PRICING A CLASS OF EXOTIC OPTIONS VIA MOMENTS AND SDP RELAXATIONS

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We present a new methodology for the numerical pricing of a class of exotic derivatives such as Asian or barrier options when the underlying asset price dynamics are modelled by a geometric Brownian motion or a number of mean-reverting processes of interest. This methodology identifies derivative prices with infinite-dimensional linear programming problems involving the moments of appropriate measures, and then develops suitable finite-dimensional relaxations that take the form of semi-definite programs (SDP) indexed by the number of moments involved. By maximising or minimising appropriate criteria, monotone sequences of both upper and lower bounds are obtained. Numerical investigation shows that very good results are obtained with only a small number of moments. Theoretical convergence results are also established.

Key Words: exotic options, options pricing, moments of measures, semidefinite programming

1. INTRODUCTION

We propose a new methodology for calculating the prices of several exotic options of European type. For concreteness, our analysis focuses on fixed-strike, arithmetic-average Asian and down-and-out barrier call option payoff structures. However, it can easily be modified to account for other payoff structures including double-barrier knockout and Parisian call options, or their "put" counterparts. The study of exotic options has been a major research area in mathematical finance, and the literature is abundant in results such as exact formulas or numerical approximation techniques when the underlying asset price dynamics are modelled by a geometric Brownian motion (GBM). The approach that we propose also addresses the cases arising when the underlying price dynamics are modelled by processes such as an Ornstein-Uhlenbeck process as in Vasicek's model, a standard square-root process with mean-reversion as in the Cox-Ingersoll-Ross (CIR) model,

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and others (see Remark 2.1 in Section 2), which are of particular interest in the fixed-income and the commodity markets.

The objective of this paper is to introduce the new approach rather than present an exhaustive study of its range of applicability relative to different payoff structures or underlying asset price dynamics, which is significantly large. In fact, encouraged by the quality of our numerical results, we expect that our approach will become a standard tool in financial engineering. Therefore, we do not attempt a comprehensive survey of the literature on the valuation of the options we focus on here, but we restrict ourselves to a few remarks only. If we assume that the underlying asset follows a GBM, then there exists a closed form analytic expression for the price of a down-and-out barrier call option (e.g., see Musiela and Rutkowski (1997, Section 9.6)). On the other hand, this is not the case for arithmetic-average Asian options. Such options are important for both their practical importance and their theoretical interest, and their analysis has attracted significant interest in the literature. Approaches to the approximate pricing of Asian options when the underlying is modelled by a GBM include quasi-analytic techniques based on Edgeworth and Taylor expansions, and the like (e.g., Turnbull and Wakeman (1991)), methods derived by means of probabilistic techniques (Curran (1992), Rogers and Shi (1995)), the numerical solution of appropriate PDE's (e.g., Rogers and Shi (1995)) and Monte Carlo simulations (e.g., Glasserman, Heidelberger and Shahabuddin (1999)). Most of these approximation techniques can be adapted to account for the pricing of Asian options when the underlying follows other diffusion processes such as a mean-reverting process. However, despite their practical importance, such extensions are still at their early stages.

The approach that we develop in this paper derives from the methodology of moments introduced by Dawson (1980) for the analysis of geostochastic systems modelled as solutions, called *stochastic measure diffusions*, of measure-valued martingale problems with a view to a range of applications in areas such as statistical physics, population, ecology, epidemic modelling, and others. In the present context, the idea can be described informally as follows. First, one identifies the price of a European style option with a linear combination of moments of suitably defined measures. One then exploits the martingale property of certain associated stochastic integrals to derive an infinite system of linear equations involving the moments of the measures considered. Thus, one relates the value of an option with the solution of an infinite-dimensional linear programming (LP) problem the variables of which identify with the moments of certain measures. The next step is to obtain finite-dimensional relaxations by restricting the infinite-dimensional LP problem to one that involves only a finite number of moments. For this step to make sense, one has to introduce extra constraints, called *moment conditions*, that reflect necessary conditions for a set of scalars to be identified with moments of a measure supported on a given set. Depending on the choice of moment conditions (see Section 4 below), one can end up with an LP problem or a semi-definite programming (SDP) problem. In either case, by maximising (resp. minimising) the resulting problems, one obtains upper (resp. lower) bounds for the value of the option under consideration, the quality of which is enhanced as the number of moments increases.

A similar methodology has been used by Helmes, Röhl and Stockbridge (2001) to evaluate the moments of certain exit time distribution and change-point detection problems. These authors used moment conditions that resulted in LP problems, and, although they did not establish the convergence of their algorithm, they obtained very good numerical results. Also, Schwerer (2001) considered the evaluation of moments of the steady-state distribution of a reflected Brownian motion. Her analysis provided bounds by means of appropriate LP relaxations, and established the convergence of her algorithm.

Recently, a related, though fundamentally different, approach leading to the computation of bounds for the prices of certain European options has been proposed by Boyle and Lin (1997), Bertsimas and Popescu (2000, 2002), and Gotoh and Konno (2002). These authors assume a non-parametric framework in which *only* the first n moments of the underlying asset's distribution at maturity are known. Their work was extended in a more general framework by Zuluaga and Peña (2005), and by Han, Li, Sun and Sun (2005) (see also the references therein).

The contributions of this paper are multi-fold, and can be summarised as follows. First, we introduce the approach described above in the area of pricing a class of exotic derivatives of interest in the financial and the commodity markets. The scope of our methodology is not restricted in the Black and Scholes world because it can as well be applied when the underlying asset price dynamics are modelled by a number of mean-reverting diffusions that have been considered in the mathematical finance literature and are of particular interest in the theory used in the interest rate and the commodity markets. Also, one of its important aspects is that it provides monotone sequences of *both* upper *and* lower bounds, which can be used to control the maximal possible approximation error.

In contrast to Helmes, Röhl and Stockbridge (2001), and Schwerer (2001), the objective criteria that we wish to evaluate here are *not* anymore expressed in terms of moments of the underlying process' distribution at a given (stopping) time, which gives rise to a situation that requires additional modelling effort. Moreover, we consider moment conditions that derive from the semi-definite positivity property of *moment* and *localising* matrices (see Section 4). As a result, we are faced with the solution of SDP instead of LP problems. SDP problems are convex optimisation problems that, given a required precision, can, in principle, be solved in time that is polynomial in the problem's size, and standard software solvers are now available, including public domain ones. It is worth noting that, instead of considering moment conditions leading to SDP problems, we could as well have considered moment conditions leading to LP problems. Although we did not test the efficiency of the two possibilities against each other in the present context, our experience suggests the superiority of the SDP approach (for results in this direction, see Lasserre and Prieto-Rumeau (2004)).

We also prove the convergence of our algorithms to the value of the objective criterion under certain conditions. The most important of these assumptions requires that the underlying asset's distribution at the derivative's maturity is uniquely determined by its moment sequence. With regard to the underlying asset dynamics considered in this paper, the Ornstein-Uhlenbeck process as in Vasicek's model and the standard square-root process with mean reversion as in the CIR model satisfy this condition because the normal and the non-central χ^2 distributions are uniquely determined by their moment sequences. On the other hand, we cannot guarantee the convergence of our algorithms to the exact value of the options under consideration if we assume that the underlying follows a GBM because the lognormal distribution is *not* moment-determinate (see also Example 3.3 for a further illustration of this issue). In all cases, our numerical investigation has revealed that our method yields very good results with only a small number of moments, and compares favourably with standard approximation techniques against which we have tested it. In particular, our lower bounds for fixed-strike, arithmetic-average Asian options on a GBM outperform the Curran (1992) lower bound.

At this point, we should note that what appears as a possible drawback of our method, namely the lack of our algorithm's convergence when the underlying asset dynamics are modelled by a GBM, actually provides a further contribution of the paper. Indeed, our analysis reveals that the moment determinancy of the underlying probability distributions plays a fundamental role when addressing the qualitative nature of bounds for option prices using moment methods. In particular, it has obvious implications for the quality of the bounds derived by Bertsimas and Popescu (2000, 2002), Gotoh and Konno (2002), and the other related references discussed above.

The paper is organised as follows. In Section 2, we present the derivative structures on which our analysis focuses. In Section 3, we develop the theoretical background from which our methodology derives, and we explore certain of its ramifications. Section 4 is concerned with the issue of moment conditions pertaining to the relaxations developed in Section 3. Sections 5 and 6 present how the general method can be applied to the pricing of derivatives with Asian and barrier payoff structures, respectively. Finally, our numerical investigation is presented in Section 7 and some proofs are postponed in an appendix in Section 8.

2. A CLASS OF OPTIONS

Fix a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual conditions and supporting a standard, one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a number of derivatives the underlying asset price process of which satisfies a stochastic differential equation (SDE) of the form

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x_0 \in \mathcal{I}. \tag{2.1}$$

Here, $\mathcal{I}$ is either $(0,+\infty)$ or $\mathbb{R}$, and $b, \sigma : \mathcal{I} \to \mathcal{I}$ are given functions such that (2.1) has a unique strong solution with values in $\mathcal{I}$, for all $t \geq 0$, P-a.s.. In particular, we assume that the underlying asset price process X is given by one of the following three models.

Model 1: $b(x) := bx$, $\sigma(x) := \sigma x$ and $\mathcal{I} = (0,+\infty)$, for some constants $b, \sigma \in \mathbb{R}$.

This is the familiar *geometric Brownian motion* underlying the Black and Scholes model.

Model 2: $b(x) := \gamma(\theta - x)$, $\sigma(x) := \sigma$ and $\mathcal{I} = \mathbb{R}$, for some constants $\gamma, \theta, \sigma \in \mathbb{R}$.

This mean-reverting diffusion is an *Ornstein-Uhlenbeck process*, which appears, for instance, in Vasicek's interest rate model.

Model 3: $b(x) := \gamma(\theta - x)$, $\sigma(x) := \sigma\sqrt{x}$ and $\mathcal{I} = (0,+\infty)$, for some constants $\gamma, \theta, \sigma \in \mathbb{R}$ such that $\gamma\theta > \frac{1}{2}\sigma^2$.

This diffusion models the short rate dynamics assumed in the Cox-Ingersoll-Ross interest rate model. Note that the inequality $\gamma\theta > \frac{1}{2}\sigma^2$ is necessary and sufficient for the solution of (2.1) to be non-explosive, in particular, for the hitting time of 0 to be equal to ∞ with probability one.

Remark 2.1. *At this point, we note that, apart from the models above, there are several other choices for the underlying asset dynamics that we could have considered. These include the diffusions as in (2.1) with*

$$b(x) = \gamma(\theta - x) \quad \textit{and} \quad \sigma(x) = \sigma x,$$

or

$$b(x) = \gamma(\theta - x)x \quad \textit{and} \quad \sigma(x) = \sigma x,$$

which have been considered in the interest rate and the real option theories. In fact, any one-dimensional diffusion the infinitesimal generator of which maps polynomials into polynomials with the same or smaller degrees presents a choice that is compatible with our approach.

Our analysis focuses on three types of derivative payoff structures. In particular, we consider the payoffs of standard European, Asian and barrier call options (it is a totally trivial exercise to modify our analysis to account for the corresponding put options).

The value of a *European call* option written on the underlying X is given by

$$v_E(x_0) := e^{-\rho T} E\left[(X_T - K)^+\right]. \tag{2.2}$$

Here, $T > 0$ is the option's *maturity time*, K is the option's *strike price*, ρ is a constant discounting factor, and x_0 is the initial underlying asset price. Plainly, the simplest way of calculating $v_E(x_0)$ is by integrating the function $x \mapsto (x-K)^+$ with respect to the distribution of X_T (which, in the Models 1, 2 and 3 that we consider is lognormal, normal and non-central χ^2, respectively). The reason we consider this type of options is because we can easily compare our approximation results with the exact available formulas.

The value of a *fixed-strike, arithmetic-average Asian call* option written on X is given by

$$v_A(x_0) := e^{-\rho T} E\left[\left(\frac{1}{T}\int_0^T X_t\, dt - K\right)^+\right]. \tag{2.3}$$

The parameters appearing in this expression have same interpretation as above.

The price of a typical *down-and-out barrier call* option written on the underlying process X is given by

$$v_B(x_0) := e^{-\rho T} E\left[(X_T - K)^+ I_{\{\tau = T\}}\right], \tag{2.4}$$

where τ is the $(\mathcal{F}_t)$-stopping time defined by

$$\tau := \inf\{t \geq 0 \mid \; X_t \leq H\} \wedge T, \tag{2.5}$$

and $H < x_0$ is the *knockout barrier*. The other parameters are the same as in the other options considered above.

At this point, we should note that, with regard to the dynamics of Model 1 and the standard Black and Scholes theory, we assume that the probability measure P with respect to which we compute expectations in (2.2), (2.3) and (2.4) identifies with the unique so-called risk neutral probability measure and ρ is the short term interest rate. In this context, we must also have $b = \rho$, but we do not impose such a condition because it does not provide any simplification.

3. THE MOMENT APPROACH TO THE EVALUATION OF FUNCTIONALS OF DIFFUSIONS

The purpose of this section is to introduce the moment approach to the problem of approximately evaluating certain functionals of a diffusion such as the process X given by (2.1). This approach starts by identifying the value of such functionals with the solution of appropriate infinite-dimensional optimisation problems in which the controlled variables are the moments of suitably defined measures. It then proceeds to the derivation of suitable finite-dimensional relaxations of the resulting optimisation problems. These relaxations involve

(a) the restriction to a *finite number of moments*, and

(b) the replacement of the unknown moments by scalars constrained to satisfy necessary, so-called moment conditions for these scalars to be identified with *moments of measures with appropriate supports.*

3.1. **Functionals of Diffusions and Martingale Moment Conditions.**

To fix ideas, let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space satisfying the usual conditions and supporting a standard, m-dimensional $(\mathcal{F}_t)$-Brownian motion B, and consider the SDE

$$dZ_t = \beta(Z_t)\,dt + e(Z_t)\,dB_t, \quad Z_0 = z_0 \in \mathbb{R}^n, \tag{3.1}$$

where $\beta : \mathbb{R}^n \to \mathbb{R}^n$ and $e : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given deterministic functions such that (3.1) has a unique strong solution. The infinitesimal generator $\mathcal{A}$ of this diffusion is defined by

$$f \mapsto (\mathcal{A}f)(z) := \frac{1}{2}\,\mathrm{tr}\left[ee^{\mathrm{T}} D_{zz} f\right](z) + \left[\beta^{\mathrm{T}} D_z f\right](z), \quad f \in \mathcal{D}(\mathcal{A}),$$

the domain $\mathcal{D}(\mathcal{A})$ of which contains the set $C_c^2(\mathbb{R}^n)$ of all twice-continuously differentiable functions $f : \mathbb{R}^n \mapsto \mathbb{R}$ with compact support. Here, given $f \in \mathcal{D}(\mathcal{A})$, the vector $D_z f$ and the $n \times n$ matrix $D_{zz} f$ are the gradient and the Hessian of f, respectively, so that

$$D_z f := (f_{z_1}, \ldots, f_{z_n})^{\mathrm{T}} \quad \text{and} \quad D_{zz} f(i,j) := f_{z_i z_j},\ i,j = 1,\ldots,n.$$

We impose the following assumption.

Assumption 3.1. The entries of the vector $\beta(z)$ and the matrix $\left(ee^{\mathrm{T}}\right)(z)$ are polynomials in z, so that $\mathcal{A}$ maps polynomials into polynomials. Moreover,

$$\sup_{t\in[0,T]} \sum_{j=1}^{n} E\left[|Z_t^j|^k\right] < \infty, \quad \text{for all } T > 0, \quad \text{for all } k \in \mathbb{N}.$$

This assumption implies in particular that, given any polynomial $f : \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{D}(\mathcal{A})$, and the process M^f defined by

$$\begin{aligned} M_t^f &:= f(Z_t) - f(z_0) - \int_0^t (\mathcal{A}f)(Z_s)\,ds \\ &= \int_0^t \left[e^{\mathrm{T}} D_z f\right]^{\mathrm{T}} (Z_s)\,dB_s, \quad t \geq 0, \end{aligned} \tag{3.2}$$

is a square-integrable martingale.

To proceed further, fix any $(\mathcal{F}_t)$-stopping time τ that is bounded by a constant $T > 0$, P-a.s.. With regard to Doob's optional sampling theorem, observe that, under Assumption 3.1, if $f : \mathbb{R}^n \to \mathbb{R}$ is a polynomial, then

$$E\left[f(Z_\tau)\right] - f(z_0) - E\left[\int_0^\tau (\mathcal{A}f)(Z_s)\,ds\right] = 0. \tag{3.3}$$

Now, consider

• the *expected occupation measure* $\mu(\cdot) = \mu(\cdot\,; z_0)$ of the diffusion Z up to time τ that is defined by

$$\mu(B) := E\left[\int_0^\tau I_{\{Z_s \in B\}}\,ds\right], \quad B \in \mathcal{B}(\mathbb{R}^n), \tag{3.4}$$

where $\mathcal{B}(\mathbb{R}^n)$ is the Borel σ-algebra on $\mathbb{R}^n$, and

- the *exit location measure* $\nu(\cdot) = \nu(\cdot\,; z_0)$ that is defined by

$$\nu(B) := P\left(Z_\tau \in B\right), \quad B \in \mathcal{B}(\mathbb{R}^n), \tag{3.5}$$

which is the probability distribution of Z_τ.

One may then rewrite (3.3) as

$$\int_{\mathbb{R}^n} f(z)\,\nu(dz) - f(z_0) - \int_{\mathbb{R}^n} (\mathcal{A}f)(z)\,\mu(dz) = 0, \tag{3.6}$$

which is called the *basic adjoint equation* (e.g., see Helmes, Röhl and Stockbridge (2001)), and characterises the measures μ and ν associated with the generator $\mathcal{A}$ (see Kurtz and Stockbridge (1998)).

Given a multi-index $\alpha \in \mathbb{N}^n$, if f is the monomial

$$z \mapsto f(z) = z^\alpha := \prod_{j=1}^{n} z_j^{\alpha_j},$$

then Assumption 3.1 implies that there exists a *finite* collection $\{c_\beta(\alpha)\}$ of real numbers such that $(\mathcal{A}f)(z) = \sum_\beta c_\beta(\alpha) z^\beta$, for all $z \in \mathbb{R}^n$. If we define $\{\mu^\alpha\} = \{\mu^\alpha(z_0)\}$ and $\{\nu^\alpha\} = \{\nu^\alpha(z_0)\}$ to be the moments of μ and ν, respectively, assumed to be finite, i.e.,

$$\mu^\alpha = \int_{\mathbb{R}^n} z^\alpha\,\mu(dz) < \infty \quad \text{and} \quad \nu^\alpha = \int_{\mathbb{R}^n} z^\alpha\,\nu(dz) < \infty,$$

then the basic adjoint equation (3.6) implies

$$\nu^\alpha - \sum_\beta c_\beta(\alpha)\mu^\beta = z_0^\alpha, \quad \text{for all } \alpha \in \mathbb{N}^n, \tag{3.7}$$

which is an infinite system of *linear* equations linking the moments of μ and ν.

Going back to the main objective of this section, suppose that we want to evaluate the functional $J(z_0)$ of the process Z that is defined by

$$J(z_0) := E\left[p(Z_\tau)\right] = \sum_{j=1}^{k} \int_{\mathcal{K}_j} p_j(z)\,\nu(dz), \tag{3.8}$$

where $\{\mathcal{K}_j,\ j = 1, \ldots, k\}$ is a given Borel measurable partition of $\mathbb{R}^n$,

$$p(z) := \sum_{j=1}^{k} p_j(z) I_{\mathcal{K}_j}(z), \quad z \in \mathbb{R}^n,$$

and, for all $j = 1, \ldots, k$,

$$p_j(z) := \sum_\alpha p_{j\alpha} z^\alpha, \quad z \in \mathcal{K}_j,$$

are given polynomials. If we define

$$\nu_j(\cdot) \equiv \nu_j(\cdot\,; z_0) := \nu(\cdot\,; z_0)\big|_{\mathcal{K}_j}$$

to be the restriction of ν on $\mathcal{K}_j$, for $j = 1, \dots, k$, then we can see that

$$J(z_0) = \sum_{j=1}^{k} \sum_{\alpha} p_{j\alpha} \nu_j^{\alpha},$$

which is a linear combination of the moments $\{\nu_j^\alpha\} = \{\nu_j^\alpha(z_0)\}$ of the measures ν_j.

We are now faced with two possible cases. The first one, Case I, arises when we can easily pre-compute the moments $\{\nu^\alpha\}$ of the measure ν (see Section 3.2 below). The second one, Case II, arises otherwise and involves the use of the martingale moment conditions (3.7) (see Section 3.3 below). Our treatment of European and Asian options is developed within the framework of Case I, whereas barrier options require the use of techniques associated with Case II.

3.2. **Case I.**

Suppose that the moments ν^α, for $\alpha \in \mathbb{N}^n$, are known. We bound from above and below the value $J(z_0)$ of the functional introduced in (3.8) with the maximum and the minimum, respectively, of the infinite dimensional LP problem defined by

$$\mathbb{Q}^{\mathrm{I}}(z_0) \rightarrow \begin{cases} \underset{\nu_1,\dots,\nu_k}{\text{extremise}} & \displaystyle\sum_{j=1}^{k} \sum_{\alpha} p_{j\alpha} \nu_j^{\alpha}, \\ \text{subject to} & \displaystyle\sum_{j=1}^{k} \nu_j^{\alpha} = \nu^{\alpha}, \quad \alpha \in \mathbb{N}^n, \\ & \nu_j \in \mathcal{M}(\mathcal{K}_j), \quad j = 1, \dots, k, \end{cases} \tag{3.9}$$

where $\mathcal{M}(\mathcal{K})$ is the space of all Borel measures with finite moments of all orders that are supported on a given Borel measurable set $\mathcal{K} \subseteq \mathbb{R}^n$, and where "extremise" stands for either "maximise" or "minimise".

Plainly, the constraints defining the feasible region of this LP problem are necessary conditions, and therefore,

$$\inf \mathbb{Q}^{\mathrm{I}}(z_0) \leq J(z_0) \leq \sup \mathbb{Q}^{\mathrm{I}}(z_0). \tag{3.10}$$

To proceed further, we need the following definition.

Definition 3.2. *Let ν be a measure on $\mathbb{R}^n$ with finite moments of all orders. The measure ν is said to be moment-determinate if $\nu = \mu$ whenever*

$$\int z^\alpha \, \mu(dz) = \int z^\alpha \, \nu(dz), \quad \textit{for all } \alpha \in \mathbb{N}^n,$$

for some measure μ on $\mathbb{R}^n$.

Notice that, when ν is moment-determinate, then both relations in (3.10) hold with equality. Indeed, in this case, the measure $\nu \equiv \sum_j \nu_j$ is unique, and, necessarily, $\nu_j = \nu\big|_{\mathcal{K}_j}$. In general, a probability distribution is not necessarily uniquely determined by its moment sequence. For instance, consider the following important example.

Example 3.3. *Suppose that we want to calculate*

$$J(1) = E\left[(Z_1 - K)^+\right],$$

where Z is the geometric Brownian motion given by

$$dZ_t = \tfrac{1}{2} Z_t\, dt + Z_t\, dB_t, \quad Z_0 = 1,$$

where B is a standard, one-dimensional Brownian motion. Then, Z_1 has a standard lognormal distribution, and the value of $J(1)$ can be calculated by means of a simple integration.

Now, consider the following family of probability density functions:

$$f_a(x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{1}{2}(\log x)^2}(1 + a\sin(2\pi \log x)), \quad \textit{for } x > 0,$$

parametrised by $a \in [-1, 1]$. All of these density functions identify with probability measures sharing the same moment sequence, namely the sequence $\{\exp(k^2/2),\ k \in \mathbb{N}\}$, and f_0 is the lognormal probability density function of Z_1 (e.g., see Feller (1965, p. 227)). If we define

$$V(a) := \int (x - K)^+ f_a(x) dx,$$

then V depends linearly on a, and $V(0) = J(1)$. It follows that the corresponding "gap" in (3.10) is greater than or equal to

$$\max_{a\in[-1,1]} V(a) - \min_{a\in[-1,1]} V(a) > 0.$$

For illustration purposes, if we choose $K = 1.1$, then

$$V(-1) = 0.8471, \quad V(0) = J(1) = 0.8391, \quad V(1) = 0.8311,$$

so the corresponding gap in (3.10) is of at least 0.16 *or, relative to $J(1)$, of at least* 1.9%. *Other choices of K yield relative gaps of at least* 0.4% *for $K = 1$, and* 3% *for $K = 1.2$.*

In general, it is worth noting that the size of the gap in (3.10) is associated with the value of the index of dissimilarity of the Stieltjes class $\{f_a,\ a \in [-1, 1]\}$ (see Stoyanov (2004)).

Determining whether a given distribution is moment-determinate has been an important and long-standing problem in probability theory, and is still an area of active research (e.g., see Feller (1965); for recent developments, see Stoyanov (2000, 2002), and the references therein).

Given a measure μ on $\mathbb{R}$, the Cramér condition

$$\int_{\mathbb{R}} \exp\{c|x|\}\, \mu(dx) < +\infty, \quad \text{for some } c > 0, \tag{3.11}$$

provides a sufficient condition for μ to be moment-determinate that is easy to check. With regard to the distributions associated with the dynamics described by models considered in Section 2, we note that, while the normal distribution (Model 2) and the non-central χ^2 distribution (Model 3) are moment-determinate (e.g., they verify (3.11)), the lognormal distribution (Model 1) is *not* (see also Example 3.3 above).

3.3. **Case II.**

In the more general case, when the moments of ν are not easily available, we consider the infinite dimensional LP problem defined by

(3.12)
$$\mathbb{Q}^{\mathrm{II}}(z_0) \rightarrow \begin{cases} \underset{\nu_1,\dots,\nu_k,\mu}{\text{extremise}} & \displaystyle\sum_{j=1}^{k}\sum_{\alpha} p_{j\alpha}\nu_j^{\alpha}, \\ \text{subject to} & \displaystyle\sum_{j=1}^{k}\nu_j^{\alpha} - \sum_{\beta} c_\beta(\alpha)\mu^{\beta} = z_0^{\alpha}, \quad \alpha \in \mathbb{N}^n, \\ & \mu \in \mathcal{M}(\mathbb{R}^n), \quad \nu_j \in \mathcal{M}(\mathcal{K}_j), \quad j = 1,\dots,k, \end{cases}$$

which incorporates the martingale moment conditions (3.7). Note that, for $\alpha = (0,\dots,0)$, these constraints yield

$$\nu_1 + \cdots + \nu_k \text{ is a probability measure on } \mathbb{R}^n.$$

Similarly to (3.10), since the constraints defining the feasible region of this LP problem are necessary conditions,

$$\inf \mathbb{Q}^{\mathrm{II}}(z_0) \le J(z_0) \le \sup \mathbb{Q}^{\mathrm{II}}(z_0). \tag{3.13}$$

Furthermore, note that, if the constraints defining the feasible region of this LP problem uniquely determine the moments of the measures ν_j, and if the probability measure ν is moment-determinate, then both relations in (3.13) hold with equality, and $\mathbb{Q}^{\mathrm{II}}(z_0)$ identifies the value $J(z_0)$.

3.4. **Finite-Dimensional Relaxations.**

We now address the problem of deriving finite-dimensional relaxations of the infinite-dimensional LP problem defined by (3.9) or (3.12).

Given a Borel measurable set $\mathcal{K} \subseteq \mathbb{R}^n$, let $\mathcal{M}(\mathcal{K})$ be the set of all Borel measures with support contained in $\mathcal{K}$, and with all moments finite. We define

$$\mathcal{N}_r(\mathcal{K}) := \left\{ \int x^{\alpha}\,\mu(dx) \;\middle|\; \alpha \in \mathbb{N}^n,\ |\alpha| \le 2r,\ \mu \in \mathcal{M}(\mathcal{K}) \right\}, \tag{3.14}$$

where $|\alpha| := \sum_{i=1}^{n} \alpha_i \le 2r$, and we consider a set $\mathcal{S}_r(\mathcal{K})$ that is defined by appropriate *necessary* moment conditions for scalars $\{\eta^{\alpha},\ |\alpha| \le 2r\}$ to be moments of some measure in $\mathcal{M}(\mathcal{K})$, so that

$$\mathcal{S}_r(\mathcal{K}) \supseteq \mathcal{N}_r(\mathcal{K}). \tag{3.15}$$

The issue of a precise definition of such a set $\mathcal{S}_r(\mathcal{K})$ will be clarified in Section 4 below.

If the moments of ν are known, cf. Section 3.2, then it is natural to consider the finite-dimensional relaxation $\mathbb{Q}^{\mathrm{I}}_r(z_0)$ of $\mathbb{Q}^{\mathrm{I}}(z_0)$ that is defined by

(3.16)

$$\mathbb{Q}^{\mathrm{I}}_r(z_0) \to \begin{cases} \underset{\eta_1,\ldots,\eta_k}{\text{extremise}} & \displaystyle\sum_{j=1}^{k} \sum_{\alpha} p_{j\alpha} \eta_j^{\alpha}, \\ \text{subject to} & \displaystyle\sum_{j=1}^{k} \eta_j^{\alpha} = \nu^{\alpha}, \quad |\alpha| \le 2r, \\ & \eta_j \equiv \left\{\eta_j^{\alpha},\ |\alpha| \le 2r\right\} \in \mathcal{S}_r(\mathcal{K}_j), \quad j = 1, \ldots, k. \end{cases}$$

If the moments of the measure ν are not readily available, cf. Section 3.3, we consider the finite dimensional relaxation $\mathbb{Q}^{\mathrm{II}}_r(z_0)$ of $\mathbb{Q}^{\mathrm{II}}(z_0)$ that is defined by

(3.17)

$$\mathbb{Q}^{\mathrm{II}}_r(z_0) \to \begin{cases} \underset{\eta_1,\ldots,\eta_k,\vartheta}{\text{extremise}} & \displaystyle\sum_{j=1}^{k} \sum_{\alpha} p_{j\alpha} \eta_j^{\alpha}, \\ \text{subject to} & \displaystyle\sum_{j=1}^{k} \eta_j^{\alpha} - \sum_{\beta} c_{\beta}(\alpha) \vartheta^{\alpha} = z_0^{\alpha}, \quad |\alpha| \le 2r, \\ & \eta_j \equiv \left\{\eta_j^{\alpha},\ |\alpha| \le 2r\right\} \in \mathcal{S}_r(\mathcal{K}_j), \quad j = 1, \ldots, k, \\ & \vartheta \equiv \{\vartheta^{\alpha},\ |\alpha| \le 2r\} \in \mathcal{S}_r(\mathbb{R}^n), \end{cases}$$

where the finite sets $\{c_{\beta}(\alpha)\}$ of scalars are provided from the basic adjoint equation as in (3.7). Plainly, if the expected occupation measure is supported on a set $\mathcal{K} \subset \mathbb{R}^n$ rather than $\mathbb{R}^n$, then we impose the constraint $\vartheta \in \mathcal{S}_r(\mathcal{K})$ instead of the constraint $\vartheta \in \mathcal{S}_r(\mathbb{R}^n)$, which can improve the precision of the resulting approximation.

Of course, for either of these optimisation problems to be well defined, we assume that $2r$ is greater than or equal to the maximum of $|\alpha|$, where α runs over all of the multi-indices in the finite sum that we want to extremise.

4. MOMENT CONDITIONS

The purpose of this section is to provide explicit expressions for the sets $\mathcal{S}_r(\mathcal{K})$ appearing in (3.15) and used in the finite-dimensional relaxations (3.16)–(3.17). In fact there are two approaches that are particularly attractive to this end: the linear programming (LP) and the semi-definite programming (SDP) approaches that we discuss below.

4.1. **The LP Approach.**

In the linear programming (LP) approach, developed notably in Helmes, Röhl and Stockbridge (2001), the moment conditions defining the sets $\mathcal{S}_r(\mathcal{K})$ appearing in (3.15) are *linear constraints* linking the moment variables. These constraints reflect only necessary conditions, and derive from considering a finite subset of the so-called *Hausdorff* moment conditions, which are necessary and sufficient conditions for a sequence of scalars to be identified with the moments of a measure supported on a polytope (see Feller (1965), and Shohat and Tamarkin (1943)). For instance,

$$\sum_{j=0}^{r} (-1)^j \binom{r}{j} m_{j+k} \geq 0, \quad r,k = 0,1,\dots, \tag{4.1}$$

state necessary and sufficient conditions for the scalars $\{m_\alpha,\ \alpha \geq 0\}$ to be moments of a measure μ supported on the interval $[0,1]$.

The linear character of such constraints is their main attractive feature because the resulting relaxations $\mathbb{Q}_r^{\mathrm{I}}$ or $\mathbb{Q}_r^{\mathrm{II}}$, as in (3.16) or (3.17), respectively, present standard linear programs, for which many LP software solvers that can solve very large size problems are now available. On the other hand they are numerically ill-posed because of the binomial coefficients involved.

4.2. **The SDP Approach.**

More recent necessary and sufficient conditions can be expressed in terms of the positive semi-definiteness of appropriate *moment* and *localising* matrices. The resulting relaxations $\mathbb{Q}_r^{\mathrm{I}}$ or $\mathbb{Q}_r^{\mathrm{II}}$, as in (3.16) or (3.17), respectively, now become convex optimisation problems, called *semi-definite programs*, for which efficient software solvers exist. The interested reader can find details on semi-definite programming in Vandenberghe and Boyd (1996). The semi-definite programming (SDP) approach is more general than the LP approach because one can easily derive necessary (resp., necessary and sufficient) SDP moment conditions for arbitrary (resp., compact) semi-algebraic sets. Also, the SDP moment conditions are particularly attractive for measures with support on the real line, a half-line $[a,+\infty)$ or $(-\infty,b]$, or a segment $[a,b]$. Furthermore, these moment conditions have been applied with success in a variety of global non-convex optimisation problems (see Henrion and Lasserre (2003), and Lasserre (2001)).

In the context of certain diffusions models, it is shown in Lasserre and Prieto-Rumeau (2004) that the SDP approach is indeed more precise than the LP approach. Moreover, for the same required performance of the upper and lower bounds, the SDP approach required significantly fewer moments than the LP approach. On the other hand, the more recent SDP solvers are much less developed than their older LP counterparts. However, future progress in numerical analysis is bound to eliminate such a drawback.

4.3. **Moment Matrices.**

Let

$$(x^\alpha,\ |\alpha| \leq k) := \left(1, x_1, \dots, x_n, \dots, x_1^2, x_1x_2, \dots, x_1^k, x_1^{k-1}x_2, \dots, x_n^k\right), \tag{4.2}$$

be the usual basis of the space of real-valued polynomials in n variables, of degree at most k, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Given a multi-index family of scalars $\tilde{y} \equiv \{y_\alpha,\ \alpha \in \mathbb{N}^n\}$, let $\hat{y} \equiv \{\hat{y}_i,\ i \in \mathbb{N}\}$ be the sequence obtained by ordering $\tilde{y}$ so that it conforms with the indexing implied by the basis (4.2). The *moment-matrix* $M_k(\tilde{y})$ with rows and columns indexed in the basis (4.2) is then defined by

$$M_k(\tilde{y})(1,i) = M_k(\tilde{y})(i,1) = \hat{y}_{i-1}, \quad \text{for } i = 1,\dots,k+1,$$
$$M_k(\tilde{y})(1,j) = y_\alpha \quad \text{and} \quad M_k(\tilde{y})(i,1) = y_\beta \quad \Rightarrow \quad M_k(\tilde{y})(i,j) = y_{\alpha+\beta},$$

where $M_k(\tilde{y})(i,j)$ is the (i,j)-entry of the matrix $M_k(\tilde{y})$. For instance, when $n=1$, $\hat{y} = \{y_0, y_1, y_2, \dots\}$ and $M_k(\tilde{y})$ is the *Hankel* matrix

$$H_k(\tilde{y})(i,j) = y_{i+j-2}, \quad i,j = 1,\dots,k+1,$$

while, when $n=2$, $\hat{y} = \{y_{0,0}, y_{1,0}, y_{0,1}, y_{2,0}, y_{1,1}, y_{0,2}, \dots\}$ and $M_1(\tilde{y})$ reads

$$M_1(\tilde{y}) \;=\; \begin{pmatrix} y_{0,0} & y_{1,0} & y_{0,1} \\ y_{1,0} & y_{2,0} & y_{1,1} \\ y_{0,1} & y_{1,1} & y_{0,2} \end{pmatrix}.$$

In the context of this paper, moment matrices are of relevance if the family of scalars $\tilde{y} \equiv \{y_\alpha,\ \alpha \in \mathbb{N}^n\}$ considered above can be identified with the moments of a finite measure μ defined on the Borel σ-algebra on $\mathbb{R}^n$. In such a case, given any $k \in \mathbb{N}$, the moment matrix $M_k(\tilde{y})$ is positive semi-definite, denoted $M_k(\tilde{y}) \succeq 0$ (similarly, we use the notation $\succ 0$ for positive definite matrices). Indeed, for all polynomials $x \mapsto f(x)$ of degree at most k, and with vector of coefficients $(f_\alpha,\ |\alpha| \le k)$ in the basis (4.2), we have

$$\langle f, M_k(\tilde{y}) f \rangle = \int f^2 \, d\mu \ge 0.$$

Note that the converse is *not* in general true: given a moment-like matrix $M_k(\tilde{y}) \succeq 0$, the y_α's involved are not necessarily moments of some measure μ on $\mathbb{R}^n$.

4.4. **Localising Matrices.**

Given a polynomial q, we consider the set $\mathcal{K} \subseteq \mathbb{R}^n$ defined by

$$\mathcal{K} = \{x \in \mathbb{R}^n \mid \ q(x) \ge 0\}.$$

The localising matrix $M_k(q,\tilde{y})$ is defined as follows. Let $\beta(i,j)$ be the β-subscript of the (i,j)-entry of the matrix $M_k(\tilde{y})$. If the polynomial q has coefficients (q_α) in the basis (4.2), then the localising matrix is defined by

$$M_k(q,\tilde{y})(i,j) = \sum_\alpha q_\alpha y_{\beta(i,j)+\alpha}.$$

For example, if $x \mapsto q(x) := 1 - x_1^2 - x_2^2$, for $x \in \mathbb{R}^2$, then $M_1(q,\tilde{y})$ reads

$$M_1(q,\tilde{y}) = \begin{pmatrix} 1 - y_{2,0} - y_{0,2} & y_{1,0} - y_{3,0} - y_{1,2} & y_{0,1} - y_{2,1} - y_{0,3} \\ y_{1,0} - y_{3,0} - y_{1,2} & y_{2,0} - y_{4,0} - y_{2,2} & y_{1,1} - y_{3,1} - y_{1,3} \\ y_{0,1} - y_{2,1} - y_{0,3} & y_{1,1} - y_{3,1} - y_{1,3} & y_{0,2} - y_{2,2} - y_{0,4} \end{pmatrix}.$$

If the the elements of the family $\tilde{y} \equiv \{y_\alpha\}$ are the moments of some measure μ supported on $\mathcal{K}$, then $M_k(q,\tilde{y}) \succeq 0$, because, for all polynomials $x \mapsto f(x)$

of degree at most k, and with vector of coefficients $(f_\alpha, |\alpha| \leq k)$ in the basis (4.2),

$$\langle f, M_k(q,\tilde{y})f\rangle = \int f^2 q\, d\mu \geq 0,$$

Again, the converse is not true: the necessary conditions $M_k(q,\tilde{y}) \succeq 0$ and $M_k(\tilde{y}) \succeq 0$ are *not* in general sufficient to ensure that the elements of $\tilde{y}$ are the moments of some measure μ supported on $\mathcal{K}$.

In general, if $\mathcal{K}$ is a semi-algebraic set, i.e., a set of the form

$$\mathcal{K} := \{x \in \mathbb{R}^n \mid \; g_i(x) \geq 0, \text{ for all } i = 1,\ldots,l\},$$

where g_i, $i = 1,\ldots,l$, are given polynomials, then the conditions

$$M_k(\tilde{y}) \succeq 0 \quad \text{and} \quad M_k(g_i,\tilde{y}) \succeq 0, \quad i = 1,\ldots,l, \quad k = 1,2,\ldots, \tag{4.3}$$

are necessary (but not sufficient) for the elements of $\tilde{y}$ to be moments of some measure μ supported on $\mathcal{K}$. However, by a result in Putinar (1993), if $\mathcal{K}$ is compact, then, under some mild assumption, the conditions (4.3) are also sufficient.

We now state a result that applies when $n = 1$ and will be useful in the following sections. Given a vector $y = (y_0, y_1, \ldots, y_{2r})$ and a set $\mathcal{K} \subseteq \mathbb{R}$, the $\mathcal{K}$-*moment problem* investigates necessary and sufficient conditions for y to be identified with the corresponding vector of moments of some measure supported on $\mathcal{K}$. This problem is called the *truncated Hausdorff moment problem* if $\mathcal{K} = [a,b]$, and the *truncated Stieltjes moment problem* if $\mathcal{K} = [a,+\infty)$.

Theorem 4.1. *Given a vector $y = (y_0, y_1, \ldots, y_{2r}) \in \mathbb{R}^{2r+1}$, the following statements are true:*

(a) *With regard to the truncated Hausdorff moment problem,*

$$M_r(y) \succeq 0 \quad \text{and} \quad M_{r-1}(g,y) \succeq 0,$$

with $x \mapsto g(x) := (b-x)(x-a)$, are necessary and sufficient conditions for the elements of y to be the first $2r+1$ moments of a measure supported on $[a,b]$.

(b) *With regard to the truncated Stieltjes moment problem,*

$$M_r(y) \succ 0 \quad \text{and} \quad M_{r-1}(g,y) \succ 0$$

with $x \mapsto g(x) := x - a$, are sufficient conditions for the elements of y to be the first $2r+1$ moments of a measure supported on $[a,+\infty)$.

For the proof of this theorem we refer to Krein and Nudel'man (1977) or Curto and Fialkow (1991). This important result provides sufficient conditions for measures supported on non-compact sets and, as we shall shortly see, it is very useful to establish convergence theorems. This is one of the advantages of the SDP moment approach because, in the LP approach, one must consider moments of measures with supports on *compact* sets.

Remark 4.2. *Suppose that $\mathcal{K} \subseteq \mathbb{R}$ is the interval $[a, b]$, $[a, +\infty)$ or $(-\infty, a]$. With regard to the notation introduced by (3.14) in Section 3.4, given any integer $r \geq 1$, $\mathcal{N}_r(\mathcal{K})$ is the set of all vectors that provide the first $2r+1$ moments of a measure with finite moments of all orders and supported on $\mathcal{K}$. Now, given $r \geq 1$, let $\mathcal{S}_r(\mathcal{K}) \supseteq \mathcal{N}_r(\mathcal{K})$ be the set of vectors in $\mathbb{R}^{2r+1}$ defined by the necessary SDP moment conditions $M_r(y) \succeq 0$, and $M_{r-1}(g, y) \succeq 0$ with $x \mapsto g(x) := (b-x)(x-a)$, $x \mapsto g(x) := x - a$ or $x \mapsto g(x) := a - x$, respectively. Since these conditions are also sufficient in the case of the truncated Hausdorff moment problem (see part (a) of Theorem 4.1 above), it follows that*

$$\mathcal{S}_r(\mathcal{K}) = \mathcal{N}_r(\mathcal{K}), \quad \textit{for } \mathcal{K} = [a, b]. \tag{4.4}$$

With regard to the truncated Stieltjes moment problem, we define

$$\mathcal{T}_r(\mathcal{K}) := \left\{ y \in \mathbb{R}^{2r+1} \mid \ M_r(y) \succ 0 \text{ and } M_{r-1}(g, y) \succ 0 \right\}.$$

In view of the associated definitions, $\overline{\mathcal{T}_r(\mathcal{K})} = \mathcal{S}_r(\mathcal{K})$, while, part (b) of Theorem 4.1 implies that $\mathcal{T}_r(\mathcal{K}) \subseteq \mathcal{N}_r(\mathcal{K})$. However, these observations imply that

$$\overline{\mathcal{N}_r(\mathcal{K})} = \mathcal{S}_r(\mathcal{K}), \quad \textit{for } \mathcal{K} = [a, \infty) \textit{ or } \mathcal{K} = (-\infty, a]. \tag{4.5}$$

The result provided by (4.4) and (4.5) will play an instrumental role in the proofs of the convergence of our finite-dimensional schemes.

5. SDP RELAXATIONS FOR EUROPEAN AND ASIAN OPTIONS

We now turn our attention to the development of a special case of the general theory presented in the previous two sections that can account for the problem of pricing European and Asian options. The case of European options being developed in exactly the same way, and being much simpler because it considers measures involving only the process X, which are supported on subsets of the real line, we concentrate our analysis on Asian options.

Consider the issue of calculating the value $v_A(x_0)$ of an Asian call option that is given by (2.3). With regard to standard analyses of Asian options using PDE techniques (see Rogers and Shi (1995)), we consider the process Y defined by

$$Y_t = \frac{1}{T} \int_0^t X_s \, ds, \quad t \geq 0.$$

The first step in the direction of applying the theory of Sections 3–4 is to consider the process $Z = (X, Y)^{\mathrm{T}}$ and the stopping time $\tau \equiv T$. The exit location measure ν being defined as in (3.5), we can see that the value of an Asian call option given by (2.3) admits the expression

$$v_A(x_0) = E\left[(Y_T - K)^+ \right] = \int_{\mathbb{R}} (y - K)^+ \, \nu_Y(dy), \tag{5.1}$$

where $\nu_Y(dy) := \nu(\mathbb{R}, dy)$ is the Y-marginal of ν, namely the distribution of the random variable Y_T. Here, we have set the discounting factor ρ equal to 0 because its presence does not really affect any aspect of our analysis.

To express (5.1) as a linear combination of moments, we consider the restrictions ν_{Y1} and ν_{Y2} of the measure ν_Y on the sets $(-\infty, K)$ and $[K, +\infty)$, respectively. If we denote by $\{\nu^i_{Yl}\}$ the moments of the measure ν_{Yl}, where $l = 1, 2$, then we can see that (5.1) admits the expression

$$v_A(x_0) = \nu^1_{Y2} - K\nu^0_{Y2}.$$

Now, with regard to any of the Models 1, 2 or 3 considered for the underlying asset price dynamics in Section 2, it is possible to pre-compute the moments of the measure ν and, in particular, the moments of the measure ν_Y.

Example 5.1. *With reference to Model 2, given any integers $i, j \geq 0$, we can use Itô's formula and the fact that the associated stochastic integrals are martingales to compute*

$$E\left[X^i_t Y^j_t\right] = E\left[\int_0^t \left[\tfrac{1}{2}\sigma^2 i(i-1)X^{i-2}_s Y^j_s + \gamma\theta i X^{i-1}_s Y^j_s - \gamma i X^i_s Y^j_s + j X^{i+1}_s Y^{j-1}_s / T\right] ds\right].$$

If we define $\bar{\nu}^{(i,j)}(t) := E\left[X^i_t Y^j_t\right]$, then we can see that these relations imply

$$\frac{d}{dt}\bar{\nu}^{(i,j)}(t) = \tfrac{1}{2}\sigma^2 i(i-1)\bar{\nu}^{(i-2,j)}(t) + \gamma\theta i \bar{\nu}^{(i-1,j)}(t) - \gamma i \bar{\nu}^{(i,j)}(t) + j\bar{\nu}^{(i+1,j-1)}(t)/T, \qquad \bar{\nu}^{(i,j)}(0) = 0. \tag{5.2}$$

If we restrict i and j so that $0 \leq i + j \leq 2r$, for some integer $r > 0$, then (5.2) yield a closed system of linear ordinary differential equations. This system can easily be solved to compute the moments of the measure ν and, in particular, the moments $\{\nu^j_Y\} \equiv \{\bar{\nu}^{(0,j)}(T)\}$ of the measure ν_Y.

It is straightforward to see that a similar situation arises if we consider Models 1 or 3.

In view of the above observations, it is appropriate to consider relaxations $\mathbb{Q}^A_r$ that are developed as in (3.16). We are thus faced with the finite-dimensional SDP problem defined by

$$\mathbb{Q}^A_r(x_0, 0) \to \begin{cases} \underset{\eta_1,\eta_2}{\text{extremise}} & \eta^1_2 - K\eta^0_2, \\ \text{subject to} & \eta^j_1 + \eta^j_2 = \nu^j_Y, \quad j = 0, \ldots, 2r, \\ & M_r(\eta_1),\, M_{r-1}(g, \eta_1) \succeq 0, \\ & M_r(\eta_2),\, M_{r-1}(-g, \eta_2) \succeq 0. \end{cases}$$

Here, g is the polynomial defined by $x \mapsto g(x) := K - x$, and $M_r(\cdot), M_{r-1}(\cdot\,, \cdot)$ are the moment and localising matrices defined in Sections 4.3 and 4.4. Note that the last two constraints defining the feasible region of this SDP problem correspond to the necessary SDP moment conditions for $\{\eta_1^j,\ j=0,\ldots,2r\}$ and $\{\eta_2^j,\ j=0,\ldots,2r\}$ to be moments of measures supported on $(-\infty,K]$, and $[K,+\infty)$, respectively. Note that it is not a restriction to consider necessary moment conditions on $(-\infty,K]$ and $[K,+\infty)$ instead of $(-\infty,K)$ and $[K,+\infty)$, respectively, because we evaluate the expectation of the function $y\mapsto(y-K)^+$ that vanishes at $y=K$.

It is of interest to note that this SDP problem involves the moments of the measure ν_Y only (as opposed to the moments of the "full" measure ν). This is due to the fact that the objective criterion involves only the marginal ν_Y. Indeed, the joint distribution ν of $(X_T,Y_T)^{\mathrm{T}}$ is of relevance only as long as the pre-computation of ν_Y's moments is concerned (see Example 5.1 above).

The following result establishes the convergence of the relaxations considered when the underlying asset price dynamics are as in Model 2 or 3 in Section 2.

Theorem 5.2. *Suppose that the measure ν_Y is moment-determinate. Then*

$$\min\mathbb{Q}_r^A(x_0,0)\uparrow v_A(x_0)\quad\text{and}\quad\max\mathbb{Q}_r^A(x_0,0)\downarrow v_A(x_0).$$

The proof of this result can be found in Section 8.1 in the Appendix.

6. SDP RELAXATIONS FOR BARRIER OPTIONS

To put the problem of calculating the value of a barrier call option given by (2.4) in the context of the theory developed in Sections 3 and 4, we consider the time-price process $Z=(Y,X)^{\mathrm{T}}$, where $Y_t=t$ for all $t\geq 0$, and the stopping time τ defined by (2.5). It follows that the associated expected occupation measure μ of the process Z up to time τ that is defined by (3.4) is supported on $[0,T)\times(H,+\infty)$, while the exit location measure of Z, namely the distribution of Z_τ that is defined by (3.5), is supported on

$$[0,T]\times\{H\}\cup\{T\}\times[H,+\infty).$$

With regard to these definitions, the value $v_B(x_0)$ of the option is given by

$$v_B(x_0)=\int_{[0,T]\times\mathbb{R}}(x-K)^+\,\nu(dt,dx).$$

Again, without loss of generality, we have set the discounting factor $\rho=0$. To express this criterion as a linear combination of moments, we decompose the measure ν into the sum of three measures ν_1, ν_2 and ν_3, supported on

$$[0,T]\times\{H\},\quad\{T\}\times(H,K)\quad\text{and}\quad\{T\}\times[K,+\infty),$$

respectively. Note that, in effect, we may assume that the measures ν_1, ν_2 and ν_3 are supported on the subsets of the real line $[0,T]$, (H,K) and $[K,+\infty)$, respectively, which presents a significant computational simplification. It follows that the objective criterion admits the expression

$$v_B(x_0)=\int_{[0,T]\times\mathbb{R}}(x-K)^+\,\nu(dt,dx)=\nu_3^1-K\nu_3^0.$$

In the context of the problem considered here, the moments of the measures ν_1, ν_2 and ν_3 are not readily available, which suggests a relaxation $\mathbb{Q}_r^B$ as in (3.17). Such a relaxation involves the infinitesimal generator $\mathcal{A}$ of the process $Z = (Y, X)^{\mathrm{T}}$ that takes the form

$$f \mapsto (\mathcal{A}f)(t,x) := f_t(t,x) + \frac{1}{2}\sigma^2(x) f_{xx}(t,x) + b(x) f_x(t,x),$$

where b and σ are the data appearing in (2.1). With a view to the moment conditions (3.7) and the corresponding constraint in the definition (3.17) of the associated SDP problem, we first observe that, given a monomial $(t,x) \mapsto t^i x^j$,

$$\int_{\mathbb{R}^2} t^i x^j \, \nu(dt,dx) = H^j \int_0^T t^i \, \nu_1(dt) + T^i \int_H^K x^j \, \nu_2(dx) + T^i \int_K^{+\infty} x^j \nu_3(dx).$$

In the spirit of the notation used in Section 3, if we denote by $\{\nu^{(i,j)}\}$ and $\{\nu_l^i\}$ the moments of the measures ν and ν_l with $l = 1,2,3$, respectively, this relation reads

$$\nu^{(i,j)} = H^j \nu_1^i + T^i \nu_2^j + T^i \nu_3^j.$$

With regard to the term in (3.6) involving $\mathcal{A}$, it is straightforward to verify that, if $\{\mu^{(i,j)}\}$ are the moments of the measure μ, then

$$\int_{\mathbb{R}^2} (\mathcal{A}f)(t,x) \, \mu(dt,dx) = i\mu^{(i-1,j)} + \left[\tfrac{1}{2}\sigma^2 j(j-1) + bj\right] \mu^{(i,j)}, \tag{6.1}$$

if we adopt Model 1 in Section 2, and derive similar expressions for the cases arising if we consider Models 2 or 3. Furthermore, we can see that, given any of the models in Section 2, the moment conditions (3.7), which now take the form

$$H^j \nu_1^i + T^i \nu_2^j + T^i \nu_3^j - \sum_\beta c_\beta(i,j) \mu^{(i,j)} = 0, \quad \text{for } 0 \le i+j \le 2r, \tag{6.2}$$

for choices of the coefficients $c_\beta(i,j)$ that result from (6.1) or its analogues, present a *closed* system of *linear* equations.

Summarising the discussion above, we are faced with the finite-dimensional SDP:

$$\mathbb{Q}_r^B(0,x_0) \to \begin{cases} \underset{\eta_1,\eta_2,\eta_3,m}{\text{extremise}} & \eta_3^1 - K\eta_3^0, \\ \text{subject to} & \text{(6.3), (6.4) and (6.5) below.} \end{cases}$$

The constraints defining the feasible region of this SDP problem are

$$H^j \eta_1^i + T^i \eta_2^j + T^i \eta_3^j - \sum_\beta c_\beta(i,j) m^{(i,j)} = 0, \text{ for } 0 \le i+j \le 2r, \tag{6.3}$$

where the coefficients $c_\beta(i,j)$ are as in (6.2),

$$\left\{\begin{array}{l} M_r(\eta_1),\ M_{r-1}(g_1,\eta_1)\succeq 0,\\ M_r(\eta_2),\ M_{r-1}(g_2,\eta_2)\succeq 0,\\ M_r(\eta_3),\ M_{r-1}(g_3,\eta_3)\succeq 0,\\ M_r(m),\ M_{r-1}(g_4,m),\ M_{r-1}(g_5,m)\succeq 0,\end{array}\right. \tag{6.4}$$

with

$$\begin{gathered} t\mapsto g_1(t):=t(T-t),\quad x\mapsto g_2(x):=(K-x)(x-H),\\ x\ \mapsto\ g_3(x):=x-K,(t,x)\mapsto g_4(t,x):=t(T-t),\\ (t,x)\mapsto g_5(t,x):=x-K,\end{gathered}$$

and

$$|\eta_1^i|,\ |\eta_2^j|,\ |\eta_3^j|,\ |m^{(i,j)}|\leq A_r,\quad \text{for } 0\leq i+j\leq 2r, \tag{6.5}$$

for some *large* constant A_r. Note that, apart from imposing the necessary SDP moment conditions on the measures ν_l, $l=1,2,3$, and μ, we also bound the feasible region by means of (6.5). This extra constraint will be used in the proof of the convergence of our algorithm. Plainly, as far as numerical considerations are concerned, these bounds can be totally disregarded.

The next result is concerned with the convergence of our algorithm.

Theorem 6.1. *Suppose that the following conditions hold:*
(i) *the measure ν_3 is moment-determinate, and*
(ii) *the infinite system*

$$\left\{\begin{array}{ll} (6.3), & \textit{for every } i,j=0,1,\ldots,\\ (6.4), & \textit{for every } r=1,2,\ldots,\end{array}\right. \tag{6.6}$$

uniquely determines the moment sequence $\{\eta_3^j,\ j\geq 0\}$.
Then

$$\min \mathbb{Q}_r^B(0,x_0)\uparrow v_B(x_0)\quad \text{and}\quad \max \mathbb{Q}_r^B(0,x_0)\downarrow v_B(x_0).$$

For the proof of this theorem, we refer to Section 8.2 in the Appendix.

Remark 6.2. *In view of the structure of the problem considered and the sufficient condition associated with (3.11), note that Assumption (i) in the statement of Theorem 6.1 is satisfied if we assume that*

$$E(\exp\{c|X_T|\})<+\infty,\quad \textit{for some } c>0,$$

which is true for Models 2 and 3, but not for Model 1.

7. NUMERICAL EXAMPLES

In this section, we present numerical results obtained for the option pricing problems considered above using the SDP moment approach. In our numerical investigation, we used a modified version of the software GloptiPoly (see Henrion and Lasserre (2003)), which in turn uses the SDP solver SeDuMi (see Sturm (1999)). Although our approximation techniques have yielded

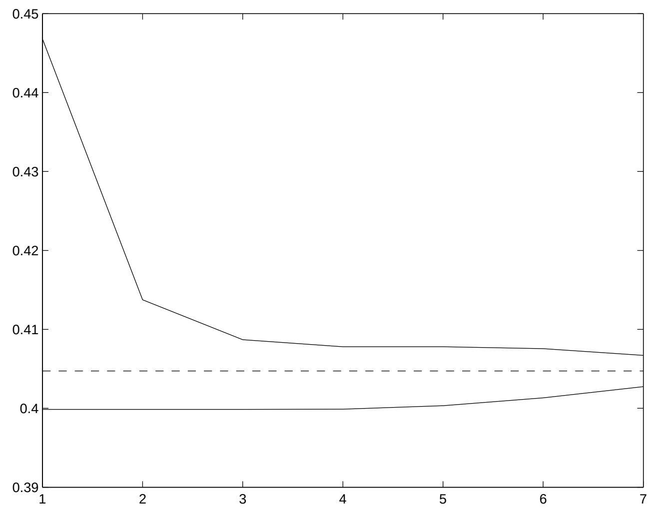

FIGURE 7.1. European Options. Model 1

very precise and strongly encouraging results, we should mention that we ran into numerical problems for certain choices of parameter values (these cases are indicated in the discussion below), despite the relatively small size of the associated SDP relaxations. Such numerical problems can possibly be attributed to the fact that SeDuMi does not perform well on sparse and degenerate problems. Other SDP solvers such as SDPT3, CSDP or SDPA might provide more stable alternatives, which is a possibility that we have not tested. However, it is not clear whether one of these solvers systematically outperforms the others, which is reflected by the fact that a new, currently available version of GloptiPoly allows the user to choose any of these SDP solvers.

In the presentation below, notice that, unless indicated, the discounting factors have been dropped.

7.1. **European Options.**

Since there exist explicit expressions for the value of European call options, we are able to evaluate in a precise way the accuracy of our numerical approximations. It turns out that the SDP approach provides tight bounds.

Figure 7.1 illustrates results for the geometric Brownian motion (Model 1). The lower bounds $\min \mathbb{Q}_r^E$ and the upper bounds $\max \mathbb{Q}_r^E$ for $1 \leq r \leq 7$, are compared to the exact value (dashed line) provided by the Black and Scholes formula. The values of the parameters in this example are

$$x_0 = 1,\ K = 0.95,\ T = 2,\ b = 0.15 \text{ and } \sigma = 0.15.$$

Despite the fact that convergence is not guaranteed for this particular case (recall Example 3.3), an accuracy of 0.5% is achieved for $r = 7$.

Similarly, for the Ornstein-Uhlenbeck process, Table 7.1 shows the precision of the SDP approach approximations for orders $r = 2, 3, 4, 5$ or, in

TABLE 7.1. European Options. Model 2

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 2$	0.87%	3.42%	8.31%	12.42%
$r = 3$	0.50%	2.77%	4.60%	6.40%
$r = 4$	0.46%	1.92%	4.18%	6.38%
$r = 5$	0.34%	1.91%	3.36%	4.42%

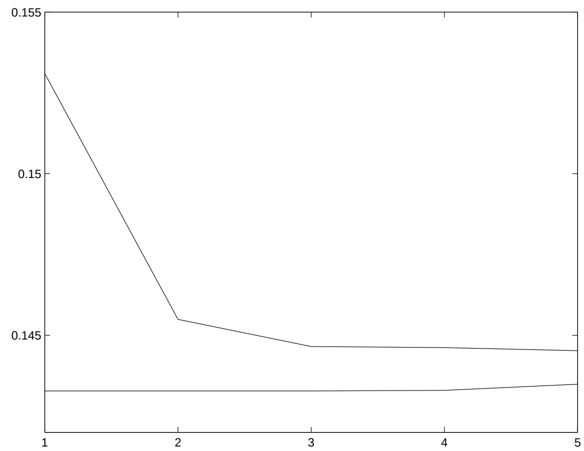

FIGURE 7.2. European Options. Model 3

other words, for a number of moments equal to $4, 6, 8, 10$. The values of the parameters are:

$$x_0 = 1,\ K = 0.95,\ T = 2,\ \gamma = 1 \text{ and } \theta = 1.1.$$

We have chosen volatilities ranging from $\sigma = 10\%$ to $\sigma = 25\%$.

Finally, upper $\{\max \mathbb{Q}_r^E\}_{r=1}^5$ and lower bounds $\{\min \mathbb{Q}_r^E\}_{r=1}^5$ are shown in Figure 7.2 when the asset price obeys the dynamics of Model 3, and for the following values of the parameters:

$$x_0 = 1,\ K = 0.95,\ T = 3,\ \gamma = 0.9,\ \theta = 1.1 \text{ and } \sigma = 0.1.$$

For $r = 5$ the relative error is 0.36%.

7.2. **Asian Options.**

We have compared the SDP approximations with the Curran (1992) bound for the fixed-strike, arithmetic-average Asian options when the underlying dynamics are the geometric Brownian motion. The Curran bound is a lower bound which is based on a conditional Jensen inequality. The method used to derive this bound is very model specific because it relies heavily on the lognormality of the marginal distributions. However, as displayed in Table 7.2, for a grid of values for (b, σ), our lower bound $\min \mathbb{Q}_5^A$

TABLE 7.2. Asian Options. Model 1

Drift $b = 0.14$

	$\sigma = 0.08$	$\sigma = 0.10$	$\sigma = 0.12$
Curran lower bound	0.16605	0.16658	0.16778
SDP lower bound	0.16642	0.16715	0.16796
SDP upper bound	0.16656	0.16772	0.16965
Relative error	0.08%	0.34%	1.01%

Drift $b = 0.16$

	$\sigma = 0.08$	$\sigma = 0.10$	$\sigma = 0.12$
Curran lower bound	0.18497	0.18518	0.18578
SDP lower bound	0.18534	0.18565	0.18704
SDP upper bound	0.18562	0.18652	0.18788
Relative error	0.15%	0.47%	0.45%

TABLE 7.3. Asian Options. Model 2

	$\sigma = 0.05$	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 1$	0.88%	3.40%	7.22%	11.94%	17.20%
$r = 2$	0.03%	0.42%	1.98%	5.31%	10.21%
$r = 3$	0.03%	0.20%	1.58%	4.69%	7.48%
$r = 4$	0.02%	0.19%	1.52%	3.95%	5.54%

improves Curran's lower bound. The other parameters are:

$$x_0 = 1,\ K = 1.05 \text{ and } T = 4.$$

To compare with the approach given by Thompson (2002), we have taken discounting into account with a discount rate $\rho = b$. Typical results are shown in Table 7.2. The relative error shown in the last line of each table is the relative length of the confidence interval with respect to its medium point: $(u-l)/(\frac{1}{2}(l+u))$, where l and u are the SDP lower and upper bound, respectively.

Tables 7.3 and 7.4 display the relative errors (computed as explained above) when the underlying asset price dynamics follow the Ornstein-Uhlenbeck and the standard square-root with mean-reversion processes, for several choices of the volatility σ, and for:

$$x_0 = 1,\ K = 0.9,\ T = 3,\ \theta = 1.2,\ \gamma = 1.1 \text{ (Table 7.3)},\ \gamma = 0.5 \text{ (Table 7.4)}.$$

Notice that very few relaxations suffice to obtain sharp bounds.

7.3. **Barrier Options.**

Table 7.4. Asian Options. Model 3

	$\sigma = 0.05$	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 5$	0.01%	0.14%	0.93%	2.80%	5.45%

Table 7.5. Barrier Options

Model 1

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 8$	2.63%	3.91%	0.52%	(1.07%)

Model 2

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 9$	1.97%	2.19%	1.36%	2.8%

Model 3

	$\sigma = 0.10$	$\sigma = 0.15$	$\sigma = 0.20$	$\sigma = 0.25$
$r = 9$	6.3%	2.85%	1.47%	0.83%

For barrier options, relative errors of the SDP approach are given in Table 7.5 for a number of values of the volatility σ. The rest of the parameters are:

$$\begin{array}{rl} \text{Model 1:} & x_0 = 1,\ K = 1,\ H = 0.8,\ T = 2,\ b = 0; \\ \text{Models 2 and 3:} & x_0 = 1,\ K = 1,\ H = 0.8,\ T = 2,\ \gamma = 1,\ \theta = 0.95. \end{array}$$

With regard to Model 1, the SDP solver ran into numerical problems for $r = 8$ and the value $\sigma = 0.25$. The relative error in the table gives the accuracy of the $r = 7$ approximation.

These results indicate that the bounds provided by the SDP approach are very tight. However, our experimentation was limited in size because of the numerical problems encountered with the SDP solver SeDuMi. As a result, we could not go beyond relaxations $\mathbb{Q}_r^B$ with $r = 8$ or $r = 9$ for barrier payoff structures.

8. APPENDIX

In this final section, we provide the proofs of the theoretical results presented in the paper.

8.1. **Proof of Theorem 5.2.**

The proof of this convergence result requires two preliminary lemmas. The following one is a standard exercise.

Lemma 8.1. *Let μ_r, for $r \geq 1$, and μ be probability measures on $\mathbb{R}$ such that $\mu_r \Rightarrow \mu$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that, for some continuous function $h : \mathbb{R} \to \mathbb{R}_+$,*

$$\lim_{|x| \to +\infty} \frac{g(x)}{h(x)} = 0 \quad \text{and} \quad \sup_{r \geq 1} \int_{\mathbb{R}} h(x)\, \mu_r(dx) < +\infty.$$

Then

$$\int_{\mathbb{R}} g(x)\, \mu_r(dx) \longrightarrow \int_{\mathbb{R}} g(x)\, \mu(dx), \quad \text{as } r \to \infty.$$

Given a Borel measure μ on $\mathbb{R}$ and a Borel set A we denote by μ_A the restriction of μ on A, namely, the measure defined by

$$B \mapsto \mu_A(B) := \mu(B \cap A), \quad B \in \mathcal{B}(\mathbb{R}).$$

Lemma 8.2. *Let μ be a finite measure on $\mathbb{R}$ which is moment-determinate. Suppose that $\{\mu_r\}$ is a sequence of finite measures on $\mathbb{R}$ whose moments converge to those of μ, that is,*

$$\lim_{r \to +\infty} \int x^\alpha\, \mu_r(dx) = \int x^\alpha\, \mu(dx), \quad \text{for all } \alpha \in \mathbb{N}. \tag{8.1}$$

Let $A \subseteq \mathbb{R}$ be a Borel set with $\mu(\partial A) = 0$ and let $\mu_{r,A}$ and μ_A be the restrictions of μ_r and μ on A, respectively. Then $\mu_{r,A} \Rightarrow \mu_A$ and the moments of $\mu_{r,A}$ converge to the moments of μ_A.

Proof. We prove the result under the assumption that μ_r and μ are probability measures. The generalisation to finite measures is straightforward.

First, we note that $\mu_r \Rightarrow \mu$ because the convergence of moments implies weak convergence of the associated moment-determinate measures (see Billingsley (1979, pp. 342–353)). Second, we prove that $\mu_{r,A} \Rightarrow \mu_A$ or, equivalently, that:

$$\int_{\mathbb{R}} f(x) I_A(x)\, \mu_r(dx) \to \int_{\mathbb{R}} f(x) I_A(x)\, \mu(dx),$$

for every bounded and continuous function f. Since the set of discontinuities of $f I_A$ has μ-measure zero, this results follows from Billingsley (1968, Theorem 5.2, p. 31).

To complete the proof, we still need to show that, for any $\alpha \in \mathbb{N}$,

$$\lim_{r \to +\infty} \int_{\mathbb{R}} x^\alpha\, \mu_{r,A}(dx) = \int_{\mathbb{R}} x^\alpha\, \mu_A(dx). \tag{8.2}$$

To this end, we fix $\alpha \in \mathbb{N}$, we consider the functions g and h defined by $g(x) := x^\alpha$ and $h(x) := 1 + x^{2\alpha}$, respectively, and we observe that

$$\sup_{r \geq 1} \int_{\mathbb{R}} h(x)\, \mu_{r,A}(dx) \leq \sup_{r \geq 1} \int_{\mathbb{R}} h(x)\, \mu_r(dx) < \infty.$$

With regard to Lemma 8.1, it follows that (8.2) is true for all $\alpha \in \mathbb{N}$, i.e., the moments of $\mu_{r,A}$ converge to the moments of μ_A. $\square$

Now we proceed to prove Theorem 5.2.

Proof. We develop the proof for the sequence $\{\min \mathbb{Q}_r^A(x_0,0),\ r \geq 1\}$. The monotonicity property follows immediately from the fact that the feasible regions decrease as r grows.

Fix any initial condition x_0, and consider any integer $r \geq 1$. The feasible region of $\mathbb{Q}_r^A(x_0,0)$ is non-empty (because the moments of the restrictions of ν_Y provide a feasible solution). Moreover, it is bounded. Indeed, from the constraints $\eta_1^j+\eta_2^j=\nu_Y^j$ we deduce that η_1^j and η_2^j are bounded for j even. It follows that the main diagonal of the positive semi-definite matrices $M_r(\eta_1)$ and $M_r(\eta_2)$ are bounded, and therefore, all of their entries are bounded as well. Since the feasible region of $\mathbb{Q}_r^A(x_0,0)$ is also closed, it follows that it is compact.

Since its feasible region is compact, $\min \mathbb{Q}_r^A(x_0,0)$ has an optimal solution. Let $\{q_{1,r}^j,\ 0 \leq j \leq 2r\}$ and $\{q_{2,r}^j,\ 0 \leq j \leq 2r\}$ be such an optimal solution. As a consequence of (4.5) in Remark 4.2, given any $\varepsilon_r > 0$, there exist measures $\nu_{1,r}$ and $\nu_{2,r}$ supported on $(-\infty,K]$ and $[K,+\infty)$, respectively, such that their moments satisfy

$$\max_{0\leq j\leq 2r} |\nu_{1,r}^j - q_{1,r}^j| \leq \varepsilon_r \quad \text{and} \quad \max_{0\leq j\leq 2r} |\nu_{2,r}^j - q_{2,r}^j| \leq \varepsilon_r.$$

If we define $\nu_{(r)} := \nu_{1,r}+\nu_{2,r}$, then we can see that

$$\int_{\mathbb{R}} (y-K)^+\, \nu_{(r)}(dy) = \nu_{2,r}^1 - K\nu_{2,r}^0$$

satisfies

$$\left| \int_{\mathbb{R}} (y-K)^+\, \nu_{(r)}(dy) - \min \mathbb{Q}_r^A(x_0,0) \right| \leq (1+K)\varepsilon_r, \tag{8.3}$$

and

$$|\nu_{(r)}^j - \nu_Y^j| \leq 2\varepsilon_r, \quad \text{for } 0 \leq j \leq 2r.$$

Now, combining the observation that the moments of $\nu_{(r)}$ converge to the moments of ν_Y as $\varepsilon_r \to 0$ with the assumption that ν_Y is moment-determinate and continuous, we can see that Lemma 8.2 implies

$$\lim_{r\to+\infty} \int_{\mathbb{R}} (y-K)^+\, \nu_{(r)}(dy) = \int_{\mathbb{R}} (y-K)^+\, \nu_Y(dy) = v_A(x_0),$$

and the result follows from (8.3). $\square$

8.2. **Proof of Theorem 6.1.**

The following lemma is a standard result and can be found in Billingsley (1979) or Diaconis (1987).

Lemma 8.3. *Let $\{\mu_k\}$ be a sequence of finite measures on $\mathbb{R}$ such that the corresponding sequences of moments are convergent to a given sequence of scalars $\{\alpha_r,\ r \geq 0\}$, i.e.,*

$$\lim_{k\to+\infty} \int x^r\, \mu_k(dx) = \alpha_r, \quad \text{for all } r \geq 0.$$

Then, there exists a measure supported on $\mathbb{R}$, *the moment sequence of which is* $\{\alpha_r,\ r \geq 0\}$.

We can now prove Theorem 6.1.

Proof. In this proof, part of which is similar to the proof of Theorem 5.2, we analyse the convergence of the sequence $\{\min \mathbb{Q}_r^B(0,x_0),\ r \geq 1\}$. It is obvious that this sequence of minima is increasing, and therefore, it suffices to establish the convergence of one subsequence.

Fix an initial condition x_0 and any $r \geq 1$. Provided that the bound A_r is large enough, the feasible region of $\mathbb{Q}_r^B(0,x_0)$ is non-empty (because the moments of the measures μ and ν_l, $l = 1, 2, 3$, provide a feasible solution), and compact. Now, let

$$\{q_{1,r}^i\}_{0 \leq i \leq 2r},\ \{q_{2,r}^j\}_{0 \leq j \leq 2r},\ \{q_{3,r}^j\}_{0 \leq j \leq 2r},\ \{q_r^{(i,j)}\}_{0 \leq i+j \leq 2r},$$

be an optimal solution of $\min \mathbb{Q}_r^B(0,x_0)$, and choose any $\varepsilon_r > 0$. With reference to (4.5) in Remark 4.2, we deduce that there exist measures $\nu_{1,r}$, $\nu_{2,r}$ and $\nu_{3,r}$, with supports on $[0,T]$, $[H,K]$ and $[K,+\infty)$, respectively, such that

$$|\nu_{1,r}^i - q_{1,r}^i|,\, |\nu_{2,r}^j - q_{2,r}^j|,\, |\nu_{3,r}^j - q_{3,r}^j| \leq \varepsilon_r, \quad 0 \leq i+j \leq 2r,$$

so that, for some constant B_r,

$$\left| H^j \nu_{1,r}^i + T^i \nu_{2,r}^j + T^i \nu_{3,r}^j - \sum_\beta c_\beta(i,j) q_r^{(i,j)} \right| \leq B_r \varepsilon_r, \tag{8.4}$$

for all i and j, and

$$|\nu_{3,r}^1 - K\nu_{3,r}^0 - \min \mathbb{Q}_r^B(0,x_0)| \leq (1+K)\varepsilon_r. \tag{8.5}$$

Now suppose that $\varepsilon_r \to 0$. Given any i and j, the sequences

$$\{\nu_{1,r}^i\},\ \{\nu_{2,r}^j\},\ \{\nu_{3,r}^j\},\ \{q_r^{(i,j)}\},$$

parametrised by $r \geq 1$ are bounded. Using a diagonal argument, we can choose a subsequence of r's such that these sequences are convergent for all i and j. To simplify the notation, let us assume that the whole sequences are converging. It follows that

$$q_r^{(i,j)} \to q^{(i,j)} \quad \text{for all } i \text{ and } j, \tag{8.6}$$

for some constants $q^{(i,j)}$, and, by Lemma 8.3, there exist measures $\tilde{\nu}_1$, $\tilde{\nu}_2$ and $\tilde{\nu}_3$ such that

$$\lim_{r \to +\infty} \nu_{l,r}^j = \tilde{\nu}_l^j, \quad \text{for } l = 1,2,3 \text{ and } j \geq 0. \tag{8.7}$$

From (8.4), (8.6) and (8.7) we conclude that $\{q^{(i,j)}\}$ and the moments of $\tilde{\nu}_l$, with $l = 1, 2, 3$, satisfy the infinite system of equations (6.6). Since this system has a unique solution (Assumption (ii) in the statement of the theorem), the moments of $\tilde{\nu}_3$ and ν_3 coincide, and therefore, $\tilde{\nu}_3 = \nu_3$ because ν_3 is moment-determinate (Assumption (i) in the statement of the theorem). The stated convergence follows from (8.5) and (8.7). $\square$

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