x}) \in [-\infty, \infty[$,*

$$
J(\mathbb{Z}_{z,x}) = \lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{T_n} e^{-\Lambda_t} Z_t \, dA_t^h \right.
$$
$$
\left. - \sum_{j=1}^{n-1} e^{-\Lambda_{T_j}} \left[g_o(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=1\}} + g_c(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right] \quad (41)
$$

and

$$
J(\mathbb{Z}_{z,x}) = z R_h(x) + \lim_{n \to \infty} \mathbb{E}_x \left[\sum_{j=1}^n e^{-\Lambda_{T_j}} \left[(R_h - g_o)(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=1\}} \right. \right.
$$
$$
\left. \left. - (R_h + g_c)(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right]. \quad (42)
$$

Proof. We fix any initial condition $(z, x) \in \{0, 1\} \times \mathcal{I}$ and any admissible switching strategy $\mathbb{Z}_{z,x} \equiv (\mathbb{S}_x, Z, T_n) \in \mathcal{A}_{z,x}$, and we note that $\lim_{n\to\infty} T_n = \infty$, $\mathbb{P}_x$-a.s., because Z is a finite variation process whose jumps all have size 1. Recalling that the total variation process $|A^h|$ of A^h is equal to $A^{|h|}$, we note that

$$
\left| \int_0^{T_n} e^{-\Lambda_t} Z_t \, dA_t^h \right| \leq \int_0^{T_n} e^{-\Lambda_t} Z_t \, d\left|A_t^h\right| = \int_0^{T_n} e^{-\Lambda_t} Z_t \, dA_t^{|h|} \leq \int_0^\infty e^{-\Lambda_t} Z_t \, dA_t^{|h|}.
$$

The last term in these inequalities has finite expectation thanks to the assumption that the measure h is (ϕ, ψ)-integrable and (17). This observation and the dominated convergence theorem imply that

$$
\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} Z_t \, dA_t^h \right] = \lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{T_n} e^{-\Lambda_t} Z_t \, dA_t^h \right] \in \mathbb{R}. \quad (43)
$$

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Similarly, we can see that the assumption that the measure $\mathcal{L}g_c$ is (ϕ, ψ)-integrable implies that

$$\mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} Z_t \, dA_t^{-\mathcal{L}g_c} \right] = \lim_{n \to \infty} \mathbb{E}_x \left[\int_0^{T_n} e^{-\Lambda_t} Z_t \, dA_t^{-\mathcal{L}g_c} \right] \in \mathbb{R}. \tag{44}$$

To proceed further, we assume that $z = 1$. Using Dynkin's formula (23), we can calculate

$$\sum_{j=1}^{2n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_j}} \left[g_o(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = -1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right]$$

$$= \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} g_o(X_{T_{2j}}) \mathbf{1}_{\{T_{2j} < \infty\}} \right] + \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j+1}}} g_c(X_{T_{2j+1}}) \mathbf{1}_{\{T_{2j+1} < \infty\}} \right]$$

$$= \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} \left[g_o(X_{T_{2j}}) + g_c(X_{T_{2j}}) \right] \mathbf{1}_{\{T_{2j} < \infty\}} \right]$$

$$+ g_c(x) + \mathbb{E}_x \left[\int_0^{T_{2n}} e^{-\Lambda_t} Z_t \, dA_t^{-\mathcal{L}g_c} \right].$$

In view of (44) and (33) in Assumption 4, the right-hand side of this expression converges in $]-\infty, \infty]$. Combining this observation with the limit

$$\lim_{n \to \infty} \mathbb{E}_x \left[e^{-\Lambda_{T_{2n}}} g_o(X_{T_{2n}}) \mathbf{1}_{\{T_{2n} < \infty\}} \right] = 0,$$

which follows from the strong transversality condition (24) and the fact that $\lim_{n \to \infty} T_n = \infty$, we can see that

$$\lim_{n \to \infty} \mathbb{E}_x \left[\sum_{j=1}^n e^{-\Lambda_{T_j}} \left[g_o(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = 1\}} + g_c(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = -1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right] \in \,]-\infty, \infty].$$

This limit and (43) imply that $J(\mathbb{Z}_{z,x}) \in [-\infty, \infty[$ as well as (41).

To see (42), we note that (21) with $\mu = h$, Dynkin's formula (23) and (43) imply that

$$\mathbb{E}_x \left[\int_0^{T_{2n-1}} e^{-\Lambda_t} Z_t \, dA_t^h \right] = \mathbb{E}_x \left[\int_0^{T_1} e^{-\Lambda_t} \, dA_t^h \right] + \sum_{j=1}^{n-1} \mathbb{E}_x \left[\int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda_t} \, dA_t^h \right]$$

$$= R_h(x) + \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} R_h(X_{T_{2j}}) \mathbf{1}_{\{T_{2j} < \infty\}} \right]$$

$$- \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j+1}}} R_h(X_{T_{2j+1}}) \mathbf{1}_{\{T_{2j+1} < \infty\}} \right].$$

$$= R_h(x) + \mathbb{E}_x \left[e^{-\Lambda_{T_{2n-1}}} R_h(X_{T_{2n-1}}) \mathbf{1}_{\{T_{2n-1} < \infty\}} \right]$$

$$+ \mathbb{E}_x \left[\sum_{j=1}^{2n-2} e^{-\Lambda_{T_j}} \left[\mathbf{1}_{\{\Delta Z_{T_j} = 1\}} - \mathbf{1}_{\{\Delta Z_{T_j} = -1\}} \right] R_h(X_{T_j}) \mathbf{1}_{\{T_j < \infty\}} \right],$$

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as well as

$$\mathbb{E}_x\left[\int_0^{T_{2n}} e^{-\Lambda_t} Z_t \, dA_t^h\right] = R_h(x) + \mathbb{E}_x\left[\sum_{j=1}^{2n-1} e^{-\Lambda_{T_j}}\left[\mathbf{1}_{\{\Delta Z_{T_j}=1\}} - \mathbf{1}_{\{\Delta Z_{T_j}=-1\}}\right] R_h(X_{T_j})\mathbf{1}_{\{T_j<\infty\}}\right],$$

These calculations, the limit

$$\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{T_{2n-1}}} R_h(X_{T_{2n-1}})\mathbf{1}_{\{T_{2n-1}<\infty\}}\right] = 0,$$

which follows from the strong transversality condition, and (41) imply (42).

Finally, the analysis when $z = 0$ follows similar steps. $\square$

4 Ramifications of our assumptions

We now consider the functions $-(R_h + g_c)/\phi$ and $(R_h - g_o)/\psi$, which appear in expression (4) of our performance criterion and will play a fundamental role in the solution of our problem, and we make the following observations. First, we note that (20) and Assumption 4 imply that

$$\lim_{x\downarrow\alpha} \frac{(R_h + g_c)(x)}{\phi(x)} = 0 \quad \text{and} \quad \lim_{x\uparrow\beta} \frac{(R_h - g_o)(x)}{\psi(x)} = 0. \tag{45}$$

Also, using (25)–(28), we can calculate

$$-\left(\frac{R_h + g_c}{\phi}\right)'_+(x) = \frac{\mathcal{W}(x)}{\phi^2(x)} \int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \tag{46}$$

and

$$\left(\frac{R_h - g_o}{\psi}\right)'_+(x) = \frac{\mathcal{W}(x)}{\psi^2(x)} \int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds). \tag{47}$$

Combining (45) and (46) with Assumption 5, we can see that we can have one of the following cases:

- In Case **A1** of Assumption 5,

$$-(R_h + g_c)(x) = 0, \quad \text{for all } x \in \mathcal{I}. \tag{48}$$

- In Case **A2** of Assumption 5,

$$\int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \le 0, \quad \text{for all } x \in \mathcal{I}, \tag{49}$$

$$-(R_h + g_c)(x) < 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing.} \tag{50}$$

- In Case **A3** of Assumption 5,

$$\int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \geq 0, \quad \text{for all } x \in \mathcal{I}, \tag{51}$$

$$-(R_h + g_c)(x) > 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad -\frac{R_h + g_c}{\phi} \text{ is increasing.} \tag{52}$$

- In Case **A4** of Assumption 5, we can have one of the following possibilities:

  A41.

$$\int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \begin{cases} \leq 0, & \text{for all } x \in \mathcal{I}, \\ > 0, & \text{for all } x \in]\alpha, \tilde{a}], \end{cases} \tag{53}$$

$$-(R_h + g_c)(x) < 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing;} \tag{54}$$

  A42. there exists a point $a^* \in]\alpha, \tilde{a}[$ such that

$$\int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \begin{cases} = 0, & \text{if } x \in]\alpha, a^*[, \\ \leq 0, & \text{if } x \in [a^*, \beta[, \end{cases} \tag{55}$$

$$-(R_h + g_c)(x) \begin{cases} = 0, & \text{for } x \in]\alpha, a^*], \\ < 0, & \text{for } x \in]a^*, \beta[, \end{cases} \quad \text{and} \quad -\frac{R_h + g_c}{\phi} \text{ is decreasing;} \tag{56}$$

  A43. there exists a point $a^* \in]\alpha, \tilde{a}[$ such that

$$\int_{]x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \begin{cases} \geq 0, & \text{if } x \in]\alpha, a^*[, \\ \leq 0, & \text{if } x \in [a^*, \beta[, \end{cases} \tag{57}$$

$$-\frac{R_h + g_c}{\phi} \text{ is} \begin{cases} \text{positive and increasing in }]\alpha, a^*[, \\ \text{decreasing in }]a^*, \beta[, \end{cases} \tag{58}$$

$$-(R_h + g_c)(a^*) > 0 \quad \text{and} \quad -(R_h + g_c)(x) < -(R_h + g_c)(a^*), \text{ for all } x \in]a^*, \beta[. \tag{59}$$

Similarly, we can see that (45), (47) and Assumption 5 imply that we can have one of the following cases:

- In Case **B1** of Assumption 5,

$$(R_h - g_o)(x) = 0, \quad \text{for all } x \in \mathcal{I}. \tag{60}$$

- In Case **B2** of Assumption 5,

$$\int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \geq 0, \quad \text{for all } x \in \mathcal{I}, \tag{61}$$

$$(R_h - g_o)(x) < 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad \frac{R_h - g_o}{\psi} \text{ is increasing.} \tag{62}$$

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- In Case **B3** of Assumption 5,

$$\int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \leq 0 \quad \text{for all } x \in \mathcal{I}, \tag{63}$$

$$(R_h - g_o)(x) > 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad \frac{R_h - g_o}{\psi} \text{ is decreasing.} \tag{64}$$

- In Case **B4** of Assumption 5, we can have one of the following possibilities:

B41.

$$\int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \begin{cases} \geq 0, & \text{for all } x \in \mathcal{I}, \\ > 0, & \text{for all } x \in [\tilde{b}, \beta[, \end{cases} \tag{65}$$

$$(R_h - g_o)(x) < 0, \text{ for all } x \in \mathcal{I}, \quad \text{and} \quad \frac{R_h - g_o}{\psi} \text{ is increasing;} \tag{66}$$

B42. there exists a point $b^* \in]\tilde{b}, \beta[$ such that

$$\int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \begin{cases} \geq 0, & \text{if } x \in]\alpha, b^*[, \\ = 0, & \text{if } x \in [b^*, \beta[, \end{cases} \tag{67}$$

$$(R_h - g_o)(x) \begin{cases} < 0, & \text{for } x \in]\alpha, b^*[, \\ = 0, & \text{for } x \in [b^*, \beta[, \end{cases} \quad \text{and} \quad \frac{R_h - g_o}{\psi} \text{ is increasing;} \tag{68}$$

B43. there exists a point $b^* \in]\tilde{b}, \beta[$ such that

$$\int_{]\alpha,x]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \begin{cases} \geq 0, & \text{if } x \in]\alpha, b^*[, \\ \leq 0, & \text{if } x \in [b^*, \beta[, \end{cases} \tag{69}$$

$$\frac{R_h - g_o}{\psi} \text{ is } \begin{cases} \text{increasing in }]\alpha, b^*[, \\ \text{positive and decreasing in }]b^*, \beta[, \end{cases} \tag{70}$$

$$(R_h - g_o)(b^*) > 0 \quad \text{and} \quad (R_h - g_o)(x) < (R_h - g_o)(b^*), \text{ for all } x \in]\alpha, b^*[. \tag{71}$$

To proceed further, we consider the cases **A42**, **A43**, **B41**, **B42** and **B43**, and the inequality

$$\frac{(R_h + g_c)(a^*)}{\phi(a^*)} < \lim_{x \uparrow \beta} \frac{(R_h - g_o)(x)}{\phi(x)}, \tag{72}$$

as well as the cases **A41**, **A42**, **A43**, **B42** and **B43**, and the inequality

$$\lim_{x \downarrow \alpha} \frac{(R_h + g_c)(x)}{\psi(x)} < \frac{(R_h - g_o)(b^*)}{\psi(b^*)}. \tag{73}$$

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A comparison of (54), (56) and (59) with (66), (68) and (71) reveals that

$$
(72) \quad \begin{cases} \text{is true in cases } \mathbf{A43\text{--}B42} \text{ and } \mathbf{A43\text{--}B43}, \\ \text{is false in cases } \mathbf{A42\text{--}B41} \text{ and } \mathbf{A42\text{--}B42}, \\ \text{may be true or false in cases } \mathbf{A43\text{--}B41} \text{ or } \mathbf{A42\text{--}B43}, \end{cases} \tag{74}
$$

and

$$
(73) \quad \begin{cases} \text{is true in cases } \mathbf{A42\text{--}B43} \text{ and } \mathbf{A43\text{--}B43}, \\ \text{is false in cases } \mathbf{A41\text{--}B42} \text{ and } \mathbf{A42\text{--}B42}, \\ \text{may be true or false in cases } \mathbf{A41\text{--}B43} \text{ or } \mathbf{A43\text{--}B42}. \end{cases} \tag{75}
$$

We consider an example that illustrates some of these possibilities in Section 7.2. Also, we note that (33) in Assumption 4 is equivalent to

$$
-(R_h + g_c)(x) < -(R_h - g_o)(x), \quad \text{for all } x \in \mathcal{I},
$$

which implies that there exists no $x \in \mathcal{I}$ such that $-(R_h + g_c)(x)$ and $(R_h - g_o)(x)$ both are non-negative. This observation implies that

$$
\begin{array}{l} \text{none of the pairs } \mathbf{A1\text{--}B1}, \, \mathbf{A1\text{--}B3}, \, \mathbf{A1\text{--}B42}, \, \mathbf{A1\text{--}B43}, \, \mathbf{A3\text{--}B1}, \, \mathbf{A42\text{--}B1}, \\ \mathbf{A43\text{--}B1}, \, \mathbf{A3\text{--}B3}, \, \mathbf{A3\text{--}B42}, \, \mathbf{A3\text{--}B43}, \, \mathbf{A42\text{--}B3} \text{ or } \mathbf{A43\text{--}B3} \text{ can occur.} \end{array} \tag{76}
$$

We can summarise this discussion by observing that our assumptions can all be satisfied only if the problem data is such that a pair in Table 1 occurs. We have organised the various pairs appearing in this table in six groups that correspond to the six possible forms that an optimal switching strategy can take. We have also used various brackets to identify pairs that appear in more than one groups, as well as the notation

$$
a_{\mathrm{A3}} = \inf \left\{ x \in \mathcal{I} \mid \mathcal{L}(R_h + g_c)(]x, \beta[) = 0 \right\} \tag{77}
$$

and

$$
b_{\mathrm{B3}} = \sup \left\{ x \in \mathcal{I} \mid \mathcal{L}(R_h - g_o)(]\alpha, x[) = 0 \right\}. \tag{78}
$$

	Table 1				
Group NA	[A1–B2] [A1–B41] A2–B41)A2–B42( )A41–B42((A2–B1) (A41–B1) A41–B2]A42–B2[ ]A42–B41[A2–B2 A41–B41 |A42–B42|				
Group O	(A2–B1) {A2–B3} (A41–B1) {A41–B3}				
Group C	[A1–B2] ⟨A3–B2⟩ [A1–B41] ⟨A3–B41⟩				
Group WO	(A2–B1))A2–B42(A2–B43)A41–B42((A41–B1) |A42–B42| {A2–B3} or {A41–B3} if $b_{B3} > \alpha$ A43–B42 or A41–B43 with (73) being false				
Group WC	[A1–B2]]A42–B2[A43–B2]A42–B41[[A1–B41] |A42–B42| ⟨A3–B2⟩ or ⟨A3–B41⟩ if $a_{A3} < \beta$ A42–B43 or A43–B41 with (72) being false				
Group S	A43–B43 A43–B42 or A41–B43 with (73) being true A42–B43 or A43–B41 with (72) being true				

It is straightforward to check that Assumption 6 excludes all of the cases **A1**, **B1**, **A42** and **B42**, as well as the cases appearing in the middle lines of Groups WO and WC of Table 1. In this context, we can see that our assumptions result in a classification of the problem data in the six mutually exclusive groups of Table 2.

	Table 2			
Group NA	A2–B2	A2–B41	A41–B2	A41–B41
Group O	A2–B3	A41–B3		
Group C	A3–B2	A3–B41		
Group WO	A2–B43 A41–B43 if (73) is false			
Group WC	A43–B2 A43–B41 if (72) is false			
Group S	A43–B43 A41–B43 if (73) is true A43–B41 if (72) is true			

We conclude this section with the following list of properties that we will need.

Lemma 2 *In cases* **A42** *and* **A43**,

$$\int_{]a^*,\beta[} \Phi(s)\,\mathcal{L}\big(R_h + g_c\big)(ds) \leq 0 \leq \int_{[a^*,\beta[} \Phi(s)\,\mathcal{L}\big(R_h + g_c\big)(ds), \tag{79}$$

$$-\int_{]\alpha,a^*]} \Psi(s)\,\mathcal{L}\big(R_h + g_c\big)(ds) \leq \frac{(R_h + g_c)(a^*)}{\phi(a^*)} \tag{80}$$

with equalities if $\mathcal{L}(R_h + g_c)(\{a^*\}) = 0$, while, in cases **B42** *and* **B43**,

$$\int_{]\alpha,b^*]} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \leq 0 \leq \int_{]\alpha,b^*[} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds), \qquad (81)$$

$$\int_{[b^*,\beta[} \Phi(s)\,\mathcal{L}(R_h - g_o)(ds) \leq -\frac{(R_h - g_o)(b^*)}{\psi(b^*)}, \qquad (82)$$

with equalities if $\mathcal{L}(R_h - g_o)(\{b^*\}) = 0$. Also, in cases **A41** *and* **A42**,

$$\lim_{x \downarrow \alpha} \frac{(R_h + g_c)(x)}{\psi(x)} = -\int_{]\alpha,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \in [0,\infty[, \qquad (83)$$

while, in cases **B41** *and* **B42**,

$$\lim_{x \uparrow \beta} \frac{(R_h - g_o)(x)}{\phi(x)} = -\int_{]\alpha,\beta[} \Psi(s)\,\mathcal{L}(R_h - g_o)(ds) \in]-\infty,0]. \qquad (84)$$

Proof. First, we note that (79) and (81) are simple consequences of (55), (57) and (67), (69), respectively. Next, we observe that (55) and (57) imply that

$$\int_{]a^*,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \leq 0,$$

with equality if and only if $\mathcal{L}(R_h + g_c)(\{a^*\}) = 0$. In view of this inequality, (19), (21) and (34), we can see that

$$\begin{aligned}
\frac{(R_h + g_c)(a^*)}{\phi(a^*)} &= \frac{R_{-\mathcal{L}(R_h + g_c)}(a^*)}{\phi(a^*)} \\
&= -\int_{]\alpha,a^*]} \Psi(s)\,\mathcal{L}(R_h + g_c)(ds) - \frac{\psi(a^*)}{\phi(a^*)}\int_{]a^*,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \\
&\geq -\int_{]\alpha,a^*]} \Psi(s)\,\mathcal{L}(R_h + g_c)(ds),
\end{aligned}$$

which establishes (80). The proof of (82) follows symmetric arguments.

To see (83), we note that (36) in Assumption 5 and (53) or (55) imply that

$$\lim_{x \downarrow \alpha} \int_{[x,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) \in]-\infty,0], \qquad (85)$$

and

$$0 \leq \int_{]\alpha,x[} \Phi(s)\,\mathcal{L}(R_h + g_c)(ds) < \infty, \quad \text{for all } x \in]\alpha,\tilde{a}[.$$

Combining these inequalities, with the identity $\Phi = \phi\Psi/\psi$ that follows from (13), and the fact that the function ϕ/ψ is decreasing, we can see that

$$\infty > \int_{]\alpha,x[} \Psi(s)\frac{\phi(s)}{\psi(s)}\,\mathcal{L}(R_h + g_c)(ds) \geq \frac{\phi(x)}{\psi(x)}\int_{]\alpha,x[} \Psi(s)\,\mathcal{L}(R_h + g_c)(ds) \geq 0,$$

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for all $x \in]\alpha, \tilde{a}[$. It follows that

$$\lim_{x \downarrow \alpha} \frac{\phi(x)}{\psi(x)} \int_{]\alpha, x[} \Psi(s) \, \mathcal{L}(R_h + g_c)(ds) = 0.$$

This limit, the expression

$$\frac{(R_h + g_c)(x)}{\psi(x)} = -\frac{\phi(x)}{\psi(x)} \int_{]\alpha, x[} \Psi(s) \, \mathcal{L}(R_h + g_c)(ds) - \int_{[x, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds),$$

and (85) establish (83). The proof of (84) follows similar reasoning. □

5   A verification theorem

In view of the existing theory on similar stochastic control problems, we expect that, when the problem data are smooth functions, the value function v identifies with a classical solution w of the HJB equation that takes the form of the coupled quasi-variational inequalities

$$\max\left\{ \frac{1}{2} \sigma^2(x) w_{xx}(z, x) + b(x) w_x(z, x) - r(x) w(z, x) + z \dot{h}(x), \right.$$
$$\left. w(1 - z, x) - w(z, x) - z g_c(x) - (1 - z) g_o(x) \right\} = 0, \tag{86}$$

which are parametrised by $z = 0, 1$. In the case that we consider in this paper, we do not assume that the problem data are smooth, so, we have to consider generalised solutions of (86).

Definition 3 A function $w : \{0, 1\} \times \mathcal{I} \mapsto \mathbb{R}$ is a solution of the HJB equation (86) if $w(z, \cdot)$ is the difference of two convex functions,

$$-\mathcal{L}w(z, \cdot) - zh \text{ is a positive measure on } (\mathcal{I}, \mathcal{B}(\mathcal{I})), \tag{87}$$
$$w(1 - z, x) - w(z, x) - z g_c(x) - (1 - z) g_o(x) \le 0, \quad \text{for all } x \in \mathcal{I}, \tag{88}$$

for $z = 0$ as well as for $z = 1$, and

$$\mathcal{L}w(0, \cdot)(\mathcal{C}_c) = \mathcal{L}w(1, \cdot)(\mathcal{C}_o) + h(\mathcal{C}_o) = 0, \tag{89}$$

where the operator $\mathcal{L}$ is defined by (12), and $\mathcal{C}_c$ and $\mathcal{C}_o$ are the open sets defined by

$$\mathcal{C}_c = \left\{ x \in \mathcal{I} \mid w(0, x) > w(1, x) - g_o(x) \right\}, \tag{90}$$
$$\mathcal{C}_o = \left\{ x \in \mathcal{I} \mid w(1, x) > w(0, x) - g_c(x) \right\}. \tag{91}$$

□

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In the context of this definition, we can make the following observations that are motivated by the existing literature in the area and link the four components composing (86) to optimal decision tactics. The sets $\mathcal{C}_c$ and $\mathcal{C}_o$ are the so-called "continuation" regions associated with the system in its closed and its open operating modes, respectively. For instance, the decision maker should take no action if the system is in its closed mode and the state process X assumes values inside $\mathcal{C}_c$. In view of (89) and Section 2.3, a solution of the HJB equation (86) should be given by

$$w(0, x) = A_c \phi(x) + B_c \psi(x)$$

and

$$w(1, x) = A_o \phi(x) + B_o \psi(x) + R_h(x),$$

for some constants $A_c, B_c, A_o, B_o \in \mathbb{R}$, which may depend on the relative location of x in $\mathcal{I}$. On the other hand, the sets $\mathcal{I} \setminus \mathcal{C}_c$ and $\mathcal{I} \setminus \mathcal{C}_o$ characterise the part of the state space in which the decision maker should take action. In particular, if, at any given time t, the system is in its closed (resp., open) mode and $X_t \in \mathcal{I} \setminus \mathcal{C}_c$ (resp., $X_t \in \mathcal{I} \setminus \mathcal{C}_o$), then the system's controller should switch the system from its closed (resp., open) mode to its open (resp., closed) one.

The following result is concerned with sufficient conditions for a solution of (86) to identify with the value function of our control problem.

Theorem 3 *Consider the stochastic control problem formulated in Section 2, and suppose that Assumptions 1–4 hold. If a function $w : \{0, 1\} \times \mathcal{I} \mapsto \mathbb{R}$ satisfies the HJB equation (86) in the sense of Definition 3,*

$$\text{the measures } \mathcal{L}w(0, \cdot) \text{ and } \mathcal{L}w(1, \cdot) \text{ are } (\phi, \psi)\text{-integrable,} \tag{92}$$

and

$$\left| w(z, \cdot) \right| \leq C \left(1 + |R_h| + |g_o| + |g_c| \right), \tag{93}$$

for some constant $C > 0$, for $z = 0, 1$, then the following statements hold true:

(a) $v(z, x) \leq w(z, x)$, for every $(z, x) \in \{0, 1\} \times \mathcal{I}$, and

(b) given an initial condition $(z, x) \in \{0, 1\} \times \mathcal{I}$, if there exists an admissible switching strategy $\mathbb{Z}_{z,x}^* \equiv (\mathbb{S}_x^*, Z^*, T_n^*) \in \mathcal{A}_{z,x}$ such that the random sets

$$\{t \geq 0 \mid Z_t^* = 0 \text{ and } X_t^* \in \mathcal{I} \setminus \mathcal{C}_c\} \quad and \quad \{t \geq 0 \mid Z_t^* = 1 \text{ and } X_t^* \in \mathcal{I} \setminus \mathcal{C}_o\} \tag{94}$$

both are countable,

$$\{t \geq 0 \mid \Delta Z_t^* = 1\} \subseteq \{t \geq 0 \mid X_t^* \in \mathcal{I} \setminus \mathcal{C}_c\} \tag{95}$$

and

$$\{t \geq 0 \mid \Delta Z_t^* = -1\} \subseteq \{t \geq 0 \mid X_t^* \in \mathcal{I} \setminus \mathcal{C}_o\}, \tag{96}$$

$\mathbb{P}_x^*$-a.s., then $w(z, x) = J(\mathbb{Z}_{z,x}^*) = v(z, x)$ and $\mathbb{Z}_{z,x}^*$ is an optimal strategy.

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Proof. We first note that, in view of (8), (20) and Assumption 4, we can see that (93) implies that

$$\lim_{x \downarrow \alpha} \frac{w(z,x)}{\phi(x)} = 0 = \lim_{x \uparrow \beta} \frac{w(z,x)}{\psi(x)}, \tag{97}$$

Now, we fix any initial condition $x \in \mathcal{I}$ and any weak solution $\mathbb{S}_x$ of the SDE (1), and we consider any strictly decreasing sequence (α_m) and any strictly increasing sequence (β_n) such that

$$\alpha_1 < x < \beta_1, \quad \lim_{m \to \infty} \alpha_m = \alpha \quad \text{and} \quad \lim_{n \to \infty} \beta_n = \beta.$$

Given $z = 0, 1$, the locally bounded function $w(z, \cdot)$ plainly satisfies the ODE (11) with $\mu = -\mathcal{L}w(z, \cdot)$. In view of (92) and Dynkin's formula (23), it follows that, given any $(\mathcal{F}_t)$-stopping time ρ,

$$\mathbb{E}_x \left[e^{-\Lambda_{\tau_{\alpha_m} \wedge \tau_{\beta_n} \wedge \rho}} w(z, X_{\tau_{\alpha_m} \wedge \tau_{\beta_n} \wedge \rho}) \right] \equiv \mathbb{E}_x \left[e^{-\Lambda_\rho} w(z, X_\rho) \mathbf{1}_{\{\rho < \tau_{\alpha_m} \wedge \tau_{\beta_n}\}} \right]$$

$$+ \mathbb{E}_x \left[e^{-\Lambda_{\tau_{\alpha_m} \wedge \tau_{\beta_n}}} w(z, X_{\tau_{\alpha_m} \wedge \tau_{\beta_n}}) \mathbf{1}_{\{\tau_{\alpha_m} \wedge \tau_{\beta_n} \leq \rho\}} \right]$$

$$= w(z,x) - \mathbb{E}_x \left[\int_0^{\tau_{\alpha_m} \wedge \tau_{\beta_n} \wedge \rho} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(z,\cdot)} \right]. \tag{98}$$

Using (10) and (97), we can see that

$$\lim_{m \to \infty} |w(z, \alpha_m)| \mathbb{E}_x \left[e^{-\Lambda_{\tau_{\alpha_m}}} \mathbf{1}_{\{\tau_{\alpha_m} < \tau_{\beta_n} \wedge \rho\}} \right] \leq \lim_{m \to \infty} |w(z, \alpha_m)| \mathbb{E}_x \left[e^{-\Lambda_{\tau_{\alpha_m}}} \right]$$

$$= \lim_{m \to \infty} \frac{|w(z, \alpha_m)| \phi(x)}{\phi(\alpha_m)}$$

$$= 0,$$

and that

$$\lim_{n \to \infty} |w(z, \beta_n)| \mathbb{E}_x \left[e^{-\Lambda_{\tau_{\beta_n}}} \mathbf{1}_{\{\tau_{\beta_n} \leq \tau_{\alpha_m} \wedge \rho\}} \right] \leq \lim_{n \to \infty} \frac{|w(z, \beta_n)| \psi(x)}{\psi(\beta_n)} = 0.$$

In light of these calculations, we can see that

$$\lim_{m,n \to \infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_{\alpha_m} \wedge \tau_{\beta_n}}} w(z, X_{\tau_{\alpha_m} \wedge \tau_{\beta_n}}) \mathbf{1}_{\{\tau_{\alpha_m} \wedge \tau_{\beta_n} \leq \rho\}} \right] = 0. \tag{99}$$

Also, (93), Assumption 4, (22) and the dominated convergence theorem imply that

$$\lim_{m,n \to \infty} \mathbb{E}_x \left[e^{-\Lambda_\rho} w(z, X_\rho) \mathbf{1}_{\{\rho < \tau_{\alpha_m} \wedge \tau_{\beta_n}\}} \right] = \mathbb{E}_x \left[e^{-\Lambda_\rho} w(z, X_\rho) \mathbf{1}_{\{\rho < \infty\}} \right], \tag{100}$$

while, (92), (17) and the dominated convergence theorem imply that

$$\lim_{m,n \to \infty} \mathbb{E}_x \left[\int_0^{\tau_{\alpha_m} \wedge \tau_{\beta_n} \wedge \rho} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(z,\cdot)} \right] = \mathbb{E}_x \left[\int_0^\rho e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(z,\cdot)} \right]. \tag{101}$$

In view of (99)–(101), we can pass to the limit as $m, n \to \infty$ in (98) to obtain

$$\mathbb{E}_x \left[e^{-\Lambda_\rho} w(z, X_\rho) \mathbf{1}_{\{\rho < \infty\}} \right] = w(z,x) - \mathbb{E}_x \left[\int_0^\rho e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(z,\cdot)} \right]. \tag{102}$$

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To proceed further, we assume that the system is in its open operating mode at time 0, i.e., that $z = 1$; the analysis of the case associated with $z = 0$ follows exactly the same steps. In particular, we consider any admissible switching strategy $\mathbb{Z}_{1,x} \in \mathcal{A}_{1,x}$, and we recall that the jumps of the associated switching process Z occur at the times composing the sequence $(T_n, \; n \geq 1)$ defined by (31) in Definition 2. For notational simplicity, we define $T_0 = 0$, and we note that $0 = T_0 \leq T_1 < T_2 < \cdots$. Iterating (102), we calculate

$$
\mathbb{E}_x \left[e^{-\Lambda_{T_{2n}}} w(0, X_{T_{2n}}) \mathbf{1}_{\{T_{2n} < \infty\}} \right]
$$

$$
= w(1, x) + \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j+1}}} \left[w(0, X_{T_{2j+1}}) - w(1, X_{T_{2j+1}}) \right] \mathbf{1}_{\{T_{2j+1} < \infty\}} \right]
$$

$$
+ \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} \left[w(1, X_{T_{2j}}) - w(0, X_{T_{2j}}) \right] \mathbf{1}_{\{T_{2j} < \infty\}} \right]
$$

$$
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(1,\cdot)} \right] - \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{T_{2j+1}}^{T_{2j+2}} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(0,\cdot)} \right]. \tag{103}
$$

Adding the term

$$
\mathbb{E}_x \left[\int_0^{T_{2n}} e^{-\Lambda_t} Z_t \, dA_t^h - \sum_{j=1}^{2n-1} e^{-\Lambda_{T_j}} \left[g_{\mathrm{o}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = 1\}} + g_{\mathrm{c}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = -1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right]
$$

$$
\equiv - \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda_t} \, dA_t^{-h} \right]
$$

$$
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j+1}}} g_{\mathrm{c}}(X_{T_{2j+1}}) \mathbf{1}_{\{T_{2j+1} < \infty\}} \right] - \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} g_{\mathrm{o}}(X_{T_{2j}}) \mathbf{1}_{\{T_{2j} < \infty\}} \right]
$$

on both sides of (103), we obtain

$$
\mathbb{E}_x \left[\int_0^{T_{2n}} e^{-\Lambda_t} Z_t \, dA_t^h - \sum_{j=1}^{2n-1} e^{-\Lambda_{T_j}} \left[g_{\mathrm{o}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = 1\}} + g_{\mathrm{c}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j} = -1\}} \right] \mathbf{1}_{\{T_j < \infty\}} \right]
$$

$$
= w(1, x) - \mathbb{E}_x \left[e^{-\Lambda_{T_{2n}}} w(0, X_{T_{2n}}) \mathbf{1}_{\{T_{2n} < \infty\}} \right]
$$

$$
- \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{T_{2j}}^{T_{2j+1}} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(1,\cdot)-h} \right] - \sum_{j=0}^{n-1} \mathbb{E}_x \left[\int_{T_{2j+1}}^{T_{2j+2}} e^{-\Lambda_t} \, dA_t^{-\mathcal{L}w(0,\cdot)} \right]
$$

$$
+ \sum_{j=0}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j+1}}} \left[w(0, X_{T_{2j+1}}) - w(1, X_{T_{2j+1}}) - g_{\mathrm{c}}(X_{T_{2j+1}}) \right] \mathbf{1}_{\{T_{2j+1} < \infty\}} \right]
$$

$$
+ \sum_{j=1}^{n-1} \mathbb{E}_x \left[e^{-\Lambda_{T_{2j}}} \left[w(1, X_{T_{2j}}) - w(0, X_{T_{2j}}) - g_{\mathrm{o}}(X_{T_{2j}}) \right] \mathbf{1}_{\{T_{2j} < \infty\}} \right].
$$

In view of (15) and the fact that w satisfies the HJB equation (86) in the sense of Definition 3, it follows that

$$\mathbb{E}_x \left[\int_0^{T_{2n}} e^{-\Lambda_t} Z_t \, dA_t^h - \sum_{j=1}^{2n-1} e^{-\Lambda_{T_j}} \left[g_{\mathrm{o}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=1\}} + g_{\mathrm{c}}(X_{T_j}) \mathbf{1}_{\{\Delta Z_{T_j}=-1\}} \right] \mathbf{1}_{\{T_j<\infty\}} \right]$$

$$\leq w(1,x) - \mathbb{E}_x \left[e^{-\Lambda_{T_{2n}}} w(0, X_{T_{2n}}) \mathbf{1}_{\{T_{2n}<\infty\}} \right]. \tag{104}$$

In view of (93), the fact that

$$\lim_{n \to \infty} \mathbb{E}_x \left[e^{-\Lambda_{T_{2n}}} w(0, X_{T_{2n}}) \mathbf{1}_{\{T_{2n}<\infty\}} \right] = 0,$$

which follows from Assumption 4 and the strong transversality condition (24), we can pass to the limit as $n \to \infty$ in (104) to obtain the inequality $J(\mathbb{Z}_{1,x}) \leq w(1,x)$, which implies that $v(1,x) \leq w(1,x)$.

Now, suppose that there exists a switching strategy $\mathbb{Z}_{1,x}^*$ that is characterised by the properties discussed in part (b) of the theorem's statement. Recalling (16), we can see that, in this case, (104) holds with equality, thanks to (88)–(89). In view of (41), we can then pass to the limit as $n \to \infty$ to obtain $J(\mathbb{Z}_{1,x}^*) = w(1,x)$, which implies that $v(1,x) \geq w(1,x)$. Combining this conclusion with the inequality $v(1,x) \leq w(1,x)$, which we have established above, we can see that $v(1,x) = w(1,x)$ and that $\mathbb{Z}_{1,x}^*$ is optimal. $\qquad\square$

6 The solution of the control problem

We now solve our control problem by constructing an explicit solution of the HJB equation (86) that satisfies the requirements of Theorem 3. To this end, we consider the various qualitatively different forms that the optimal switching strategy may take, guided by the discussion following Definition 3 and by (97) that is required by the verification theorem proved in the previous section.

To start with, the optimal strategy could involve no switchings, that is, it might be optimal to always leave the system in its original operating mode. In this case, the choice

$$w(0,\cdot) = 0 \quad \text{and} \quad w(1,\cdot) = R_h, \tag{105}$$

should provide the required solution of (86). A second possibility arises when it is optimal to irreversibly switch the system to its open operating mode at time 0, in which case,

$$w(0,\cdot) = R_h - g_{\mathrm{o}} \quad \text{and} \quad w(1,\cdot) = R_h, \tag{106}$$

should satisfy (86). Similarly, it might be optimal to irreversibly switch the system to its closed operating mode at time 0, which is associated with a solution of (86) of the form

$$w(0,\cdot) = 0 \quad \text{and} \quad w(1,\cdot) = -g_{\mathrm{c}}. \tag{107}$$

The following result, the proof of which we develop in the appendix, is concerned with conditions under which (105)–(107) indeed provide a solution of the HJB equation (86).

Lemma 4 *In the presence of Assumptions 1–5, the following statements are true:*

(I) *The function w given by (105) satisfies the HJB equation (86) in the sense of Definition 3 if and only if the problem data are such that any of the pairs in Group NA of Table 1 occurs.*

(II) *The function w given by (106) satisfies the HJB equation (86) in the sense of Definition 3 if the problem data are such that any of the pairs in Group O of Table 1 occurs.*

(III) *The function w given by (107) satisfies the HJB equation (86) in the sense of Definition 3 if the problem data are such that any of the pairs in Group C of Table 1 occurs.*

Furthermore, if Assumption 6 also holds true, then each of the above statements is true if and only if the problem data is such that one of the pairs in the corresponding groups of Table 2 occurs.

Departing from the consideration of strategies that have the simple structures considered above, the next possibility that arises is when it is optimal to wait before permanently switching the system to its open operating mode. In this case, we look for a point $b_{\mathrm{o}} \in \mathcal{I}$ such that, if the system is in its closed operating mode at time 0, then it is optimal to wait as long as the state process assumes values in the interval $]\alpha, b_{\mathrm{o}}[$, and permanently switch the system to its open operating mode as soon as the state process hits the interval $[b_{\mathrm{o}}, \beta[$. In this case, we look for a solution of the HJB equation (86) of the form given by

$$w(0, x) = \begin{cases} B\psi(x), & \text{if } x \in]\alpha, b_{\mathrm{o}}[, \\ R_h(x) - g_{\mathrm{o}}(x), & \text{if } x \in [b_{\mathrm{o}}, \beta[, \end{cases} \quad \text{and} \quad w(1, x) = R_h, \tag{108}$$

for some constant B. To determine the parameter B and the free-boundary point b_{o}, we conjecture that the inequalities

$$B\psi(b_{\mathrm{o}}) = R_h(b_{\mathrm{o}}) - g_{\mathrm{o}}(b_{\mathrm{o}}) \tag{109}$$

and

$$(R_h - g_{\mathrm{o}})'_+(b_{\mathrm{o}}) \leq B\psi'(b_{\mathrm{o}}) \leq (R_h - g_{\mathrm{o}})'_-(b_{\mathrm{o}}), \tag{110}$$

should hold. Indeed, these inequalities are the generalisation of the so-called "principle of smooth fit" that is appropriate for the analysis of our problem. Solving (109) for B and substituting for it into (110), we can see that the point b_{o} should satisfy the inequalities

$$\psi(b_{\mathrm{o}})(R_h - g_{\mathrm{o}})'_+(b_{\mathrm{o}}) - \psi'(b_{\mathrm{o}})(R_h - g_{\mathrm{o}})(b_{\mathrm{o}}) \leq 0,$$
$$\psi(b_{\mathrm{o}})(R_h - g_{\mathrm{o}})'_-(b_{\mathrm{o}}) - \psi'(b_{\mathrm{o}})(R_h - g_{\mathrm{o}})(b_{\mathrm{o}}) \geq 0,$$

which are equivalent to

$$\int_{]\alpha, b_{\mathrm{o}}]} \Psi(s) \, \mathcal{L}(R_h - g_{\mathrm{o}})(ds) \leq 0 \leq \int_{]\alpha, b_{\mathrm{o}}[} \Psi(s) \, \mathcal{L}(R_h - g_{\mathrm{o}})(ds). \tag{111}$$

Similarly, it might be optimal to wait before irreversibly switching the system to its closed operating mode, which is associated with a solution of the HJB equation (86) of the form given by

$$w(0, x) = 0 \quad \text{and} \quad w(1, x) = \begin{cases} -g_c(x), & \text{if } x \in]\alpha, a_c], \\ A\phi(x) + R_h(x), & \text{if } x \in]a_c, \beta[, \end{cases} \tag{112}$$

for some constant A and free-boundary point a_c. Following the same reasoning as above, we expect that the parameter A and the free-boundary point a_c should satisfy

$$A = -\frac{(R_h + g_c)(a_c)}{\phi(a_c)} \tag{113}$$

and

$$\int_{]a_c, \beta[} \Phi(s)\, \mathcal{L}(R_h + g_c)(ds) \leq 0 \leq \int_{[a_c, \beta[} \Phi(s)\, \mathcal{L}(R_h + g_c)(ds). \tag{114}$$

The following result is concerned with conditions under which these strategies are indeed associated with solutions of the HJB equation (86).

Lemma 5 *In the presence of Assumptions 1–5, the following statements are true:*

(I) *There exists a point b_o satisfying (111) such that the function w defined by (108), where the constant $B \geq 0$ is given by (109), satisfies the HJB equation (86) in the sense of Definition 3 if and only if the problem data is such that any of the cases in Group WO of Table 1 occurs. In particular,*
(a) *if* **B1** *holds true, then b_o is any point in $\mathcal{I}$,*
(b) *if* **B3** *holds true and $b_{B3} > \alpha$, where b_{B3} is given by (78), then b_o is any point in $]\alpha, b_{B3}]$, and*
(c) *if* **B42** *or* **B43** *holds true, then $b_o = b^*$.*

(II) *There exists a point a_c satisfying (114) such that the function w defined by (112), where the constant $A \geq 0$ is given by (113), satisfies the HJB equation (86) in the sense of Definition 3 if and only if the problem data is such that any of the cases in Group WC of Table 1 occurs. In particular,*
(a) *if* **A1** *holds true, then a_c is any point in $\mathcal{I}$,*
(b) *if* **A3** *holds true and $a_{A3} < \beta$, where a_{A3} is given by (77), then a_c is any point in $[a_{A3}, \beta[$, and*
(c) *if* **A42** *or* **A43** *holds true, then $a_c = a^*$.*

Furthermore, if Assumption 6 also holds true, then each of the above statements is true if and only if the problem data is such that one of the pairs in the corresponding groups of Table 2 occurs. In particular, if one of the pairs in Group WO of Table 2 occurs, then $B > 0$ and $b_o = b^*$, where b^* is the point associated with (69)–(71), while if one of the pairs in Group WC of Table 2 occurs, then $A > 0$ and $a_c = a^*$, where a^* is the point associated with (57)–(59).

The final possibility that arises is when it is optimal to sequentially switch the system from its open operating mode to its closed one, and vice versa. In this case, we postulate that the value function of our control problem identifies with a solution w to the HJB equation (86) that has the form given by the expressions

$$w(0, x) = \begin{cases} B\psi(x), & \text{if } x \in]\alpha, b_{\mathrm{o}}[, \\ A\phi(x) + R_h(x) - g_{\mathrm{o}}(x), & \text{if } x \in [b_{\mathrm{o}}, \beta[, \end{cases} \tag{115}$$

$$w(1, x) = \begin{cases} B\psi(x) - g_{\mathrm{c}}(x), & \text{if } x \in]\alpha, a_{\mathrm{c}}], \\ A\phi(x) + R_h(x), & \text{if } x \in]a_{\mathrm{c}}, \beta[, \end{cases} \tag{116}$$

for some constants A, B and free-boundary points a_{c}, b_{o} such that $\alpha < a_{\mathrm{c}} < b_{\mathrm{o}} < \beta$. To determine these variables, we conjecture that the inequalities

$$A\phi(a_{\mathrm{c}}) + R_h(a_{\mathrm{c}}) = B\psi(a_{\mathrm{c}}) - g_{\mathrm{c}}(a_{\mathrm{c}}), \tag{117}$$

$$A\phi'(a_{\mathrm{c}}) + (R_h)'_-(a_{\mathrm{c}}) \le B\psi'(a_{\mathrm{c}}) - (g_{\mathrm{c}})'_-(a_{\mathrm{c}}), \tag{118}$$

$$A\phi'(a_{\mathrm{c}}) + (R_h)'_+(a_{\mathrm{c}}) \ge B\psi'(a_{\mathrm{c}}) - (g_{\mathrm{c}})'_+(a_{\mathrm{c}}) \tag{119}$$

should hold at a_{c}, and the inequalities

$$B\psi(b_{\mathrm{o}}) = A\phi(b_{\mathrm{o}}) + R_h(b_{\mathrm{o}}) - g_{\mathrm{o}}(b_{\mathrm{o}}), \tag{120}$$

$$B\psi'(b_{\mathrm{o}}) \le A\phi'(b_{\mathrm{o}}) + (R_h)'_-(b_{\mathrm{o}}) - (g_{\mathrm{o}})'_-(b_{\mathrm{o}}), \tag{121}$$

$$B\psi'(b_{\mathrm{o}}) \ge A\phi'(b_{\mathrm{o}}) + (R_h)'_+(b_{\mathrm{o}}) - (g_{\mathrm{o}})'_+(b_{\mathrm{o}}) \tag{122}$$

should hold at b_{o}. An inspection of (117)–(122) reveals that, when the functions R_h, g_{c} and g_{o} are C^1, these inequalities all hold as equalities. Indeed, in this case, (117)–(122) reduce to the system of four equations that would be suggested by the so-called "principle of smooth fit", which would require that the functions $w(0, \cdot)$ and $w(1, \cdot)$ should be C^1 at the free boundary points a_{c} and b_{o}, respectively.

To proceed further, we note that (117) and (120) are equivalent to

$$A = \left(\frac{R_h(b_{\mathrm{o}}) - g_{\mathrm{o}}(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})} - \frac{R_h(a_{\mathrm{c}}) + g_{\mathrm{c}}(a_{\mathrm{c}})}{\psi(a_{\mathrm{c}})} \right) \left(\frac{\phi(a_{\mathrm{c}})}{\psi(a_{\mathrm{c}})} - \frac{\phi(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})} \right)^{-1}, \tag{123}$$

$$B = \left(\frac{R_h(b_{\mathrm{o}}) - g_{\mathrm{o}}(b_{\mathrm{o}})}{\phi(b_{\mathrm{o}})} - \frac{R_h(a_{\mathrm{c}}) + g_{\mathrm{c}}(a_{\mathrm{c}})}{\phi(a_{\mathrm{c}})} \right) \left(\frac{\psi(b_{\mathrm{o}})}{\phi(b_{\mathrm{o}})} - \frac{\psi(a_{\mathrm{c}})}{\phi(a_{\mathrm{c}})} \right)^{-1}. \tag{124}$$

Furthermore, in view of the identities (25)–(28), we can see that (117)–(119) imply the system of inequalities

$$\int_{]a_{\mathrm{c}},\beta[} \Phi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds) \le -B \le \int_{[a_{\mathrm{c}},\beta[} \Phi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds), \tag{125}$$

$$-\int_{]\alpha,a_{\mathrm{c}}]} \Psi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds) \le -A \le -\int_{]\alpha,a_{\mathrm{c}}[} \Psi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds), \tag{126}$$

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while (120)–(122) imply the system of inequalities

$$-\int_{]b_\mathrm{o},\beta[} \Phi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds) \le B \le -\int_{[b_\mathrm{o},\beta[} \Phi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds), \tag{127}$$

$$\int_{]\alpha,b_\mathrm{o}]} \Psi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds) \le A \le \int_{]\alpha,b_\mathrm{o}[} \Psi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds). \tag{128}$$

It follows that the free boundary points $a_\mathrm{c} < b_\mathrm{o}$ should satisfy the system of inequalities

$$q_\phi^{\mathcal{C}}(a_\mathrm{c}, b_\mathrm{o}) \le 0 \le q_\phi^{\mathcal{O}}(a_\mathrm{c}, b_\mathrm{o}), \tag{129}$$

$$q_\psi^{\mathcal{C}}(a_\mathrm{c}, b_\mathrm{o}) \le 0 \le q_\psi^{\mathcal{O}}(a_\mathrm{c}, b_\mathrm{o}), \tag{130}$$

where

$$q_\phi^{\mathcal{O}}(u, v) = -\int_{]u,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_{]v,\beta[} \Phi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds), \tag{131}$$

$$q_\phi^{\mathcal{C}}(u, v) = -\int_{[u,\beta[} \Phi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_{[v,\beta[} \Phi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds), \tag{132}$$

$$q_\psi^{\mathcal{O}}(u, v) = -\int_{]\alpha,u[} \Psi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_{]\alpha,v[} \Psi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds) \tag{133}$$

and

$$q_\psi^{\mathcal{C}}(u, v) = -\int_{]\alpha,u]} \Psi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_{]\alpha,v]} \Psi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds). \tag{134}$$

Furthermore, it is straightforward to check that, in the presence of Assumption 7, if a solution $(a_\mathrm{c}, b_\mathrm{o})$ of the system of inequalities (129)–(130) is such that $a_\mathrm{o} < \tilde{a}$ and $\tilde{b} < b_\mathrm{c}$, then this solution satisfies the system of equations

$$q_\phi(a_\mathrm{c}, b_\mathrm{o}) = 0 \quad \text{and} \quad q_\psi(a_\mathrm{c}, b_\mathrm{o}) = 0, \tag{135}$$

where the functions q_ϕ and q_ψ are given by

$$q_\phi(u, v) = -\int_u^\beta \Phi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_v^\beta \Phi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds) \tag{136}$$

and

$$q_\psi(u, v) = -\int_\alpha^u \Psi(s)\,\mathcal{L}(R_h + g_\mathrm{c})(ds) + \int_\alpha^v \Psi(s)\,\mathcal{L}(R_h - g_\mathrm{o})(ds) \tag{137}$$

for $u \in]\alpha, \tilde{a}[$ and $v \in]\tilde{b}, \beta[$.

The following result is concerned with conditions under which there exist points $a_\mathrm{c} < b_\mathrm{o}$ in $\mathcal{I}$ that satisfy (129)–(130) and the corresponding function w defined by (115)–(116) satisfies the HJB equation (86).

Lemma 6 *In the presence of Assumptions 1–5, the following statements are true:*

(I) There exist points $a_\mathrm{c} < b_\mathrm{o}$ in $\mathcal{I}$ satisfying the system of inequalities (129)–(130) if the problem data is such that any of the cases in Group S of Table 1 (or Table 2 if Assumption 6 is also made) occurs. In particular, there exists a solution $(a_\mathrm{c}, b_\mathrm{o})$ of (129)–(130) such that

$$a_\mathrm{c} \in \begin{cases}]\alpha, \tilde{a}[, & \text{if } \mathbf{A41} \text{ is satisfied,} \\ [a^*, \tilde{a}[, & \text{if } \mathbf{A42} \text{ or } \mathbf{A43} \text{ is satisfied,} \end{cases} \tag{138}$$

and

$$b_\mathrm{o} \in \begin{cases}]\tilde{b}, \beta[, & \text{if } \mathbf{B41} \text{ is satisfied,} \\ ]\tilde{b}, b^*], & \text{if } \mathbf{B42} \text{ or } \mathbf{B43} \text{ is satisfied.} \end{cases} \tag{139}$$

(II) If Assumptions 6 and 7 also hold true, then the system of equation (135) has a solution $(a_\mathrm{o}, b_\mathrm{c})$ such that $a_\mathrm{o} \in]\alpha, \tilde{a}[$ and $b_\mathrm{o} \in]\tilde{b}, \beta[$ if and only if the problem data is such that any of the cases in Group S of Table 2 occurs, in which case, this solution is unique.

(III) In either of the two cases above, the function w defined by (115)–(116) with A and B given by (123) and (124), respectively, satisfies the HJB equation (86) in the sense of Definition 3.

We can now establish our main result.

Theorem 7 *Consider the optimal sequential switching problem formulated in Section 2, and suppose that its data satisfy Assumptions 1–5. We have the following six cases corresponding to the six groups in Table 1 (or Table 2 if Assumption 6 is also satisfied):*

(a) if the problem data is such that any of the cases in Group NA of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (105);

(b) if the problem data is such that any of the cases in Group O of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (106);

(c) if the problem data is such that any of the cases in Group C of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (107);

(d) if the problem data is such that any of the cases in Group WO of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (108), where the constant B is given by (109) and b_o satisfies (111);

(e) if the problem data is such that any of the cases in Group WC of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (112), where the constant A is given by (113) and a_c satisfies (114);

(f) if the problem data is such that any of the cases in Group S of Table 1 (or Table 2) occurs, then the value function v identifies with the function w given by (115)–(116), where the points $a_\mathrm{c} < b_\mathrm{o}$ satisfy the system of inequalities (129)–(130) and A, B are given by (123), (124).

Optimal switching strategies associated with each of these cases are constructed in the proof below.

Proof. In view of Lemmas 4, 5 and 6, the function w associated with each case satisfies the HJB equation (86) in the sense of Definition 3. Also, it is straightforward to check that, in all cases, w satisfies (92) and (93) in the verification Theorem 3. In view of these observations, we only need to construct a switching strategy $\mathbb{Z}_{z,x}^* = (\mathbb{S}_x^*, Z^*, T_n^*)$ that possesses the properties required by part (b) of Theorem 3. To this end, we fix any initial condition $(z, x) \in \{0, 1\} \times \mathcal{I}$ and any weak solution $\mathbb{S}_x^* = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, X, W)$ of the SDE (1), and we discuss the construction of the switching process Z^*, the jumps of which occur at the times composing the sequence (T_n^*), in what follows.

In Case (a), the sets $\mathcal{C}_c$ and $\mathcal{C}_o$ defined by (90) and (91) in Definition 3 are given by $\mathcal{C}_c = \mathcal{C}_o = \mathcal{I}$, and the switching process $Z^* \equiv z$, which is associated with $T_n^* = \infty$, for all $n \geq 1$, is the required one because both of the sets in (94) are empty and the inclusions in (95)–(96) are trivially true.

In Case (b), $\mathcal{C}_c = \emptyset$ and $\mathcal{C}_o = \mathcal{I}$, and the switching process Z^* given by

$$Z_t^* = z\mathbf{1}_{\{0\}}(t) + \mathbf{1}_{]0,\infty[}(t)$$

is optimal because the sets in (94) contain at most one element, while (95)–(96) plainly hold.

In Case (d), $\mathcal{I} \setminus \mathcal{C}_c = [b_o, \beta[$, $\mathcal{I} \setminus \mathcal{C}_o = \emptyset$, and the switching process Z^* given by

$$Z_t^* = z\mathbf{1}_{[0,T_1^*]}(t) + \mathbf{1}_{]T_1^*,\infty[}(t),$$

where $T_1^* = \inf\{t \geq 0 \mid X_t \geq b_o\}$, has all of the required properties.

The constructions that are appropriate for Cases (c) and (e) are mirror images of the constructions associated with Cases (b) and (d) above, respectively.

In Case (f), $\mathcal{I} \setminus \mathcal{C}_c = [b_o, \beta[$ and $\mathcal{I} \setminus \mathcal{C}_o =]\alpha, a_c]$. If $z = 1$, then the switching process Z^* given by

$$Z_t^* = \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{\infty} 1_{]T_{2j+1}^*, T_{2j}^*]}(t),$$

where the $(\mathcal{F}_t)$-stopping times T_n^*, $n \geq 1$ are defined recursively by

$$T_{2n+1}^* = \inf\{t \geq T_{2n}^* \mid X_t \leq a_c\}, \quad n = 0, 1, 2, \ldots,$$
$$T_{2n}^* = \inf\{t \geq T_{2n-1}^* \mid X_t \geq b_o\}, \quad n = 1, 2, \ldots,$$

where we have set $T_0^* = 0$, provides an optimal choice because the sets in (94) both are countable, while the inclusions in (95)–(96) both hold. Note that Z^* is indeed a finite variation process because $T_n^* \to \infty$, $\mathbb{P}_x$-a.s.. To see this claim, we use the definition (3) of the discounting factor Λ, the strong Markov property of the process X and (10) to obtain

$$\mathbb{E}_x\left[e^{-\Lambda_{T_{2n+1}^*}}\right] = \mathbb{E}_x\left[e^{-\Lambda_{T_{2n}^*}}\mathbb{E}_x\left[\exp\left(-\int_0^{T_{2n+1}^* - T_{2n}^*} r(X_{s+T_{2n}^*})\,ds\right) \mid \mathcal{F}_{T_{2n}^*}\right]\right]$$

$$= \mathbb{E}_x\left[e^{-\Lambda_{T_{2n}^*}}\mathbb{E}_{b_o}\left[\exp\left(-\int_0^{\tau_{a_c}} r(X_s)\,ds\right)\right]\right]$$

$$= \frac{\phi(b_o)}{\phi(a_c)}\mathbb{E}_x\left[e^{-\Lambda_{T_{2n}^*}}\right].$$

Similarly, we can see that

$$\mathbb{E}_x \left[e^{-\Lambda_{T_{2n}^*}} \right] = \frac{\psi(a_\mathrm{c})}{\psi(b_\mathrm{o})} \mathbb{E}_x \left[e^{-\Lambda_{T_{2n-1}^*}} \right].$$

These calculations and the dominated convergence theorem imply that

$$\mathbb{E}_x \left[\lim_{n\to\infty} e^{-\Lambda_{T_{2n+1}^*}} \right] = \lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{T_1^*}} \right] \left(\frac{\phi(b_\mathrm{o})\psi(a_\mathrm{c})}{\phi(a_\mathrm{c})\psi(b_\mathrm{o})} \right)^n = 0,$$

the second equality following from the facts that ϕ (resp., ψ) is strictly decreasing (resp., increasing) and $a_\mathrm{c} < b_\mathrm{o}$. This conclusion contradicts the possibility that $\mathbb{P}_x \left(\lim_{n\to\infty} T_n^* < \infty \right) > 0$.

Finally, in Case (f), if $z = 0$, then the optimal switching process Z^* can be constructed in a similar fashion. $\qquad\square$

7 Examples

7.1 The model studied by Duckworth and Zervos [DZ01]

Suppose that $\mathcal{I} =]0, \infty[$, that X is the geometric Brownian motion given by

$$dX_t = bX_t \, dt + \sigma X_t \, dW_t,$$

for some constants b, σ, and that the discounting rate r is a constant. In this case,

$$\phi(x) = x^m \quad \text{and} \quad \psi(x) = x^n,$$

where the constants $m < 0 < n$ are given by

$$m, n = \frac{-\left(\mu - \frac{1}{2}\sigma^2\right) \pm \sqrt{\left(\mu - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 r}}{\sigma^2},$$

and both of $\alpha \equiv 0$ and $\beta \equiv \infty$ are natural boundary points. Assume that the measure $h \in \mathcal{I}_{\phi,\psi}$ is absolutely continuous with Radon-Nikodym derivative with respect to the Lebesgue measure denoted by $\dot{h}$, where $\dot{h}$ is an increasing function such that $\lim_{x\to\infty} \dot{h}(x) = \infty$. Also, suppose that $g_\mathrm{o}(x) = K_\mathrm{o}$ and $g_\mathrm{c}(x) = K_\mathrm{c}$, for some constants K_o, K_c with $K_\mathrm{o} + K_\mathrm{c} > 0$. In this context, Assumptions 1–4 and (39)–(40) in Assumption 5 are all satisfied. Also, the measures $\mathcal{L}(R_h + g_\mathrm{c})$ and $\mathcal{L}(R_h - g_\mathrm{o})$ are absolutely continuous,

$$\mathcal{L}(R_h + g_\mathrm{c})(dx) = \left(-\dot{h}(x) - rK_\mathrm{c}\right) dx \quad \text{and} \quad \mathcal{L}(R_h - g_\mathrm{o})(dx) = \left(-\dot{h}(x) + rK_\mathrm{o}\right) dx.$$

In view of the inequality $-\dot{h}(x) - rK_\mathrm{c} < -\dot{h}(x) + rK_\mathrm{o}$ and the fact that $\lim_{x\to\infty} \dot{h}(x) = \infty$, we can check that we can have only one of the following three cases (see also Theorem 5 in Duckworth and Zervos [DZ01]).

DZ1 If $0 \leq \inf_{x>0} \dot{h}(x) - rK_{\mathrm{o}}$, then cases **A2** and **B3** in Assumption 5 occur, and we are in the context of Group O in Tables 1 and 2.

DZ2 If $\inf_{x>0} \dot{h}(x) - rK_{\mathrm{o}} < 0 \leq \inf_{x>0} \dot{h}(x) + rK_{\mathrm{c}}$, then cases **A2** and **B4** in Assumption 5 occur, with

$$\tilde{b} = \sup \left\{ x > 0 \mid \dot{h}(x) - rK_{\mathrm{o}} \leq 0 \right\} > 0. \tag{140}$$

In particular, case **B43** occurs, which puts us in the context of Group WO in Tables 1 and 2. The claim that case **B43** rather than either of the cases **B41** or **B43** occurs follows from (30) and a simple inspection of (66) and (68).

DZ3 If $\inf_{x>0} \dot{h}(x) + rK_{\mathrm{c}} < 0$, then the requirements of cases **A4** and **B4** in Assumption 5 are satisfied, with

$$\tilde{a} = \inf \left\{ x > 0 \mid \dot{h}(x) + rK_{\mathrm{c}} \geq 0 \right\} > 0,$$

and with $\tilde{b} > \tilde{a}$ being given by (140). In view of (30), we can see that cases **A43** and **B43** occur, which puts us in the context of Group S of Tables 1 and 2. Also, it is worth noting that both of Assumptions 6 and 7 are satisfied, so the uniqueness claims made by part (II) of Lemma 6 hold true.

7.2 Square-root mean reverting process

Suppose that $\mathcal{I} =]0, \infty[$, and that X is the square-root mean-reverting process given by

$$dX_t = (2 - X_t) \, dt + \sqrt{2X_t} \, dW_t.$$

This SDE is a special case of the one modelling the short rate in the Cox-Ingersoll-Ross interest rate model, and X satisfies Assumptions 1 and 2. Also, let

$$r(x) = 2, \quad \text{for all } x > 0,$$

which is a choice compatible with Assumption 3. The associated ODE (2.3) takes the form of

$$xf''(x) + (2 - x)f'(x) - 2f(x) = 0,$$

which is a special case of Kumer's equation. The functions ϕ and ψ that span the solution space of this equation and satisfy (6)-(8) are given by

$$\phi(x) = U(2, 2; x) \quad \text{and} \quad \psi(x) = {}_1F_1(2, 2; x) \equiv e^x,$$

where U and ${}_1F_1$ are confluent hypergeometric functions (see Abramowitz and Stegun [AS72, Chapter 13]). In view of the computation

$$\mathbb{E}_x \left[\int_0^\infty e^{-2t} X_t \, dt \right] = \int_0^\infty e^{-2t} \mathbb{E}_x [X_t] \, dt = \int_0^\infty e^{-2t} \left[2 + (x - 2)e^{-t} \right] dt = \frac{x + 1}{3},$$

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and (18), we can see that, if we choose the measure h to be absolutely continuous with Radon-Nikodym derivative with respect to the Lebesgue measure $(dh/dx)(x) = x$, then

$$R_h(x) = \frac{x+1}{3} \quad \text{and} \quad \mathcal{L}R_h(dx) = -x\,dx,$$

while, if we choose

$$g_c(x) = \frac{2x+\delta}{3} \quad \text{and} \quad g_o = \frac{-(e-1)x+1}{3},$$

where $e = 2.73\ldots$ is the base of the natural logarithms, then

$$\mathcal{L}g_c(dx) = \frac{-6x+2(2-\delta)}{3}\,dx \quad \text{and} \quad \mathcal{L}g_o(dx) = \frac{3(e-1)x-2e}{3}\,dx.$$

We can also verify that these choices satisfy all of of the requirements of Assumption 4, providing that $\delta > -1$. Furthermore, noting that

$$\mathcal{L}(R_h + g_c)(dx) = \frac{-9x+2(2-\delta)}{3}\,dx \quad \text{and} \quad \mathcal{L}(R_h - g_o)(dx) = \frac{-3ex+2e}{3}\,dx,$$

we can see that cases **A4** and **B4** of Assumption 5 occur with

$$0 < \tilde{a} = \frac{2(2-\delta)}{9} \leq \frac{2}{3} = \tilde{b},$$

as long as $\delta \in [-1, 2[$. In view of the consideration above, we can see that the choices of the problem data that we have made satisfy Assumptions 1–7, provided that

$$\delta \in \,]-1, 2[.$$

Also, we can calculate

$$\lim_{x\downarrow 0} \frac{(R_h + g_c)(x)}{\psi(x)} = \frac{1+\delta}{3} \quad \text{and} \quad \frac{(R_h - g_o)(b^*)}{\psi(b^*)} = \frac{1}{3},$$

where $b^* = 1$ is the point appearing in (69)–(71) of case **B43** discussed in Section 4. These calculations reveal that, in the special case that we consider here, (73) is true (resp., false) if $\delta \in \,]-1, 0[$ (resp., $\delta \in [0, 2[$).

Acknowledgements

Appendix: proofs of results in Section 6

Proof of Lemma 4. By construction, the function w given by (105) satisfies the HJB equation (86) if and only if

$$w(1, \cdot) - w(0, \cdot) - g_{\mathrm{o}} \leq 0 \quad \text{and} \quad w(0, \cdot) - w(1, \cdot) - g_{\mathrm{c}} \leq 0,$$

which is equivalent to

$$R_h - g_{\mathrm{o}} \leq 0 \quad \text{and} \quad -R_h - g_{\mathrm{c}} \leq 0.$$

However, a simple inspection of the lists in Section 4 reveals that these inequalities hold true if and only if the problem data is as in the corresponding statement of the lemma.

To establish part (II) of the lemma, we note that the function w defined by (106) satisfies (86) if and only if

$$-\mathcal{L}(R_h - g_{\mathrm{o}}) \text{ is a positive measure} \tag{141}$$

and

$$w(0, \cdot) - w(1, \cdot) - g_{\mathrm{c}} \leq 0. \tag{142}$$

Inequality (142) is plainly equivalent to $-g_{\mathrm{o}} - g_{\mathrm{c}} \leq 0$, which is true by assumption. On the other hand, inequality (141) holds if and only in the problem data is such that **B1** or **B3** is satisfied, which gives rise to the cases in Group O of Table 1, or the cases in Group O of Table 2 if Assumption 6 is also made.

Finally, we can use symmetric arguments to prove all claims associated with part (III) of the lemma. $\qquad\qquad\square$

Proof of Lemma 5. To prove part (I) of the lemma, we first observe that, if there exists a point b_{o} satisfying (111), then the associated function w defined by (108) will satisfy the HJB equation (86) if and only if

$$\operatorname{supp}\left[\mathcal{L}w(0, \cdot)\right]^+ \cap [b_{\mathrm{o}}, \beta[= \emptyset, \tag{143}$$

$$w(0, x) - w(1, x) - g_{\mathrm{c}}(x) \leq 0, \quad \text{for all } x \in \mathcal{I}, \tag{144}$$

$$w(1, x) - w(0, x) - g_{\mathrm{o}}(x) \leq 0, \quad \text{for all } x \in \,]\alpha, b_{\mathrm{o}}[. \tag{145}$$

In view of the inequality

$$\mathcal{L}w(0, \cdot)(\{b_{\mathrm{o}}\}) = \frac{1}{2}\sigma^2(b_{\mathrm{o}})\left[(R_h - g_{\mathrm{o}})'_+(b_{\mathrm{o}}) - B\psi'(b_{\mathrm{o}})\right]$$
$$\leq 0,$$

which follows from (109) and (110), we can see that (143) holds true if and only if

$$\text{the restriction of } \mathcal{L}(R_h - g_{\mathrm{o}}) \text{ in } \left(]b_{\mathrm{o}}, \beta[, \mathcal{B}(]b_{\mathrm{o}}, \beta[)\right) \text{ is a positive measure.} \tag{146}$$

An inspection of the conditions in Assumption 5 and the various associated cases appearing in Section 4 reveals that there exists a point $b_{\mathrm{o}} \in \mathcal{I}$ that satisfies (111) and is such that (146) holds true if and only if we are in the context of one of the cases (I.a), (I.b) or (I.c) in the statement of the lemma.

Now, for $x \geq b_{\mathrm{o}}$, (144) is equivalent to $-(g_{\mathrm{c}}+g_{\mathrm{o}})(x) \leq 0$, which is true by Assumption (33). Furthermore, we can use the definition (108) of w and the expression for B provided by (109) to verify that (144), for $x < b_{\mathrm{o}}$, and (145) are equivalent to

$$\frac{(R_h + g_{\mathrm{c}})(x)}{\psi(x)} \geq \frac{(R_h - g_{\mathrm{o}})(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})} \geq \frac{(R_h - g_{\mathrm{o}})(x)}{\psi(x)}, \quad \text{for all } x \in]\alpha, b_{\mathrm{o}}[. \tag{147}$$

The second of these inequalities holds with equality in cases (I.a) and (I.b) in the statement of this lemma, thanks to (47). In case (I.c), it follows immediately from the fact that the function $(R_h - g_{\mathrm{o}})/\psi$ is increasing in $]\alpha, b^*[$ (see (68) and (70)).

To complete the proof of part (I) of the lemma, we need to establish conditions under which the first inequality in (147) holds true in the context of one of the cases (I.a), (I.b) or (I.c) in the statement of this lemma. To this end, we use (27) to calculate

$$\left(\frac{R_h + g_{\mathrm{c}}}{\psi}\right)'_+ (x) = \frac{\mathcal{W}(x)}{\psi^2(x)} \int_{]\alpha, x]} \Psi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds).$$

When the problem data is such that case **A2** in Assumption 5 holds, this calculation implies that the function $(R_h + g_{\mathrm{c}})/\psi$ is decreasing in $\mathcal{I}$. Combining this observation with the inequality

$$\frac{(R_h + g_{\mathrm{c}})(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})} > \frac{(R_h - g_{\mathrm{o}})(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})}, \tag{148}$$

which follows from (33) in Assumption 4, we can see that the first of the two inequalities in (147) is satisfied. On the other hand, when the problem data is such that case **A4** in Assumption 5 holds, this calculation implies that there exists a point $\gamma \in]\tilde{a}, \beta[$ such that the function $(R_h + g_{\mathrm{c}})/\psi$ is increasing in $]\alpha, \gamma[$ and decreasing in $]\gamma, \beta[$. However, this observation and inequality (148) imply that the first of the inequalities in (147) is satisfied if and only if

$$\lim_{x \downarrow \alpha} \frac{(R_h + g_{\mathrm{c}})(x)}{\psi(x)} \geq \frac{(R_h - g_{\mathrm{o}})(b_{\mathrm{o}})}{\psi(b_{\mathrm{o}})}.$$

In cases **B42** and **B43**, in which $b_{\mathrm{o}} = b^*$, this inequality holds true if and only if the inequality in (73) is not true. Combining all these consideration with the fact that we are in the context of one of the cases (I.a), (I.b) or (I.c) in the statement of the lemma, (75), and the restrictions on the possible pairings given by (76), we can see that the first inequality in (147) holds true if and only if the problem data is such that one of the cases in Group WO of Table 1 occurs.

Finally, the proof of part (II) of the lemma follows arguments that are symmetric to the ones we have developed above to establish part (I). $\qquad \square$

Proof of Lemma 6. We start by assuming that the problem data is such that **A4** and **B4** in Assumption 5 are satisfied. Given any $v \in \mathcal{I}$, we can see that

$$q_\phi^{\mathcal{O}}(u_2, v) - q_\phi^{\mathcal{O}}(u_1, v) = \int_{]u_1, u_2]} \Phi(s)\, \mathcal{L}(R_h + g_{\mathrm{c}})(ds) \begin{cases} \geq 0, & \text{if } u_1 < u_2 < \tilde{a}, \\ \leq 0, & \text{if } \tilde{a} < u_1 < u_2, \end{cases}$$

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the inequalities following because of (36) in Assumption 5. In view of this calculation and a similar one with $q_\phi^\mathcal{C}$, we can see that, given any $v \in \mathcal{I}$,

$$\text{the functions } u \mapsto q_\phi^\mathcal{O}(u,v) \text{ and } u \mapsto q_\phi^\mathcal{C}(u,v) \text{ are } \begin{cases} \text{increasing in }]\alpha, \tilde{a}[, \\ \text{decreasing in }]\tilde{a}, \beta[, \end{cases} \tag{149}$$

In the same way, we can see that, given any $u \in \mathcal{I}$,

$$\text{the functions } v \mapsto q_\phi^\mathcal{O}(u,v) \text{ and } v \mapsto q_\phi^\mathcal{C}(u,v) \text{ are } \begin{cases} \text{decreasing in }]\alpha, \tilde{b}[, \\ \text{increasing in }]\tilde{b}, \beta[. \end{cases} \tag{150}$$

given any $v \in \mathcal{I}$,

$$\text{the functions } u \mapsto q_\psi^\mathcal{O}(u,v) \text{ and } u \mapsto q_\psi^\mathcal{C}(u,v) \text{ are } \begin{cases} \text{decreasing in }]\alpha, \tilde{a}[, \\ \text{increasing in }]\tilde{a}, \beta[, \end{cases} \tag{151}$$

and, given any $u \in \mathcal{I}$,

$$\text{the functions } v \mapsto q_\psi^\mathcal{O}(u,v) \text{ and } v \mapsto q_\psi^\mathcal{C}(u,v) \text{ are } \begin{cases} \text{increasing in }]\alpha, \tilde{b}[, \\ \text{decreasing in }]\tilde{b}, \beta[. \end{cases} \tag{152}$$

We can also see that (36) and (37) in Assumption 5 imply that

$$q_\psi^\mathcal{C}(u,v) - q_\psi^\mathcal{O}(u,v) = -\Psi(u)\mathcal{L}(R_h + g_\mathrm{c})(\{u\}) + \Psi(v)\mathcal{L}(R_h - g_\mathrm{o})(\{v\})$$
$$\leq 0, \quad \text{for all } u < \tilde{a} \leq \tilde{b} < v, \tag{153}$$

and that

$$q_\phi^\mathcal{C}(u,v) - q_\phi^\mathcal{O}(u,v) \leq 0, \quad \text{for all } u < \tilde{a} \leq \tilde{b} < v. \tag{154}$$

Noting that each of the cases in Group S of Table 1 or Table 2 combines either **B43** with one of **A41**, **A42** or **A43**, or **A43** with one of **B41**, **B42** or **B43**, we prove all claims regarding the solvability of the system of inequalities (129)–(130) when case **B43** prevails; the proofs of the corresponding claims when **A43** prevails follow symmetric arguments. In the context of case **B43**, we start by observing that (81) implies that

$$\lim_{u \downarrow \alpha} q_\psi^\mathcal{C}(u, b^*) \leq 0 \leq \lim_{u \downarrow \alpha} q_\psi^\mathcal{O}(u, b^*), \tag{155}$$

the first of which inequalities and (151) imply that

$$q_\psi^\mathcal{C}(u, b^*) \leq 0, \quad \text{for all } u \in]\alpha, \tilde{a}[. \tag{156}$$

Also, (37) and (39) in Assumption 5 imply that

$$q_\psi^\mathcal{C}(u, \tilde{b}) = -\int_{]\alpha,u]} \Psi(s)\, \mathcal{L}(g_\mathrm{c} + g_\mathrm{o})(ds) + \int_{]u,\tilde{b}]} \Psi(s)\, \mathcal{L}(R_h - g_\mathrm{o})(ds)$$
$$> 0, \quad \text{for all } u \leq \tilde{a}. \tag{157}$$

Now, (156), (157), (152) and the right-continuity of $v \mapsto q_\psi^c(u,v)$ imply that, given any $u \in]\alpha, \tilde{a}[$, there exists a point $\underline{l}(u) \in]\tilde{b}, b^*]$ such that

$$q_\psi^c(u,v) \begin{cases} > 0, & \text{for all } v \in [\tilde{b}, \underline{l}(u)[, \\ \leq 0, & \text{for all } v \in [\underline{l}(u), b^*]. \end{cases} \tag{158}$$

On the other hand, (153), (158) and the left-continuity of $v \mapsto q_\psi^{\mathcal{O}}(u,v)$ imply that, given any $u \in]\alpha, \tilde{a}[$, there exists a point $\overline{l}(u) \in [\underline{l}(u), b^*]$ such that

$$q_\psi^{\mathcal{O}}(u,v) \begin{cases} \geq 0, & \text{for all } v \in [\tilde{b}, \overline{l}(u)], \\ < 0, & \text{for all } v \in]\overline{l}(u), b^*]. \end{cases} \tag{159}$$

We now verify that the function $\overline{l}$ is decreasing and left-continuous. Given $u_1 < u_2 < \tilde{a}$, we can use (159) and (36) in Assumption 5 to observe that

$$q_\psi^{\mathcal{O}}(u_1, \overline{l}(u_2)) = q_\psi^{\mathcal{O}}(u_2, \overline{l}(u_2)) + \int_{[u_1, u_2[} \Psi(s) \, \mathcal{L}(R_h + g_c)(ds)$$
$$\geq 0. \tag{160}$$

In view of (159), this inequality implies that $\overline{l}(u_2) \leq \overline{l}(u_1)$, which proves that $\overline{l}$ is decreasing. To establish the left-continuity of $\overline{l}$, we fix any $\hat{u} \in]\alpha, \tilde{a}[$. If there exists $\varepsilon > 0$ such that $\overline{l}(u) = \overline{l}(\hat{u})$, for all $u \in]\hat{u} - \varepsilon, \hat{u}]$, then $\overline{l}$ is plainly left-continuous at $\hat{u}$. In view of the fact that $\overline{l}$ is decreasing, we can therefore assume that $\overline{l}(u) > \overline{l}(\hat{u})$, for all $u < \hat{u}$, and use (159) to obtain

$$0 \leq \lim_{u \uparrow \hat{u}} q_\psi^{\mathcal{O}}(u, \overline{l}(u))$$
$$= -\int_{]\alpha, \hat{u}[} \Psi(s) \, \mathcal{L}(R_h + g_c)(ds) + \int_{]\alpha, \lim_{u \uparrow \hat{u}} \overline{l}(u)]} \Psi(s) \, \mathcal{L}(R_h - g_o)(ds)$$
$$= \lim_{v \downarrow \lim_{u \uparrow \hat{u}} \overline{l}(u)} q_\psi^{\mathcal{O}}(\hat{u}, v). \tag{161}$$

The positivity of the last limit in these calculations and (159) imply that $\lim_{u \uparrow \hat{u}} \overline{l}(u) \leq \overline{l}(\hat{u})$. On the other hand, the fact that $\overline{l}$ is decreasing implies that $\lim_{u \uparrow \hat{u}} \overline{l}(u) \geq \overline{l}(\hat{u})$. These inequalities imply that $\lim_{u \uparrow \hat{u}} \overline{l}(u) = \overline{l}(\hat{u})$, and the left-continuity of $\overline{l}$ follows.

We will also need the inequality

$$\Phi(\hat{u}) \, \mathcal{L}(R_h + g_c)(\{\hat{u}\}) + \int_{[\lim_{u \downarrow \hat{u}} \overline{l}(u), \overline{l}(\hat{u})[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \geq 0 \tag{162}$$

to be true for all $\hat{u} \in]\alpha, \tilde{a}[$ such that $\overline{l}(u) < \overline{l}(\hat{u})$ for all $u > \hat{u}$. To prove this result, we fix any such $\hat{u} \in]\alpha, \tilde{a}[$, and we use (158) and the fact that the function $\overline{l} \geq \underline{l}$ is decreasing to obtain

$$0 \leq -\lim_{u \downarrow \hat{u}} q_\psi^c(u, \overline{l}(u))$$
$$= \int_{]\alpha, \hat{u}]} \Psi(s) \, \mathcal{L}(R_h + g_c)(ds) - \int_{]\alpha, \lim_{u \downarrow \hat{u}} \overline{l}(u)[} \Psi(s) \, \mathcal{L}(R_h - g_o)(ds).$$

Also, we note that (159) yields

$$0 \leq - \int_{]\alpha,\hat{u}[} \Psi(s) \mathcal{L}(R_h + g_{\mathrm{c}})(ds) + \int_{]\alpha,\bar{l}(\hat{u})[} \Psi(s) \mathcal{L}(R_h - g_{\mathrm{o}})(ds).$$

Adding these inequalities side by side, we obtain

$$\Psi(\hat{u}) \mathcal{L}(R_h + g_{\mathrm{c}})(\{\hat{u}\}) + \int_{[\lim_{u\downarrow\hat{u}} \bar{l}(u), \bar{l}(\hat{u})[} \Psi(s) \mathcal{L}(R_h - g_{\mathrm{o}})(ds) \geq 0.$$

In view of the identity $\Psi = \psi\Phi/\phi$, which follows from (13), and the fact that the function ψ/ϕ is increasing, it follows that

$$
\begin{aligned}
0 &\leq \frac{\psi(\hat{u})}{\phi(\hat{u})} \Phi(\hat{u}) \mathcal{L}(R_h + g_{\mathrm{c}})(\{\hat{u}\}) + \int_{[\lim_{u\downarrow\hat{u}} \bar{l}(u), \bar{l}(\hat{u})[} \frac{\psi(s)}{\phi(s)} \Phi(s) \mathcal{L}(R_h - g_{\mathrm{o}})(ds) \\
&\leq \frac{\psi(\tilde{a})}{\phi(\tilde{a})} \Phi(\hat{u}) \mathcal{L}(R_h + g_{\mathrm{c}})(\{\hat{u}\}) + \frac{\psi(\tilde{a})}{\phi(\tilde{a})} \int_{[\lim_{u\downarrow\hat{u}} \bar{l}(u), \bar{l}(\hat{u})[} \Phi(s) \mathcal{L}(R_h - g_{\mathrm{o}})(ds),
\end{aligned}
$$

which establishes (162).

A simple inspection of (158) and (159) reveals that, given any $u \in]\alpha, \tilde{a}[$, $q_\psi^{\mathcal{C}}(u,v) \leq 0 \leq q_\psi^{\mathcal{O}}(u,v)$, for all $v \in [\underline{l}(u), \overline{l}(u)]$. In particular, if we set $l(u) = \overline{l}(u)$, then we obtain a function l such that

$$l(u) \in]\tilde{b}, b^*] \quad \text{and} \quad q_\psi^{\mathcal{C}}(u, l(u)) \leq 0 \leq q_\psi^{\mathcal{O}}(u, l(u)), \quad \text{for all } u \in]\alpha, \tilde{a}[. \tag{163}$$

Also, we note that

$$l \equiv \overline{l} \text{ is left-continuous and decreasing, and } \lim_{u\downarrow\alpha} l(u) = b^*. \tag{164}$$

The limit here is a simple consequence of the left-continuity of $l \equiv \overline{l}$, (159) and (155).

In general, the function l can have jumps as well as intervals of constancy. However,

$$\text{if Assumptions 6 and 7 also hold, then } l \text{ is continuous and strictly decreasing.} \tag{165}$$

To see this claim, we note that, in the presence of Assumptions 6 and 7, $q_\psi^{\mathcal{O}} \equiv q_\psi^{\mathcal{C}} \equiv q_\psi$, where q_ψ is given by (137), and the function $u \mapsto q_\psi(u,v)$ is continuous and strictly decreasing in $]\alpha, \tilde{a}[$, while the function $v \mapsto q_\psi(u,v)$ is continuous and strictly decreasing in $]\tilde{b}, \beta[$. In view of this observation and the arguments leading to (158)–(159), we can see that, given any $u \in]\alpha, \tilde{a}[$, there exists a unique point $l(u) \in]\tilde{b}, b^*[$ such that

$$q_\psi(u,v) \begin{cases} > 0, & \text{if } v \in [\tilde{b}, l(u)[, \\ = 0, & \text{if } v = l(u), \\ < 0, & \text{if } v \in]l(u), b^*]. \end{cases} \tag{166}$$

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The resulting function l satisfies (160) with strict inequality, which implies that l is strictly decreasing. We can also use (166) and a straightforward adaptation of the arguments associated with (161) to verify that l is continuous.

To proceed further, we consider the functions $u \mapsto q_\phi^{\mathcal{O}}(u, l(u))$ and $u \mapsto q_\phi^{\mathcal{C}}(u, l(u))$. Given $\hat{u} \in]\alpha, \tilde{a}[$, we use the fact that l is decreasing to calculate

$$
\begin{aligned}
\lim_{u \downarrow \hat{u}} q_\phi^{\mathcal{O}}(u, l(u)) &= -\int_{]\hat{u}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \lim_{u \downarrow \hat{u}} \int_{]l(u), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \\
&= q_\phi^{\mathcal{O}}(\hat{u}, l(\hat{u})) + \lim_{u \downarrow \hat{u}} \int_{]l(u), l(\hat{u})]} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \\
&\leq q_\phi^{\mathcal{O}}(\hat{u}, l(\hat{u})),
\end{aligned}
\tag{167}
$$

where the inequality follows from (37) in Assumption 5. Also, we can use the left-continuity of l (see (164)) to calculate

$$
\begin{aligned}
\lim_{u \uparrow \hat{u}} q_\phi^{\mathcal{O}}(u, l(u)) &= -\int_{[\hat{u}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \int_{]l(\hat{u}), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \\
&= q_\phi^{\mathcal{O}}(\hat{u}, l(\hat{u})) - \Phi(\hat{u}) \, \mathcal{L}(R_h + g_c)(\{\hat{u}\}) \\
&\leq q_\phi^{\mathcal{O}}(\hat{u}, l(\hat{u})),
\end{aligned}
\tag{168}
$$

where the inequality follows from (36) in Assumption 5. If there exists $\varepsilon > 0$ such that $l(u) = l(\hat{u})$, for all $u \in]\hat{u} - \varepsilon, \hat{u}]$, then

$$
\lim_{u \uparrow \hat{u}} q_\phi^{\mathcal{C}}(u, l(u)) = q_\phi^{\mathcal{C}}(\hat{u}, l(\hat{u})),
\tag{169}
$$

while, if $l(u) > l(\hat{u})$, for all $u < \hat{u}$, then we can use the left-continuity of l to calculate

$$
\begin{aligned}
\lim_{u \uparrow \hat{u}} q_\phi^{\mathcal{C}}(u, l(u)) &= -\int_{[\hat{u}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \int_{]l(\hat{u}), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \\
&= q_\phi^{\mathcal{C}}(\hat{u}, l(\hat{u})) - \Phi(l(\hat{u})) \, \mathcal{L}(R_h - g_o)(\{l(\hat{u})\}) \\
&\geq q_\phi^{\mathcal{C}}(\hat{u}, l(\hat{u})),
\end{aligned}
\tag{170}
$$

where the inequality follows from (37) in Assumption 5. Similarly, if there exists $\varepsilon > 0$ such that $l(u) = l(\hat{u})$, for all $u \in [\hat{u}, \hat{u} + \varepsilon[$, then

$$
\begin{aligned}
\lim_{u \downarrow \hat{u}} &q_\phi^{\mathcal{C}}(u, l(u)) \\
&= -\int_{]\hat{u}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \int_{[l(\hat{u}), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds) \\
&= q_\phi^{\mathcal{C}}(\hat{u}, l(\hat{u})) + \Phi(\hat{u}) \, \mathcal{L}(R_h + g_c)(\{\hat{u}\}) \\
&\geq q_\phi^{\mathcal{C}}(\hat{u}, l(\hat{u})),
\end{aligned}
\tag{171}
$$

where the inequality follows from (36) in Assumption 5, while, if $l(u) < l(\hat{u})$, for all $u > \hat{u}$, then

$$
\lim_{u \downarrow \hat{u}} q_\phi^c(u, l(u))
$$

$$
= - \int_{]\hat{u}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \int_{[\lim_{u \downarrow \hat{u}} l(u), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds)
$$

$$
= q_\phi^c(\hat{u}, l(\hat{u})) + \Phi(\hat{u}) \, \mathcal{L}(R_h + g_c)(\{\hat{u}\}) + \int_{[\lim_{u \downarrow \hat{u}} l(u), l(\hat{u})[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds)
$$

$$
\geq q_\phi^c(\hat{u}, l(\hat{u})), \tag{172}
$$

where the inequality follows from the fact that the function $l \equiv \bar{l}$ satisfies (162).

The calculations in (167)–(168) imply that the function $u \mapsto q_\psi^{\mathcal{O}}(u, l(u))$ is upper semicontinuous, while the calculations in (169)–(172) imply that the function $u \mapsto q_\psi^c(u, l(u))$ is lower semicontinuous. It follows that the sets

$$
\mathcal{E}_o = \left\{ u \in]\alpha, \tilde{a}[\mid q_\phi^{\mathcal{O}}(u, l(u)) \geq 0 \right\} \quad \text{and} \quad \mathcal{E}_c = \left\{ u \in]\alpha, \tilde{a}[\mid q_\phi^c(u, l(u)) \leq 0 \right\}
$$

are closed in $]\alpha, \tilde{a}[$ if we endow this interval with the trace of the usual topology on $\mathbb{R}$. Also, the inequality (154) implies that $]\alpha, \tilde{a}[\setminus \mathcal{E}_o \subseteq \mathcal{E}_c$. This inclusion and the definitions of the sets $\mathcal{E}_o$, $\mathcal{E}_c$ imply that, if

$$
\inf \mathcal{E}_o > \alpha \quad \text{and} \quad \mathcal{E}_o \neq \emptyset, \tag{173}
$$

then $\min \mathcal{E}_o \in \mathcal{E}_c$, and the points a_o, b_c defined by

$$
a_c = \min \mathcal{E}_o \in]\alpha, \tilde{a}[\quad \text{and} \quad b_o = l(a_o) \in]\tilde{b}, b^*]
$$

satisfy the system of inequalities (129)–(130). To see that the claim $\min \mathcal{E}_o \in \mathcal{E}_c$ is indeed true, we consider any sequence (x_n) in $]\alpha, \min \mathcal{E}_o[$ such that $\lim_{n \to \infty} x_n = \min \mathcal{E}_o$. Since $]\alpha, \tilde{a}[\setminus \mathcal{E}_o \subseteq \mathcal{E}_c$, $x_n \in \mathcal{E}_c$, for all n. It follows that $\lim_{n \to \infty} x_n \in \mathcal{E}_c$ because $\mathcal{E}_c$ is closed.

In view of the definition of the set $\mathcal{E}_o$, (173) will follow if we prove that

$$
\lim_{u \downarrow \alpha} q_\phi^{\mathcal{O}}(u, l(u)) < 0 \quad \text{and} \quad \lim_{u \uparrow \tilde{a}} q_\phi^{\mathcal{O}}(u, l(u)) > 0. \tag{174}
$$

The second of these inequalities follows immediately from the observation that

$$
\lim_{u \uparrow \tilde{a}} q_\phi^{\mathcal{O}}(u, l(u)) = - \int_{[\tilde{a}, \beta[} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) + \lim_{u \uparrow \tilde{a}} \int_{]l(u), \beta[} \Phi(s) \, \mathcal{L}(R_h - g_o)(ds)
$$

$$
= \lim_{u \uparrow \tilde{a}} \left(- \int_{[\tilde{a}, l(u)]} \Phi(s) \, \mathcal{L}(R_h + g_c)(ds) - \int_{]l(u), \beta[} \Phi(s) \, \mathcal{L}(g_c + g_o)(ds) \right), \tag{175}
$$

the fact that $\lim_{u \uparrow \tilde{a}} l(u) \leq b^* < \beta$, and (36), (40) in Assumption 5.

To establish the first inequality in (174), we have to distinguish between two cases. If the problem data is such that case **A42** or case **A43** is satisfied, then we can use (82), (55) or

(57), according to the case, and (71) to calculate

$$\lim_{v\uparrow b^*} q_\phi^{\mathcal{O}}(u,v) = -\int_{]u,\beta[} \Phi(s)\,\mathcal{L}(R_h+g_c)(ds) + \int_{[b^*,\beta[} \Phi(s)\,\mathcal{L}(R_h-g_o)(ds)$$

$$\le -\int_{]u,\beta[} \Phi(s)\,\mathcal{L}(R_h+g_c)(ds) - \frac{(R_h-g_o)(b^*)}{\psi(b^*)}$$

$$< 0, \qquad \text{for all } u \in]\alpha,a^*[.$$

Combining this inequality with the fact that the function $v \mapsto q_\phi^{\mathcal{O}}(u,v)$ is increasing in $]\tilde{b},\beta[$ (see (150)) and the fact that $l:]\alpha,\tilde{a}[\to]\tilde{b},b^*]$, we can see that that $q_\phi^{\mathcal{O}}(u,l(u)) < 0$, for all $u \in]\alpha,a^*[$, which establishes the first inequality in (174) as well as the claim that $a_c \ge a^*$ in (138). On the other hand, if the problem data is such that case **A41** prevails, then we can use (82), (83) and the fact that $\lim_{u\downarrow\alpha} l(u) = b^*$ to calculate

$$\lim_{u\downarrow\alpha} q_\phi^{\mathcal{O}}(u,l(u)) = -\int_{]\alpha,\beta[} \Phi(s)\,\mathcal{L}(R_h+g_c)(ds) + \int_{[b^*,\beta[} \Phi(s)\,\mathcal{L}(R_h-g_o)(ds)$$

$$\le \lim_{x\downarrow\alpha} \frac{(R_h+g_c)(x)}{\psi(x)} - \frac{(R_h-g_o)(b^*)}{\psi(b^*)},$$

which proves that, if the problem data is such that the pair **A41**–**B43** occurs, then the first of the inequalities in (174) is satisfied if (73) is true.

The analysis up to this point has established all of the claims made in part (I) of the lemma. In view of this analysis and the fact that, in the presence of Assumptions 6 and 7, $q_\phi^{\mathcal{O}} \equiv q_\phi^{\mathcal{C}} = q_\phi$, where q_ϕ is given by (136), part (II) of the lemma will follow immediately from (174) if we prove that the function $u \mapsto q_\phi(u,l(u))$ is strictly increasing in $]\alpha,\tilde{a}[$. To this end, we fix any points $u_1 < u_2$ in $]\alpha,\tilde{a}[$. Using (13), (136), (165) and (166), we calculate

$$\big[q_\phi(u_2,l(u_2)) - q_\phi(u_1,l(u_1)) \big] / \phi(\tilde{a})$$

$$= \int_{u_1}^{u_2} \frac{\phi(s)}{\phi(\tilde{a})} \frac{1}{\sigma^2(s)\mathcal{W}(s)} \mathcal{L}(R_h+g_c)(ds) + \int_{l(u_1)}^{l(u_2)} \frac{\phi(s)}{\phi(\tilde{a})} \frac{1}{\sigma^2(s)\mathcal{W}(s)} \mathcal{L}(R_h-g_o)(ds)$$

and

$$0 = \big[q_\psi(u_2,l(u_2)) - q_\psi(u_1,l(u_1)) \big] / \psi(\tilde{a})$$

$$= -\int_{u_1}^{u_2} \frac{\psi(s)}{\psi(\tilde{a})} \frac{1}{\sigma^2(s)\mathcal{W}(s)} \mathcal{L}(R_h+g_c)(ds) - \int_{l(u_1)}^{l(u_2)} \frac{\psi(s)}{\psi(\tilde{a})} \frac{1}{\sigma^2(s)\mathcal{W}(s)} \mathcal{L}(R_h-g_o)(ds).$$

These calculations, fact that the strictly positive function ϕ (resp., ψ) is decreasing (resp.,

increasing), (36)–(37) in Assumption 5 and Assumption 6 imply that

$$
\begin{aligned}
\big[q_\phi(u_2, l(u_2)) &- q_\phi(u_1, l(u_1)) \big] / \phi(\tilde{a}) \\
&= \int_{u_1}^{u_2} \left[\frac{\phi(s)}{\phi(\tilde{a})} - \frac{\psi(s)}{\psi(\tilde{a})} \right] \frac{1}{\sigma^2(s) \mathcal{W}(s)} \mathcal{L}(R_h + g_c)(ds) \\
&\quad + \int_{l(u_1)}^{l(u_2)} \left[\frac{\phi(s)}{\phi(\tilde{a})} - \frac{\psi(s)}{\psi(\tilde{a})} \right] \frac{1}{\sigma^2(s) \mathcal{W}(s)} \mathcal{L}(R_h - g_o)(ds). \\
&> 0,
\end{aligned}
$$

which proves that the function $u \mapsto q_\phi(u, l(u))$ is strictly increasing in $]\alpha, \tilde{a}[$, and part (II) of the lemma follows.

It remains to prove part (III) of the lemma. To this end, we note that, by construction, the function w given by (115)–(116) will satisfy the HJB equation (86) in the sense of Definition 3 if and only if

$$
\text{supp} \left[\mathcal{L}w(1, \cdot) + h \right]^+ \cap \,]\alpha, a_c] = \emptyset, \tag{176}
$$

$$
\text{supp} \left[\mathcal{L}w(0, \cdot) \right]^+ \cap [b_o, \beta[= \emptyset, \tag{177}
$$

$$
w(1, x) - w(0, x) - g_o(x) \leq 0, \quad \text{for all } x \in \,]\alpha, b_o], \tag{178}
$$

and

$$
w(0, x) - w(1, x) - g_c(x) \leq 0, \quad \text{for all } x \in [a_c, \beta[. \tag{179}
$$

To prove (176) and (177), we first observe that, since R_h satisfies the ODE $\mathcal{L}R_h + h = 0$, where the operator $\mathcal{L}$ is defined by (12),

$$
\frac{1}{2}\sigma^2(x) \left[(R_h)'_+(x) - (R_h)'_-(x) \right] + h(\{x\}) = 0, \quad \text{for all } x \in \mathcal{I}.
$$

In view of this identity and the definition (116) of $w(1, \cdot)$, we can see that

$$
\begin{aligned}
\left[\mathcal{L}w(1, \cdot) + h \right] (\{a_c\}) &= \frac{1}{2}\sigma^2(a_c) \left[(A\phi + R_h)'_+(a_c) - (B\psi - g_c)'_-(a_c) \right] + h(\{a_c\}) \\
&= \frac{1}{2}\sigma^2(a_c) \left[A\phi'(a_c) + (R_h)'_-(a_c) - B\psi'(a_c) + (g_c)'_-(a_c) \right] \\
&\leq 0, \tag{180}
\end{aligned}
$$

where the inequality follows from (118). Also, we can use the definition (115) of $w(0, \cdot)$ and (122) to calculate

$$
\begin{aligned}
\mathcal{L}w(0, \cdot)(\{b_o\}) &= \frac{1}{2}\sigma^2(b_o) \left[(A\phi + R_h - g_o)'_+(b_o) - B\psi'(b_o) \right] \\
&\leq 0. \tag{181}
\end{aligned}
$$

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Combining (180) with the fact that the restrictions of the measures $\mathcal{L}w(1,\cdot)+h$ and $\mathcal{L}(R_h+g_c)$ on $\big(]\alpha, a_c[, \mathcal{B}(]\alpha, a_c[)\big)$ are equal, we can see that (176) follows from (36) in Assumption 5 and the fact that $a_c < \tilde{a}$. Similarly, (181), the fact that the restrictions of the measures $\mathcal{L}w(0,\cdot)$ and $\mathcal{L}(R_h - g_o)$ on $\big(]b_o, \beta[, \mathcal{B}(]b_o, \beta[)\big)$ are equal, (37) in Assumption 5 and the fact that $\tilde{b} < b_o$ imply (177). Also, it is straightforward to see that (178) (resp., (179)) is equivalent to $g_c(x) + g_o(x) \geq 0$ when $x \leq a_c$ (resp., when $b_o \leq x$), which is true thanks to (33) in Assumption 4.

To prove (178) for $x \in]a_c, b_o]$, we have to show that

$$\xi(x) := A\frac{\phi(x)}{\psi(x)} - B + \frac{(R_h - g_o)(x)}{\psi(x)} \leq 0, \quad \text{for all } x \in [a_c, b_o]. \tag{182}$$

To this end, we use (47) and (9) to calculate

$$\xi'_+(x) = \frac{\mathcal{W}(x)}{\psi^2(x)} \left[-A + \int_{]\alpha, x]} \Psi(s)\, \mathcal{L}(R_h - g_o)(ds) \right]. \tag{183}$$

In view of the second inequality in (128), we can see that

$$-A + \int_{]\alpha, x]} \Psi(s)\, \mathcal{L}(R_h - g_o)(ds) \geq -\int_{]x, b_o[} \Psi(s)\, \mathcal{L}(R_h - g_o)(ds),$$

which, combined with (37) in Assumption 5 and the strict positivity of the function $\mathcal{W}$, implies that

$$\xi'_+(x) \geq 0, \quad \text{for all } x \in [\tilde{b}, b_o[. \tag{184}$$

Also, we can see that (37) in Assumption 5 and (183) imply that the function $\psi^2\xi'_+/\mathcal{W}$ is increasing in $]\alpha, \tilde{b}]$ and decreasing in $]\tilde{b}, \beta[$. It follows that either $\xi'_+(x) \geq 0$, for all $x \in [a_c, b_o[$, or there exists a point $\gamma \in]a_c, \tilde{b}]$ such that $\xi'_+(x) < 0$, for all $x \in [a_c, \gamma[$, and $\xi'_+(x) \geq 0$, for all $x \in [\gamma, b_o[$. Combining either of these two cases with the calculations

$$\xi(a_c) = -\frac{g_c(a_c) + g_o(a_c)}{\psi(a_c)} < 0 \quad \text{and} \quad \xi(b_o) = 0,$$

which follow from the construction of w, we can see that the inequality (182) holds true. Finally, the proof of (179) for $x \in [a_c, b_o[$ follows similar symmetric arguments. $\qquad\square$

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