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In the unpublished papers of Thomas Harriot (BL Add Ms 6782 f. 67) there is a remarkable calculation. It concerns what is now known as the continuous compounding of interest, and was probably written before 1620. This article describes the background to Harriot’s calculation, and its significance. The general solution of Harriot’s problem depended on some of the important mathematical developments in seventeenth century, such as logarithms and infinite series. By 1727 John Arbuthnot was able to give a recognizably modern account of it, but the final step was the work of Leonhard Euler.

Interest upon interest

By the end of the middle ages the ethical and religious objections to usury had been set aside. Not only was it acceptable to charge interest on a loan of money, it was also acceptable to charge interest upon interest, or what is now called compound interest. Pegolotti’s *La pratica della mercatura* (Evans 1936) contains tables for compound interest on 100 lire for up to twenty years at various annual rates. This manual was compiled around 1340, but the first extant manuscript is dated 1471–2.

It must be stressed that the arithmeticians of this era did not have the convenience of our modern notation. If asked to work out the yield from \( \ell \) lire invested at \( r\% \) for \( n \) years, we would immediately write down a formula like

\[
\ell \left( 1 + \frac{r}{100} \right)^n.
\]

Pegolotti and his skilled arithmeticians did not use a formula of this kind. But they did understand the sequence of operations that it represents, and they knew how to do the calculations involved at each step. In other words, their approach was algorithmic.

The Italians also knew another way of doing the calculations represented by a formula like \( (1 + x)^n \). For example, calculating \( (1 + x)^3 \) is the same as calculating \( 1 + 3x + 3x^2 + x^3 \). A manuscript from c. 1395, apparently used in the Italian abacus schools, contains several problems about compound interest (Franci and Toti Rigatelli 1988). One example asks for the rate of interest required so that 100 lire will yield 150 lire after three years. The equivalence noted above prompted the author of the manuscript to remark that the problem leads to a special kind of cubic equation, in which the coefficients satisfy a certain relationship. The conclusion is that some cubic equations can be solved by a single cube-root operation. The occurrence of such ideas in the commercial literature of the fifteenth century foreshadows the great algebraic discoveries of the sixteenth century.
Throughout the sixteenth century the financial transactions involved in commerce became increasingly complicated. The period of a loan was often not an exact number of years, partial payments or repayments were allowed, and the rate of interest varied from place to place. The practice of assuring income by means of an annuity created similar problems. Consequently there was a demand in mercantile circles for tables produced by persons skilled in arithmetic.

The *Tafelen van Interest* of Simon Stevin (1582) contained the results of calculations involving compound interest, discount, and annuities. A work of a different kind was the same author’s *De Thiende* (1585). In this book he advocated the use of decimal fractions, and explained why they were superior for computational purposes to the ‘vulgar’ fractions then in use. His notation was not fully developed, but it was an idea whose time had come, and decimals soon became the standard tool of arithmetic. The fact that Stevin had previously produced commercial tables led Augustus de Morgan (1847, 27) to conclude that this was the reason for his advocacy of decimal fractions: it was a direct result of the convenience of their use in tables of compound interest.

Stevin’s work on decimals was soon translated into French, and was read by the English mathematician Thomas Harriot. As is now well known, he produced an enormous amount of original mathematics, none of which was published in his own lifetime. In these manuscripts he routinely used decimal fractions (but not the decimal point), and notation that is similar in many respects to what we use today.

Harriot’s manuscripts are currently being made available as part of the European Cultural Heritage Online (ECHO) initiative; see also Stedall (2007). Among them there is one (Figure 1) that deals with what happens when compound interest is added ‘continuously’, a topic that was later to be linked with some significant mathematical discoveries of the seventeenth century.

Harriot denotes the annual rate of interest by $1/b$; for example $b = 20$ would correspond to 5%. So one pound invested for one year will yield $1 + 1/b$ pounds. If the investment lasts two years, with interest compounded annually, the yield will be $(1 + 1/b)^2$. Similarly, after seven years the yield will be $(1 + 1/b)^7$. Harriot knew that this could be calculated by means of the expansion

$$1 + 7/b + 21/b^2 + 35/b^3 + 35/b^4 + 21/b^5 + 7/b^6 + 1/b^7,$$

where the numbers 1, 7, 21, 35, 35, 21, 7, 1 are known as *binomial coefficients*. There are two simple methods of calculating the binomial coefficients. The first is by means of the *arithmetical triangle*, which was well known in the sixteenth century. Harriot was obviously familiar with it, because it occurs in many places in his manuscripts (Beery 2009; Beery and Stedall 2009). This method depends on the relationship between the coefficients appearing in $(1 + x)^m$ and those in $(1 + x)^{m-1}$. The second method depends on the fact that the binomial coefficients can be interpreted as numbers of combinations. This leads to an algorithm in which the coefficient of $x^r$ in $(1 + x)^m$ is obtained from the coefficient of $x^{r-1}$, a rule that was known to the Indian mathematicians of an earlier era (Plofker 2007, 446). Harriot was the first to write down this rule as a formula in the modern style: the coefficient of $x^r$ in the expansion of $(1 + x)^m$ is

$$\frac{m(m - 1) \cdots (m - r + 1)}{1 \cdot 2 \cdots r}.$$
The stimulus for Harriot’s work on compound interest was the observation that if interest is added more frequently than once a year, but at the same equivalent rate, then the yield will be greater. This fact was well known in commercial practice (Lewin 1981). In algebraic language, if the interest is added twice yearly at the rate of $1/2b$, then at the end of one year the amount is $(1 + 1/2b)^2$, which is slightly more that $1 + 1/b$. If interest is added three times a year, the amount is $(1 + 1/3b)^3$, which is greater still, and so on. The question is: what happens when the compounding is done with greater and greater frequency, but at the same equivalent rate? In particular, can the yield become arbitrarily large?

The sum of interest upon interest continually for every instant in seven years with the principal of 100$^c$ after the rate of 10 in the 100 for the year.

The stimulus for Harriot’s work on compound interest was the observation that if interest is added more frequently than once a year, but at the same equivalent rate, then the yield will be greater. This fact was well known in commercial practice (Lewin 1981). In algebraic language, if the interest is added twice yearly at the rate of $1/2b$, then at the end of one year the amount is $(1 + 1/2b)^2$, which is slightly more that $1 + 1/b$. If interest is added three times a year, the amount is $(1 + 1/3b)^3$, which is greater still, and so on. The question is: what happens when the compounding is done with greater and greater frequency, but at the same equivalent rate? In particular, can the yield become arbitrarily large?
Harriot supposes that the interest is added \( n \) times per annum, at the equivalent rate of \( 1/nb \), for seven years. In that situation, one pound invested for seven years will yield \((1 + 1/nb)^7\) pounds. If \( n \) is allowed to become arbitrarily large, the result is continuous compounding. Harriot first applied the binomial formula to the cases \( n = 1, 2, 3 \), writing down the expansions of

\[
b^2(1 + 1/b)^7, \quad b^2(1 + 1/2b)^{14}, \quad b^2(1 + 1/3b)^{21}.
\]

He then considered the expansion of \( b^2(1 + 1/nb)^{7n} \) for general \( n \), putting \( b = 10 \) so that the annual rate was 10% (the maximum legal rate at that time) and the multiplier \( b^2 \) conveniently represented 100 pounds. For example, the fourth term is

\[
\frac{(7n - 2)(7n - 1)(7n)}{n^3 \cdot 6b}.
\]

When \( b = 10 \) and \( n \) is large this is approximately \( 7^3/60 = 343/60 \), as shown in the manuscript. The later terms become small very quickly, due to the presence in the denominator of powers of 10 and factorials. Thus Harriot correctly inferred that the yield does not increase unboundedly. He calculated that 100 pounds invested at 10% for seven years, with continuous compounding, will yield 201 pounds 7 shillings and 6 pence, plus .06205 pence. Remarkably, this answer is very close to the true value, which is 201 pounds 7 shillings and 6 pence, plus .06458... pence. The phrase ‘not 7/100’ in Harriot’s manuscript suggests that he had estimated the sum of the terms that he ignored, and was confident that the true value was very close to his.

We now know that the true value is given by a simple and elegant formula, which can be implemented by pressing a few keys on an electronic calculator, but this method was not available to Harriot. The object of this article is to describe how the relevant mathematics was developed.

**The hyperbolic logarithm**

The arithmetical algorithms for multiplication and division are significantly more intricate than those for addition and subtraction. In the early seventeenth century the time was ripe for an invention which, in modern terms, we describe on the following lines. For every number \( s \) assign a corresponding number \( L(s) \), in such a way that multiplying \( s \) and \( t \) corresponds to adding \( L(s) \) and \( L(t) \):

\[
L(st) = L(s) + L(t).
\]

If tables of \( s \) and \( L(s) \) are available, then \( st \) can be found by first adding \( L(s) \) and \( L(t) \), and then finding the number \( st \) that corresponds to the sum.

The first person to devise a workable system of this kind was a Scottish nobleman, John Napier. His system was published in 1614. It was quite complicated, and was designed specifically for calculations in trigonometry. But to Napier goes the credit for the original invention, and for the word *logarithm*. (For the avoidance of doubt it must be stressed that Napier did not invent the so-called *natural* logarithm, although this claim can be found in some older books on the history of the subject.)

A few years later Henry Briggs, professor at Gresham College in London, took up Napier’s idea. When the two men met they discussed the best way to set up a system that would facilitate the making of the necessary tables. Since multiplying by 1 has no effect, this should be represented by adding 0: in other words, the logarithm
of 1 should be 0. But what number should be chosen so that its logarithm is 1? Since numbers and fractions were now being written in decimal notation, the obvious choice was 10. That was the origin of what became known as common logarithms. For nearly 400 years tables of common logarithms were in constant use in all branches of science where arithmetical calculations were required.

Compiling tables of common logarithms was hard work, but their potential usefulness was immediately recognized, and by the end of the 1620s such tables were available for all integers up to 100,000. In practical calculations the final step is to find a number whose logarithm is known, and this can be done by scanning the table of logarithms. But it is more convenient to use a table drawn up specifically for that purpose—a table of antilogarithms. An extensive table of this kind was compiled by John Pell and William Warner in the 1640s, using methods suggested by Harriot (Malcolm and Stedall 2005). Nowadays we are accustomed to the idea that if \( y \) is the common logarithm of \( x \) then \( x = 10^y \), so the Pell–Warner table is just a table of values of the exponential function \( 10^y \), but that was not how it was seen in the 1640s. One of the themes of this article is the gradual growth of understanding of the inverse functional relationship between logarithms and exponentials.

The tables of logarithms and antilogarithms proved so useful that there was considerable interest in alternative ways of calculating them. The key idea sprang from a discovery of Gregoire de Saint-Vincent (1647). Saint-Vincent wanted to go down in history by solving the classical problem of squaring the circle, but in that aim he failed (understandably). So it was a humbler problem on areas that has led to his lasting fame. While trying to calculate the area under the hyperbola \( xy = 1 \) Gregoire discovered a curious fact: the area between the ordinates \( x = p \) and \( x = q \) depends only on the ratio \( p/q \). He gave two proofs of this property, both of which are essentially correct.

A pupil of Gregoire’s, Alphonse Antonio de Sarasa, noticed that the ratio property is equivalent to an even simpler one. If \( A(s) \) denotes the area between the ordinates 1 and \( s \), then

\[
A(st) = A(s) + A(t).
\]

In other words, the area under the hyperbola satisfies the law of the logarithm, the very rule that makes logarithms such a useful practical device. Henceforth the name hyperbolic logarithm was applied to the area \( A(s) \).

A significant advance was the publication of a book by Nicolaus Mercator (1668) containing rules for calculating hyperbolic logarithms by means of an infinite series. The publication of Mercator’s book alarmed Isaac Newton, because he had previously discovered a similar method, although he had not published it. In 1671 he wrote up his results in a memoir intended for publication, but in the event that too was not published until after his death. In his 1671 manuscript (Whiteside 1969, 231–233) Newton remarked on the relationship between his results and the calculation of common logarithms. After calculating the hyperbolic logarithm of 2 to many decimal places, he observed that such calculations could be applied to the construction of what he called ‘the canon of logarithms’. By this he meant tables of common logarithms. He stated that the common logarithm of any number can be obtained by multiplying its hyperbolic logarithm by

\[
0.43429 44819 03251 8,
\]

which is the reciprocal of the hyperbolic logarithm of 10.
The inverse problem

Although Newton’s 1671 manuscript was not published at the time, parts of it gradually entered the public domain. In 1676, in the first of his famous letters to Leibniz/Oldenbourg (Turnbull 1960), Newton stated his binomial theorem for fractional exponents in the form of an infinite series for the expression \((P + PQ)^{m/n}\). He asserted that, taking the first term to be \(A = P^{m/n}\), the successive terms \(B, C, D, \ldots\) can be calculated recursively by the rule

\[
B = \frac{m}{n} AQ, \quad C = \frac{m-n}{2n} BQ, \quad D = \frac{m-2n}{3n} CQ, \ldots.
\]

If the parameters are chosen suitably, the terms get small very quickly. That was essentially Harriot’s method, although in his case the exponent was an integer and the series was finite (but its length was unbounded).

Newton thought it necessary to explain the meaning of the exponent \(m/n\) at some length, by relating it to \(n\)th roots and \(m\)th powers, where \(n\) and \(m\) are integers. This adds weight to the suggestion that the notion of ‘raising a number to the power \(y\)’ (and the corresponding functional relationship) was not fully understood at that time. However, Newton had made another discovery that was to be crucial in setting up the theory of exponents.

One of Newton’s favourite methods for solving an equation was to write one of the variables in the form of a power series in another variable, and then ‘equate coefficients’. This will work (sometimes) even when the original relationship is itself a power series, such as the series for the hyperbolic logarithm of \(1 + x\) obtained by Mercator and Newton himself:

\[
y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.
\]

In this case \(x = 0\) when \(y = 0\), so we can try \(x = ay + by^2 + cy^3 + dy^4 + \cdots\). Substituting for \(x\) on the right-hand side, and equating coefficients of powers of \(y\), we obtain a set of equations for the coefficients \(a, b, c, d, \ldots\). These can be solved, giving \(a = 1, b = \frac{1}{2}, c = \frac{1}{6}, d = \frac{1}{24}\), and so on. It follows that \(1 + x\), the antilogarithm of \(y\) is equal to

\[
1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots,
\]

where the coefficient of \(y^r\) is \(1/(1.2.3\ldots r)\). In this way Newton obtained what we now call the exponential series.

Newton’s technique was published in John Wallis’s Treatise of Algebra (1685). Wallis gave many examples and, in particular, in Example VII of Chapter XCV (p 343) he showed how ‘a Logarithm being given, we may find the Number to which it belongs’. He chose points \(B = (0, a)\) and \(D = (0, x)\) and stated, without proof, an infinite series for the length \(BD = x - a\) in terms of the area \(z\) of the region between \(BD\) and the hyperbola \(xy = ab\):

\[
BD = \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \cdots.
\]

Putting \(a = b = 1\) we obtain the exponential series.
In the 1680s Jakob (James) Bernouilli also studied the theory of infinite series. His derivation of the exponential series was not based on inverting the logarithmic series, but arose from the mathematics of continuous compounding—the method first used by Harriot. It is acknowledged (Hofmann 2008) that in the winter of 1690–91 Bernouilli obtained the exponential series from the binomial series ‘in a bold but formally unsatisfactory manner’. His papers were collected in the *De seriebus infinitis*, published posthumously in 1713. In the discussion of continuous compounding (p 303) the exponential series appears in the form

\[ 1 + \frac{x}{s} + \frac{x^2}{1.2.s^2} + \frac{x^3}{1.2.3.s^3} + \cdots. \]

At this point Bernouilli does not identify the expression as the inverse of a hyperbolic logarithm, although elsewhere in his papers he had observed that the exponential series can be regarded as the inverse of the logarithmic series.

Another derivation of the exponential series from the binomial series, but arising from a different problem, is due to Abraham de Moivre. In Problem V of his *Doctrine of chances* (1718), de Moivre set out to find the ratio \( x/q \), where \( q \) can be arbitrarily large and \( x \) is related to \( q \) by the equation

\[ 1 + \frac{1}{q} = 2. \]

His method was to expand the left-hand side by ‘Sir Isaac Newton’s theorem’, obtaining

\[ 1 + \frac{x}{q} + \frac{x(x-1)}{1.2.q^2} + \frac{x(x-1)(x-2)}{1.2.3.q^3} + \cdots. \]

When \( x/q \) is fixed, but \( q \) is allowed to be large, he obtained the series

\[ 1 + \frac{x}{q} + \frac{x^2}{2q^2} + \frac{x^3}{6q^3} + \cdots. \]

He remarked that the sum of this series is ‘the number whose hyperbolic logarithm is \( x/q \)’. Since the sum must be 2 in this case, the answer is that \( x/q \) is the hyperbolic logarithm of 2. The calculation shows that de Moivre knew that he needed Newton’s version of the binomial theorem, since the exponent \( x \) might not be an integer. Also, he understood the relationship between the exponential series and the hyperbolic logarithm.

**Continuous compounding according to John Arbuthnot**

It is clear from the evidence given above that, in the early years of the eighteenth century, mathematicians were familiar with the exponential series

\[ 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \cdots. \]

However, its sum was still being described by the cumbersome phrase ‘the number whose hyperbolic logarithm is \( x \)’. An instance of this practice, specifically referring to continuous compounding, occurs in a little-known book written by a slightly better-known mathematician, John Arbuthnot (1727).
Arbuthnot was born in Scotland and moved to London in 1691. Here he translated a book by Huygens on games of chance and, incidentally, introduced the word *probability* into the language of mathematics. He was also qualified in medicine, and was appointed physician to Queen Anne in 1705. He became a Fellow of the Royal Society, and was a member of the commission appointed to investigate the Newton–Leibniz controversy. His book *Tables of ancient coins, weights and measures* is mainly concerned with Greek and Roman weights and measures, a subject that had previously engaged several other scholars. The book was highly regarded by some contemporaries, but its central contention, concerning the mass of the Roman denarius, is now discredited in numismatic circles.

For all its failings, Arbuthnot’s book is a mine of information on the writings of classical authors. For example, ‘Chapter 22, Of the interest of money’, begins with a list of references to the various rates of interest charged in Roman times. Then, with a sudden change of key, Arbuthnot writes:

A monthly interest is higher than an annual one of the same rate, because it operates by compound interest. This suggests to me the following Problem. The rate per annum being given, to find the greatest Sum which is to be made of one Pound, supposing the interest payable every indivisible moment of time.

His argument proceeds as follows. If the interval of compounding is $t$, we require $(1 + rt)^{1/2}$ which ‘by Newton’s Theorem’ is

$$1 + r + \frac{1 - t}{2} r^2 + \frac{1 - 3t + 2t^2}{6} r^3 + \frac{1 - 6t + 11t^2 - 6t^3}{24} r^4 + \ldots.$$ 

Supposing that $t$ denotes ‘an indivisible Moment of time, and therefore it is equal to nothing’, he obtains

$$1 + r + \frac{r^2}{2} + \frac{r^3}{6} + \frac{r^4}{24} + \ldots.$$ 

He works out the answer for 10,000,000 pounds invested for one year at 6%, obtaining 10,618,365.4 pounds. Finally, he remarks that:

the Problem is likewise solv’d by a Table of Logarithms, as follows. Multiply $r$ into \(0.43429448\ldots\), viz. the Reciprocal of the Hyperbolick *Logarithm* of 10; and the product will be the Logarithm of the number requir’d, which will be found by the Common Tables.

Essentially, Arbuthnot had identified the sum of his series as ‘the number whose hyperbolic logarithm is $r$’. But he could not identify it as $e^r$, because the symbol $e$ was still unknown. It is thought that $e$ was first used by Leonhard Euler in an unpublished letter written in 1728, the year after Arbuthnot’s book was published.

In 1731 Euler referred to it as ‘the number whose hyperbolic logarithm is 1’, and it first occurs in print in the first volume of his *Mechanica* (1736). By introducing this notation, Euler dispelled the mists that had formed over many decades. It was now clear that the sum of the exponential series with parameter $x$ (the number whose hyperbolic logarithm is $x$) is the $x$th power of the sum of the series when $x = 1$ (the number $e$). In other words, the exponential function is the inverse of the
logarithmic function. Harriot’s result on continuous compounding is an instance of the rule that, for all \( x \),

\[
\left(1 + \frac{x}{n}\right)^n
\]

approaches \( e^x \) as \( n \) becomes large. This is such a neat result that Euler chose to define \( e^x \) in this way.

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