The Second Subconstituent of some Strongly Regular Graphs

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Abstract

This is a report on a failed attempt to construct new graphs X^h that are strongly regular with parameters $((h^4+3h^2+4)/2,\ h^2+1,\ 0, 2)$. The approach is based on the assumption that the second subconstituent of X^h has an equitable partition with four parts. For infinitely many odd prime power values of h we construct a graph G^h that is a plausible candidate for the second subconstituent. Unfortunately we also show that the corresponding X^h is strongly regular only when $h = 3$, in which case the graph is already known.

1. Introduction

We should like to be able to construct graphs X that have the following properties:

- X is regular with degree k;
- X is triangle-free;
- any two non-adjacent vertices have just two common neighbours.

Standard calculations with eigenvalues and multiplicities show that $k-1$ must be a square h^2, with h not congruent to 0 modulo 4. In the standard terminology [5,11], X is a strongly regular graph with parameters

$$((h^4+3h^2+4)/2,\ h^2+1,\ 0, 2).$$

Although there are infinitely many possibilities, only a few graphs are known, even when the number of common neighbours is allowed to be an arbitrary constant $c \neq 2$ [10]. The topic is particularly interesting because the known graphs are associated with remarkable groups.

For each vertex v of X we denote by $X_1(v)$, $X_2(v)$ the sets of vertices at distance $1, 2$ respectively from v. We call the graph induced by $X_2(v)$ the *second subconstituent* of X. We shall usually write it as X_2, although there is no reason why it should be independent of v. The results in [2,11] establish that X_2 is a connected graph of degree $k-2$ with diameter 2 or 3. Furthermore, the only numbers that can be eigenvalues of X_2 are: $k-2$, -2 and the eigenvalues λ_1, λ_2 of X.

There are three known examples: $h = 1, 2, 3$ corresponding to $k = 2, 5, 10$. When $k = 2$ we have the 4-cycle. When $k = 5$ we have a graph with 16 vertices known as the Clebsch graph, a name suggested by Coxeter [6] because the graph represents a geometrical configuration discussed by Clebsch. For this graph X_2 is the Petersen graph. When $k = 10$ we have the Gewirtz graph with 56 vertices. It can be represented by taking the vertices to be a set of 56 ovals in PG(2, 4), and making two vertices adjacent when the corresponding ovals are disjoint. (An historical note about this graph is appended to this paper.) The algebraic properties of the Gewirtz graph have been studied in detail by Brouwer and Haemers [4], and a list of the 56 ovals may be found at [12]. Using this list, it can be verified that the 45 vertices of X_2 are partitioned with respect to their distance from a given vertex $w \in X_2$ as $\{w\} \cup P \cup S$, where $|P| = 8$ and $|S| = 36$. The 36 vertices are of two types. One set Q of 16 vertices has the property that each is adjacent to 1 vertex in P, while the complementary set R of 20 vertices is such that each is adjacent to 2 vertices in P. Further analysis shows that the partition $\{w\} \cup P \cup Q \cup R$ is equitable [11, p.195], with the intersection numbers given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 8 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 5 & 5 & 2 \end{pmatrix}.$$

The eigenvalues of this intersection matrix are $8, 2, -2, -4$. The numbers 2 and -4 are the eigenvalues of the Gewirtz graph, while 8 ($= k - 2$) and -2 are the only other eigenvalues permitted by the general theorem mentioned above. On this basis, it seems worthwhile to investigate possible generalizations.

2. Properties of the second subconstituent

We begin by constructing a suitable intersection matrix for the second subconstituent, for a general value of k.

Let $G = (V, E)$ be a graph with vertex-set $V = K^{(2)}$, the set of unordered pairs of elements of a set K, where $|K| = k$, and suppose the edge-set E is defined so that the following conditions hold.

C1 *For each vertex $ab \in V$ there is a partition of V with four parts,*

$$V = \{ab\} \cup P_{ab} \cup Q_{ab} \cup R_{ab}$$

such that $P_{ab} = \{cd \mid \{ab, cd\} \in E\}$ and $Q_{ab} = \{cd \mid |ab \cap cd| = 1\}$.

C2 *This partition is equitable with intersection matrix*

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k-2 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & k-5 & k-5 & k-8 \end{pmatrix}.$$

The fact that P_{ab} and Q_{ab} are disjoint implies that if $\{ab, cd\}$ is an edge, then a, b, c, d are distinct. It follows from the definition of Q_{ab} that $|Q_{ab}| = 2(k-2)$. The other parameters then imply that

$$|P_{ab}| = k - 2, \quad |R_{ab}| = \frac{1}{2}(k-2)(k-5).$$

It is easy to see that a graph G with the given properties would be triangle-free, regular with degree $k - 2$, and have diameter 2.

The conditions are clearly meaningful only when $k \geq 8$. Since we know that the corresponding strongly regular graphs can exist only when $k = h^2 + 1$, and that they do exist when $k = 2$ and $k = 5$, we shall assume that $k \geq 10$ in what follows.

Theorem 1 Let G be a graph satisfying conditions **C1** and **C2**, and let C be the bipartite graph (claw) with vertex-set $\{*\} \cup K$. Denote by $G \oplus C$ the graph formed from the disjoint union of G and C by adding edges joining each vertex ab in G to the vertices a and b in C. Then $G \oplus C$ is a strongly regular graph with degree k, it is triangle-free, and each pair of non-adjacent vertices has just 2 common neighbours.

Proof Since G is regular with degree $k-2$, $G \oplus C$ is regular with degree k. Since C is a claw, there are no triangles containing the vertex $*$. A triangle containing the vertex $a \in K$ would have to contain vertices ab and ab' in G, but these vertices are not adjacent. Finally, a triangle containing the vertex ab would lie wholly in G, but G is triangle-free.

It remains to check that any two non-adjacent vertices in $G \oplus C$ have exactly two common neighbours. If the two vertices are $*$ and ab, the neighbours are a and b, and if the two vertices are a, b, the common neighbours are $*$ and ab. If the two vertices are of the form ab and ac, then $ac \in Q_{ab}$, and the neighbours are a and the unique vertex in P_{ab} that is adjacent to ac. If the two vertices are of the form ab and cd, where $cd \in R_{ab}$, then the neighbours are the two vertices in P_{ab} that are adjacent to cd. □

3. Construction of triangle-free graphs G^q

We attempt to construct graphs satisfying conditions **C1** and **C2**. We know that $k-1$ must be a square, say h^2. Let $h = q$, where q is a prime power, so that $k = q^2 + 1$. Take K to be the set of points on the projective line $PG(1, q^2)$, that is

$$K = \mathbb{F}_{q^2} \cup \{\infty\} = \langle t \rangle \cup \{0, \infty\}.$$

Here $\mathbb{F}_{q^2}$ is the finite field of order q^2, ∞ is the conventional 'point at infinity', and t is a primitive element of the field, so that $\langle t \rangle$ is a cyclic group of order $q^2 - 1 = k - 2$. The group $\mathrm{PGL}(2, q^2)$ of projective linear transformations acts 3-transitively on K, and hence transitively on the unordered pairs ab in $K^{(2)}$, and the stabilizer of the pair 0∞ is generated by $x \mapsto tx$ and $x \mapsto x^{-1}$. When q is an odd prime power (so that k is even) its orbits on $K^{(2)}$ are as follows:

$$\{0\infty\}, \quad O_0 = \{0x \mid x \in \langle t \rangle\} \cup \{\infty x \mid x \in \langle t \rangle\},$$

and $\frac{1}{2}(k-2)$ orbits of the form

$$O_v = \{vx\, x \mid x \in \langle t \rangle\} \quad (v = t, t^2, \ldots, t^{(k-2)/2}).$$

The orbit O_0 has size $2(k-2)$, the orbits O_v $(v \neq -1)$ have size $k-2$, and the orbit O_{-1} has size $(k-2)/2$. Note that when q is a power of 2 the orbit-partition takes a slightly different form: this is consistent with the fact that no construction can work when $k-1 = h^2$ with $h \equiv 0 \pmod 4$, by the feasibility conditions. (The exceptional case $q = 2$ has already been covered.)

The idea of the following construction is to define a graph with vertex-set $V = K^{(2)}$ such that for a suitable value of u, the partition postulated in condition **C1** (taking the vertex ab to be 0∞) is given by

$$P = O_u, \qquad Q = O_0, \qquad R = \bigcup_{v \neq u,0} O_v.$$

The construction depends on the *cross-ratio*, which is defined for any points $a, b, c, d \in K$ by the rule

$$(ab|cd) = \frac{(a-c)(b-d)}{(a-d)(b-c)},$$

with the usual conventions about ∞. The cross-ratio is 1 if and only if $a = b$ or $c = d$ or both, and so this value does not occur when ab and cd are in $V = K^{(2)}$. Given the unordered pair of unordered pairs ab and cd, the cross-ratio $(ab|cd)$ takes only two values ρ and ρ^{-1}, which it is convenient to write in the form $(ab|cd) = \rho^{\pm}$.

Let $V = K^{(2)}$, and given $u \in \langle t \rangle, u \neq \pm 1$ define E_u to be the set of pairs $\{ab, cd\}$ such that $(ab|cd) = u^{\pm}$. Since $(0\infty \mid ux\, x) = u$, it follows that in the graph $G_u = (V, E_u)$ the set of vertices adjacent to 0∞ is the orbit O_u, as defined above.

We consider the possibility that, for a suitable value of u, G_u is a graph in which the partition given above is equitable, with the intersection matrix M as in condition **C2**. The first step is to ensure that $M_{PP} = 0$, which means that G_u is triangle-free.

Lemma Let

$$\Omega \ = \ \{v \in K \mid v = (x + x^{-1} - 1)^{\pm} \text{ for some } x \in \langle t \rangle, x \neq 1\}.$$

Then the graph G_u is triangle-free if and only if u is not in Ω.

Proof The group $\mathrm{PGL}(2, q^2)$ acts as a group of automorphisms of G_u since it preserves cross-ratios, and so G_u is vertex-transitive. Hence we need only consider the possibility of triangles containing a given vertex, say 0∞. The stabilizer of 0∞ contains $x \mapsto tx$ and $x \mapsto 1/x$, and so we can assume that two edges of the triangle are $\{0\infty, u\ 1\}$ and $\{0\infty, ux\ x\}$, with $x \neq 1$. Now, the vertices $u\ 1$ and $ux\ x$ are adjacent if and only if

$$\frac{(ux-u)(u-1)}{(ux-1)(x-u)} = u^{\pm}.$$

After some rearrangement, this reduces to

$$u = (x + x^{-1} - 1)^{\pm}.$$

In other words, there is a triangle if and only if u is in Ω.

□

Since $x + x^{-1} - 1$ is symmetrical with respect to inverting x, the set Ω can be found by calculating at the values of $x + x^{-1} - 1$ for $x = t^j$, $j = 1, 2, 3, \ldots, (k-2)/2$. For example, when the field is $\mathbb{F}_9$ with the primitive element t satisfying $t^2 + t + 2 = 0$, we have the table

j	x	$x + x^{-1} - 1$	$(x + x^{-1} - 1)^{-1}$
1	t	$2t$	$2 + 2t$
2	$1 + 2t$	2	2
3	$2 + 2t$	$1 + t$	t
4	2	0	∞

Since $1 + 2t$ and its inverse $2 + t$ are not in Ω, we conclude that the graph G_{1+2t} is triangle-free.

Generally, the special orbit O_u could be any one of the O_v except O_0 and O_{-1}, thus $\{u, u^{-1}\}$ could be any one of the $(k-4)/2$ pairs $\{t^j, t^{-j}\}$, $j = 1, 2, \ldots, (k-4)/2$. As a working definition let us say that u is *admissible* if (1) $u =\neq 0, \infty, -1, 1$, and (2) $\{u, u^{-1}\} \notin \Omega$. At first sight it appears that as many as $(k-2)/2$ pairs are not admissible, because they are in Ω, but fortunately things are not so bad.

Theorem 2 Suppose that q is an odd prime power and there is an element ζ in $\mathbb{F}_{q^2}$ such that $\zeta^2 = 3$. Then there is at least one u of the form t^j with $j \in \{1, 2, \ldots, (k-4)/2\}$ such that u is not in Ω, and hence the graph G_u is triangle-free.

Proof When q is odd q^2 is congruent to 1 mod 4, and so there is an element ι such that $\iota^2 = -1$. Then

$$\iota + \iota^{-1} - 1 = (\iota + \iota^{-1} - 1)^{-1} = -1,$$

which means that the 'pair' $\{-1,-1\}$ occurs in Ω. But $u=-1$ is not admissible anyway, and so the number of non-admissible pairs in Ω is effectively reduced to at most $(k-4)/2$.

Similarly, if we can find an x such that $x+x^{-1}-1=0$ then the pair $\{0,\infty\}$ will occur in Ω, and since this pair is also not admissible, the number of non-admissible pairs in Ω will be reduced to at most $(k-6)/2$. It is easy to see that this happens if there is an element $\zeta \in \mathbb{F}_{q^2}$ such that $\zeta^2=3$. In that case, let

$$\theta = 2^{-1}(1+\iota\zeta), \quad \text{so that} \quad \theta^{-1} = 2^{-1}(1-\iota\zeta) \quad \text{and} \quad \theta+\theta^{-1}-1=0.$$

Hence an admissible u must exist. □

For example, in the field $\mathbb{F}_{25}$ with the primitive element t satisfying $t^2+t+2=0$, we have $\iota=2$, $\zeta=4+3t$, $\theta=2+3t$. Hence the pair $\{2+3t, 4+2t\}$ is admissible. A complete check shows that there are two other admissible pairs $\{1+2t, 2+4t\}$, and $\{3+t, 4+3t\}$.

It is easy to see that there are infinitely many fields $\mathbb{F}_{q^2}$ which contain an element with $\zeta^2=3$, for example by applying the law of quadratic reciprocity. We do not pursue this matter, since the final step is to rule out all fields except $\mathbb{F}_9$, where explicit calculation shows that the construction works.

4. Failure of the construction in general

We now know that in many cases a graph G_u can be constructed satisfying the condition $M_{PP}=0$. But it remains to check that the other entries of M are correct. In fact, several of them can be verified, but it turns out that the condition $M_{PR}=2$ cannot be satisfied in general.

Theorem 3 Let G_u be defined for an odd prime power q as in Section 3, and let the partition $\{0\infty\} \cup P \cup Q \cup R$ of the vertices of G_u be as stated there. Then this partition is equitable with an intersection matrix of the form required by condition **C2** only when $q=3$.

Proof We shall show that the condition $M_{PR}=2$ cannot hold, except when $q=3$.

A typical vertex in R is $vw\,w$ where $v \neq u, 0, \infty, 1$ and $w \in \langle t \rangle$. This vertex is adjacent to the vertices $ux\,x$ in P for which $(ux\,x \mid vw\,w) = u$ or u^{-1}. These two equations can be written as quadratics in x:

$$ux^2-(1+u)vwx+vw^2=0, \qquad ux^2-(1+u)wx+vw^2=0,$$

and their discriminants are

$$\Delta^+ = w^2((1+u)^2v^2 - 4uv), \qquad \Delta^- = w^2((1+u)^2 - 4uv).$$

If there are just two solutions for x then either (1) exactly one of Δ^+, Δ^- is a square in $\mathbb{F}_{q^2}$, or (2) Δ^+ and Δ^- are both zero.

Consider first the case $v = -1$. Here $\Delta^+ = \Delta^-$ and so their common value must be zero. That is,

$$(1 + u^2) + 4u = 0.$$

Then for all $v \neq -1$

$$\Delta^+ = w^2(-4uv^2 - 4uv) = vw^2(-4u - 4uv) = v\Delta^-.$$

Hence in order that exactly one of Δ^+, Δ^- is a square, v must be a non-square, and this must hold for all the orbits $O_v \subseteq R$ except O_{-1}. In the case $q = 3$ we chose $u = 1 + t = t^2$, so $R = O_t \cup O_{t^3} \cup O_{-1}$, and the condition is satisfied. But for $q \geq 5$ there must be at least one square among the relevant values of v and the condition cannot be satisfied. □

Historical note on the Gewirtz graph

Gewirtz discussed his graph in two papers published in 1969 [8,9]. Brouwer [3] says that the graph was discovered by Sims, and calls it the Sims-Gewirtz graph.

My own interest in strongly regular graphs dates from the late 1960s, when I was told by John McKay about the exciting discoveries of new simple groups. The topic (but not the Gewirtz graph) is mentioned in a paper I gave at the 1969 Oxford Conference [1]. I do not wish to claim any originality for myself, but I am fairly sure that I initially derived my knowledge of the Gewirtz graph from a 1965 paper of W.L. Edge 'On some implications of the geometry of the 21-point plane' [7]. In that paper the three sets of 56 ovals in PG(2, 4) are clearly described, with the critical property that any one of the sets of 56 has the property that two of them intersect in 0 or 2 points.

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