Solving Stochastic Jump Differential Equations

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We present a reducibility criterion for stochastic differential equations driven by both a Wiener process and a Poisson random measure to equations with linear diffusion coefficients having explicit solutions or reducible to ordinary equations. We construct jump analogues of continuous diffusions satisfying equations of the type mentioned above and show that the obtained processes have the same supports of marginal distributions as the initial processes. We also illustrate the action of this method on some diffusions.

1 Introduction

Stochastic differential equations play a central role in the theory of stochastic processes and are often used in the modelling of various random processes in nature. In the case of Lipschitz linearly growing coefficients, by means of Picard approximations one can construct a pathwise unique process called a *strong solution* of the given stochastic differential equation. But in a special case of linear diffusion coefficients it can admit an explicit solution (see Gard [7; Chapter IV]) or at least can be reduced to an ordinary differential equation (see Øksendal [11; Chapter V]). Such stochastic differential equations will be called *solvable* (in a closed form).

Often defined as strong solutions of stochastic differential equations, diffusion processes are widely used in stochastic modelling. Bachélier [1] constructed a discrete pre-image of Brownian motion for the description of stock prices on a financial market. Uhlenbeck and Ornstein [15] used their process for the analysis of velocity of a particle in a fluid under the bombardment by molecules. Samuelson [13] introduced geometric Brownian motion for modelling the behavior of financial assets. After Vasiček [16] applied Ornstein-Uhlenbeck type process for the description of interest rate evolution, a dozen of diffusion models was used for the modelling interest rates (see Björk [3] or Shiryaev [14; Chapter III, Section 4] for a review). Also diffusions processes appear, e.g., in models of nonlinear filtering (see Liptser and Shiryaev [10; Chapter IX]) and in stochastic population modelling (see Øksendal [11; Chapter V, Example 5.15]).

In the recent years jump processes were also used for modelling the behavior of assets on financial markets (see Björk, Kabanov, and Runggaldier [4] and references therein). Defined as the stochastic exponent of a jump-diffusion process, generalized geometric Brownian motion was applied for the description of stock prices evolution. Barndorff-Nielsen [2] proposed the idea for generalizing diffusion processes by means of changing the driving Wiener process by

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a Lévy process and defined the so-called background driven Ornstein-Uhlenbeck type process. For a review of jump-diffusion models and their applications in financial mathematics we refer to Runggaldier [12]. The use of jump-diffusion processes for the interest rates modelling was also pointed out in Shiryaev [14; Chapter III, Section 4].

In the present paper we consider stochastic differential equations driven by both a Wiener process and a Poisson random measure with finite intensity (further called stochastic jump differential equations) and study the question of finding closed form solutions for such equations. We also propose a method for constructing the jump analogues of diffusions satisfying stochastic jump differential equations which are reducible to the equations solvable in a closed form by means of smooth invertible transformations.

The paper is organized as follows. In Section 2, by means of the arguments in [7; Chapter IV] we show that a linear stochastic jump differential equation admits an explicit solution, and using the method from [11; Chapter V, Example 5.16] we read that a stochastic equation with Lipschitz linearly growing drift and special type linear diffusion coefficients by both continuous and jump parts can be reduced to an ordinary differential equation by means of introducing an integrating factor process. In Section 3 we propose an enlargement of the set of solvable stochastic jump differential equations by means of smooth invertible transformations. Following the arguments in [7; Chapter IV], we prove a sought-after reducibility criterion for such equations and give the related transformations for some diffusions. In Section 4 we describe our method for construction of jump analogues for diffusions being closed form solutions of related stochastic jump differential equations. We show that the constructed jump-diffusion processes have the same supports of marginal distributions as the initial continuous processes and illustrate the action of the method on some diffusions.

2 Solvable stochastic jump differential equations

Suppose that on some complete stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ there exist a standard Wiener process $W = (W_t)_{t\geq 0}$ and a homogeneous Poisson random measure $\mu(dt, dv)$ on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\mathbb{R})$ with the intensity measure (compensator) $\nu(dt, dv) = dt \otimes F(dv)$, where $F(dv)$ is a positive finite measure on $\mathbb{R}$ such that $F(\{0\}) = 0$ and $F(\mathbb{R}) < \infty$ (see, e.g., [8; Chapter II, Section 1]), and W and μ are assumed to be independent.

2.1. Let us consider the stochastic differential equation

$$dX_t = \beta(t, X_t)dt + \gamma(t, X_t)dW_t + \int \delta(t, X_{t-}, v)\mu(dt, dv), \quad X_0 = x, \tag{2.1}$$

where $\beta(t, x)$, $\gamma(t, x)$ and $\delta(t, x, v)$ are some Borel functions on $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, respectively, the integral is taken over $\mathbb{R}$, and for all $t \geq 0$ we have

$$\int_0^t \left(|\beta(s, X_s)| + \gamma^2(s, X_s) + \int |\delta(s, X_s, v)|F(dv) \right) ds < \infty \quad (P - \text{a.s.}). \tag{2.2}$$

We will assume that the functions $\beta(t, x)$ and $\gamma(t, x)$ satisfy Lipschitz and linear growth conditions, i.e., there exists a constant $C > 0$ such that

$$|\beta(t, x) - \beta(t, x')| + |\gamma(t, x) - \gamma(t, x')| \leq C|x - x'|, \tag{2.3}$$

$$|\beta(t, x)| + |\gamma(t, x)| \leq C(1 + |x|) \tag{2.4}$$

for all $t \geq 0$ and $x, x' \in \mathbb{R}$. Note that condition $F(\mathbb{R}) < \infty$ implies $\nu([0,t] \times A) < \infty$ for all $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$, and thus, the Poisson random measure μ has only finite number of jumps on each finite time interval (P-a.s.). Therefore, according to the arguments in [8; Chapter III, Section 2], we conclude that under conditions (2.3) - (2.4) equation (2.1) has a unique strong solution, i.e., there exists a (pathwise) unique process $X = (X_t)_{t \geq 0}$ satisfying (2.1).

Let us suppose that $\gamma(t,x) = \gamma_0(t) + \gamma_1(t) \cdot x$ and $\delta(t,x,v) = \delta_0(t,v) + \delta_1(t,v) \cdot x$ for all $t \geq 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$ and some continuous functions $\gamma_i(t)$ and $\delta_i(t,v)$, $i = 0, 1$, satisfying

$$\delta_1(t,v) > -1 \tag{2.5}$$

for $v \in \mathbb{R}$ and

$$\int_0^t \int \left(|\delta_0(t,v)| + |\delta_1(t,v)| + \frac{|\delta_1(t,v)| + \delta_1^2(t,v)}{1 + \delta_1(t,v)} + |\log(1 + \delta_1(s,v))| \right) F(dv) ds < \infty \tag{2.6}$$

for all $t \geq 0$. Then the equation (2.1) takes the form

$$dX_t = \beta(t, X_t)dt + (\gamma_0(t) + \gamma_1(t)X_t)dW_t + \int (\delta_0(t,v) + \delta_1(t,v)X_{t-})\mu(dt,dv), \quad X_0 = x. \tag{2.7}$$

2.2. First, following the arguments in [7; Chapter IV], we show that if

$$\beta(t,x) = \beta_0(t) + \beta_1(t) \cdot x \tag{2.8}$$

for all $t \geq 0$ and $x \in \mathbb{R}$, then the stochastic differential equation (2.7) can be solved in the explicit form. For this, let us introduce the *integrating factor* process $Z = (Z_t)_{t \geq 0}$ defined as

$$Z_t = \exp\left(\int_0^t \frac{\gamma_1^2(s)}{2} ds - \int_0^t \gamma_1(s) dW_s - \int_0^t \int \log(1 + \delta_1(s,v))\mu(ds,dv) \right). \tag{2.9}$$

Then applying Itô's formula ([8; Chapter I, Theorem 4.57]) to (2.9), we get that the process Z has the expression

$$dZ_t = Z_{t-} \left(\gamma_1^2(t)dt - \gamma_1(t)dW_t - \int \frac{\delta_1(t,v)}{1 + \delta_1(t,v)} \mu(dt,dv) \right), \quad Z_0 = 1, \tag{2.10}$$

and hence, from (2.7) with (2.8) it follows that the process $F = (F_t)_{t \geq 0}$,

$$F_t = e^{-\int_0^t \beta_1(s)ds} Z_t X_t, \tag{2.11}$$

admits the representation

$$\begin{aligned} dF_t =& e^{-\int_0^t \beta_1(s)ds}[Z_{t-}dX_t + X_{t-}dZ_t + d\langle Z^c, X^c \rangle_t + \Delta Z_t \Delta X_t - Z_{t-}X_{t-}\beta_1(t)dt] \\ =& e^{-\int_0^t \beta_1(s)ds} \Bigg[Z_{t-} \Bigg((\beta_0(t) + \beta_1(t)X_{t-})dt + (\gamma_0(t) + \gamma_1(t)X_{t-})dW_t \\ &+ \int (\delta_0(t,v) + \delta_1(t,v)X_{t-})\mu(dt,dv) \Bigg) \\ &+ Z_{t-}X_{t-} \left(\gamma_1^2(t)dt - \gamma_1(t)dW_t - \int \frac{\delta_1(t,v)}{1 + \delta_1(t,v)} \mu(dt,dv) \right) \\ &- Z_{t-}(\gamma_0(t)\gamma_1(t) + \gamma_1^2(t)X_{t-})dt - Z_{t-}X_{t-} \int \frac{\delta_1^2(t,v)}{1 + \delta_1(t,v)} \mu(dt,dv) - Z_{t-}X_{t-}\beta_1(t)dt \Bigg] \\ =& e^{-\int_0^t \beta_1(s)ds} Z_{t-} \left([\beta_0(t) - \gamma_0(t)\gamma_1(t)]dt + \gamma_0(t)dW_t + \int \delta_0(t,v)\mu(dt,dv) \right). \end{aligned} \tag{2.12}$$

Therefore, from (2.11) and (2.12) we obtain that the process $X = (X_t)_{t\geq 0}$ given by

$$X_t = Z_t^{-1}\left(e^{\int_0^t \beta_1(s)ds}x + \int_0^t e^{\int_s^t \beta_1(u)du} Z_s[\beta_0(s) - \gamma_0(s)\gamma_1(s)]ds \right. \tag{2.13}$$
$$\left. + \int_0^t e^{\int_s^t \beta_1(u)du} Z_s\gamma_0(s)dW_s + \int_0^t e^{\int_s^t \beta_1(u)du} Z_{s-}\int \delta_0(s,v)\mu(ds,dv)\right)$$

is a (unique strong) solution of the equation (2.7) with (2.8).

Example 2.1. (Ornstein-Uhlenbeck type process with jumps [2]) Suppose that in (2.7) with (2.8) we have $\gamma_1(t) = \delta_1(t,v) = 0$ for all $t \geq 0$ and $v \in \mathbb{R}$. Then the solution X is given by (2.13) with $Z_t = 1$ for all $t \geq 0$.

Example 2.2. (Generalized geometric Brownian motion [12]) Suppose that in (2.7) with (2.8) we have $\beta_0(t) = \gamma_0(t) = \delta_0(t,v) = 0$ for all $t \geq 0$ and $v \in \mathbb{R}$. Then the solution X has the form $X_t = xZ_t^{-1}$, $t \geq 0$.

2.3. Now, following the arguments in [11; Chapter V, Example 5.16], we show that if

$$\gamma_0(t) = \delta_0(t,v) = 0 \tag{2.14}$$

for all $t \geq 0$ and $v \in \mathbb{R}$, then the stochastic differential equation (2.7) can be reduced to an ordinary differential equation. For this, we apply Itô's formula to the process $G = (G_t)_{t\geq 0}$,

$$G_t = Z_t X_t, \tag{2.15}$$

and using (2.7) with (2.10) and (2.14), we get the representation

$$\begin{aligned} dG_t =& Z_{t-}dX_t + X_{t-}dZ_t + d\langle Z^c, X^c\rangle_t + \Delta Z_t \Delta X_t \\ =& Z_{t-}\left(\beta(t,X_{t-})dt + \gamma_1(t)X_{t-}dW_t + X_{t-}\int \delta_1(t,v)\mu(dt,dv)\right) \\ &+ Z_{t-}X_{t-}\left(\gamma_1^2(t)dt - \gamma_1(t)dW_t - \int \frac{\delta_1(t,v)}{1+\delta_1(t,v)}\mu(dt,dv)\right) \\ &- Z_{t-}X_{t-}\gamma_1^2(t)dt - Z_{t-}X_{t-}\int \frac{\delta_1^2(t,v)}{1+\delta_1(t,v)}\mu(dt,dv) \\ =& Z_{t-}\beta(t, Z_{t-}^{-1}G_{t-})dt. \end{aligned} \tag{2.16}$$

Therefore, if $\beta(t,x)$ satisfy conditions (2.3) - (2.4), then the (unique strong) solution X is determined from (2.15), where for all $\omega \in \Omega$ the process $G(\omega) = (G_t(\omega))_{t\geq 0}$ is a unique solution of the ordinary differential equation

$$dG_t(\omega) = Z_t(\omega)\beta(t, Z_t^{-1}(\omega)G_t(\omega))dt, \quad G_0(\omega) = x. \tag{2.17}$$

Example 2.3. (Equation with power drift and linear diffusion [11; Example 5.16 (d)]) Suppose that in (2.7) with (2.14) we have $\beta(t,x) = x^\alpha$, $t \geq 0$, $x > 0$, and $\alpha < 1$. Then the solution X is given by

$$X_t = Z_t^{-1}\left(x^{1-\alpha} + (1-\alpha)\int_0^t Z_s^{1-\alpha}ds\right)^{1/(1-\alpha)}.$$

3 Reducibility to solvable equations

3.1. Let us consider the stochastic differential equation

$$dY_t = \eta(t, Y_t)dt + \sigma(t, Y_t)dW_t + \int \theta(t, Y_{t-}, v)\mu(dt, dv), \quad Y_0 = y, \tag{3.1}$$

where $\eta(t, y)$, $\sigma(t, y)$ and $\theta(t, y, v)$ are some Borel functions on $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, respectively, the integral is taken over $\mathbb{R}$, and for all $t \geq 0$ we have

$$\int_0^t \left(|\eta(s, Y_s)| + \sigma^2(s, Y_s) + \int |\theta(s, Y_s, v)| F(dv) \right) ds < \infty \quad (P - \text{a.s.}). \tag{3.2}$$

Suppose that $f(t, y)$ is an *invertible* function from the class $C^{1,2}(\mathbb{R}_+, \mathbb{R})$, i.e., there exists a function $g(t, x)$ such that $f(t, g(t, x)) = x$ and $g(t, f(t, y)) = y$ for all $t \geq 0$, $x, y \in \mathbb{R}$. Then, denoting $X_t = f(t, Y_t)$, by means of Itô's formula we get that

$$\begin{aligned} dX_t =& f_t(t, Y_t)dt + f_y(t, Y_t)dY_t + (f_{yy}(t, Y_t)/2)d\langle Y^c, Y^c\rangle_t \\ &+ [f(t, Y_{t-} + \Delta Y_t) - f(t, Y_{t-}) - f_y(t, Y_{t-})\Delta Y_t] \\ =& [f_t(t, Y_t) + f_y(t, Y_t)\eta(t, Y_t) + (f_{yy}(t, Y_t)/2)\sigma^2(t, Y_t)]dt \\ &+ f_y(t, Y_t)\sigma(t, Y_t)dW_t + \int [f(t, Y_{t-} + \theta(t, Y_{t-}, v)) - f(t, Y_{t-})]\mu(dt, dv), \end{aligned} \tag{3.3}$$

and thus, using $Y_t = g(t, X_t)$, we see that the equation (3.1) is reduced to (2.1) with

$$\beta(t, x) = f_t(t, g(t, x)) + \eta(t, g(t, x))f_y(t, g(t, x)) + \sigma^2(t, g(t, x))f_{yy}(t, g(t, x))/2, \tag{3.4}$$

$$\gamma(t, x) = \sigma(t, g(t, x))f_y(t, g(t, x)), \tag{3.5}$$

$$\delta(t, x, v) = f(t, g(t, x) + \theta(t, g(t, x), v)) - f(t, g(t, x)). \tag{3.6}$$

Therefore, if the functions $\beta(t, x)$ and $\gamma(t, x)$ from (3.4) and (3.5) satisfy conditions (2.3) - (2.4), then by virtue of invertibility of the function $f(t, y)$ we conclude that the equation (3.1) has a unique (strong) solution $Y = (Y_t)_{t\geq 0}$.

From (3.3) it is easily seen that if $f(t, y)$ solves the equations

$$f_t(t, y) + \eta(t, y)f_y(t, y) + \sigma^2(t, y)f_{yy}(t, y)/2 = \beta(t, f(t, y)), \tag{3.7}$$

$$\sigma(t, y)f_y(t, y) = \gamma_0(t) + \gamma_1(t)f(t, y), \tag{3.8}$$

$$f(t, y + \theta(t, y, v)) - f(t, y) = \delta_0(t, v) + \delta_1(t, v)f(t, y), \tag{3.9}$$

with some continuous functions $\beta(t, x)$, $\gamma_i(t)$ and $\delta_i(t, v)$, $i = 0, 1$, $t \geq 0$, $x, y \in \mathbb{R}$, $v \in \mathbb{R}$, satisfying conditions (2.2), (2.3) - (2.4) and (2.5) - (2.6), then the equation (3.1) is reduced to the equation (2.7), which is solvable in a closed form under conditions (2.8) or (2.14).

EXAMPLE 3.1. (Black-Karasinski process [5]) Suppose that in (3.1) we have $\eta(t, y) = y(\eta_0(t) + \eta_1(t) \log y)$, $\sigma(t, y) = \sigma_0(t)y$ and $\theta(t, y, v) = 0$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t, y) = \log y$, $y > 0$, (with the inverse $g(t, x) = e^x$, $x \in \mathbb{R}$) reduces (3.1) to the equation (2.7) with (2.8), where $\beta_0(t) = \eta_0(t) - \sigma_0^2(t)/2$, $\beta_1(t) = \eta_1(t)$, $\gamma_0(t) = \sigma_0(t)$, $\gamma_1(t) = \delta_i(t, v) = 0$, $i = 0, 1$, for all $t \geq 0$ and $v \in \mathbb{R}$ (see Example 2.1).

EXAMPLE 3.2. (Stochastic population model [11; Chapter V, Example 5.15]) Suppose that in (3.1) we have $\eta(t,y) = \eta_0(t)y(\eta_1(t) - y)$, $\eta_0(t) > 0$, $\eta_1(t) > 0$, $\sigma(t,y) = \sigma_0(t)y$ and $\theta(t,y,v) = 0$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$. Then the function $f(t,y) = 1/y$, $y > 0$, (with the inverse $g(t,x) = 1/x$, $x > 0$) reduces (3.1) to the equation (2.7) with (2.8), where $\beta_0(t) = \eta_0(t)$, $\beta_1(t) = \sigma_0^2(t) - \eta_0(t)\eta_1(t)$, $\gamma_1(t) = -\sigma_0(t)$, $\gamma_0(t) = \delta_i(t,v) = 0$, $i = 0,1$, for all $t \geq 0$ and $v \in \mathbb{R}$.

REMARK 3.1. Observe that in Examples 3.1 and 3.2 the function $\eta(t,y)$ does not satisfy the condition (2.4), but we see that the equation (3.1) has a unique solution, since it is reducible to the linear equation (2.7) with (2.8).

3.2. In the rest of the section we describe the transformations of (3.1) to solvable equations in the *time-homogeneous case*, i.e., we suppose that $\eta(t,y) = \eta(y)$, $\sigma(t,y) = \sigma(y)$, $\theta(t,y,v) = \theta(y,v)$, $\beta(t,x) = \beta(x)$, $\gamma_i(t) = \gamma_i$, $\delta_i(t,v) = \delta_i(v)$, $i = 0,1$, and $f(t,y) = f(y)$, $g(t,x) = g(x)$ for all $t \geq 0$, $x,y \in \mathbb{R}$ and $v \in \mathbb{R}$. We will assume that $\sigma(y) \neq 0$ for $y \in \mathbb{R}$, all the derivatives below exist and the integrals are well-defined.

Solving the homogeneous variant of the equation (3.8), in particular, we get that

$$f(y) = ce^{\gamma_1 r(y)} - \gamma_0/\gamma_1 \tag{3.10}$$

is turned out to be a solution when $\gamma_1 \neq 0$, and

$$f(y) = \gamma_0 r(y) \tag{3.11}$$

when $\gamma_1 = 0$, where $c \neq 0$ is some constant and

$$r(y) = \int^y \frac{dy'}{\sigma(y')}. \tag{3.12}$$

Following the arguments from [7; Chapter IV, pages 115-116], namely, substituting the expression (3.10) + (3.12) for $f(y)$ into (3.7) with (2.8), we get that

$$[\gamma_1 p(y) + \gamma_1^2/2 - \beta_1]e^{\gamma_1 r(y)} = (\gamma_1\beta_0 - \gamma_0\beta_1)/(c\gamma_1), \tag{3.13}$$

where

$$p(y) = \eta(y)/\sigma(y) - \sigma_y(y)/2. \tag{3.14}$$

Differentiating (3.13) and using (3.14), we get $[(\gamma_1 p(y) + \gamma_1^2/2 - \beta_1)/\sigma(y) + p_y(y)]\gamma_1 e^{\gamma_1 r(y)} = 0$, wherefrom after multiplying both parts by $\sigma(y)e^{-\gamma_1 r(y)}/\gamma_1$ and differentiating again we see that $\gamma_1 p_y(y) + [\sigma p_y]_y(y) = 0$. Hence, we conclude that if (3.7) - (3.8) are satisfied then (for all $y \in \mathbb{R}$)

$$\text{either} \quad p_y(y) = 0 \quad \text{or} \quad \left([\sigma p_y]_y/p_y\right)_y(y) = 0. \tag{3.15}$$

Substituting the expression (3.11) + (3.12) for $f(y)$ into (3.7), we see that if $\gamma_1 = 0$, then the choice of $f(y)$ as in (3.11) similarly leads to the reducibility condition $[\sigma p_y]_y(y) = 0$, implying also (3.15).

Conversely, if $p_y(y) = 0$ for all $y \in \mathbb{R}$, then there exists $\gamma_1 \neq 0$ and $\beta_1 \in \mathbb{R}$ such that $f(y)$ has the form (3.10) with $\gamma_0 = 0$, and (3.7) is satisfied with (2.8) and $\beta_0 = 0$. If $p_y(y) \neq 0$, but $[\sigma p_y]_y(y) = 0$ for all $y \in \mathbb{R}$, then $f(y)$ has the form (3.11) with an arbitrary $\gamma_0 \neq 0$. If $p_y(y) \neq 0$ and $[\sigma p_y]_y(y) \neq 0$, but the latter part of (3.15) is satisfied for all $y \in \mathbb{R}$, then $f(y)$ has the form (3.10) with $\gamma_0 = 0$ and $\gamma_1 = -[\sigma p_y]_y(y)/p_y(y)$.

By analogy with the arguments above, substituting the expression (3.10) + (3.12) for $f(y)$ into (3.9), we get that

$$[q(y,v) - (1+\delta_1(v))]\, e^{\gamma_1 r(y)} = (\gamma_1\delta_0(v) - \gamma_0\delta_1(v))/(c\gamma_1), \tag{3.16}$$

where

$$q(y,v) = \gamma_1 \exp\left(\int_y^{y+\theta(y,v)} \frac{dy'}{\sigma(y')}\right). \tag{3.17}$$

Then, differentiating (3.16), we get that $[(q(y,v) - (1+\delta_1(v)))/\sigma(y) + q_y(y,v)/\gamma_1]\gamma_1 e^{\gamma_1 r(y)} = 0$, wherefrom after multiplying both parts by $\sigma(y)e^{-\gamma_1 r(y)}/\gamma_1$ and differentiating again we see that $\gamma_1 q_y(y,v) + [\sigma q_y]_y(y,v) = 0$. Hence, we conclude that if (3.8) - (3.9) are satisfied, then (for all $y \in \mathbb{R}$ and $v \in \mathbb{R}$)

$$\text{either} \quad q_y(y,v) = 0 \quad \text{or} \quad ([\sigma q_y]_y/q_y)_y(y,v) = 0. \tag{3.18}$$

Substituting the expression (3.11) + (3.12) for $f(y)$ into (3.9), we see that if $\gamma_1 = 0$, then the choice of $f(y)$ as in (3.11) similarly leads to the reducibility condition

$$([\sigma q_y]/q)_y(y,v) = 0. \tag{3.19}$$

Conversely, if $q_y(y,v) = 0$ for all $y \in \mathbb{R}$ and $v \in \mathbb{R}$, then there exists $\gamma_1 \neq 0$ and $\beta_1 \in \mathbb{R}$ such that $f(y)$ has the form (3.10) with $\gamma_0 = 0$, and (3.9) is satisfied with (2.8) and $\delta_0(v) = 0$ for all $v \in \mathbb{R}$. If $q_y(y,v) \neq 0$, but $([\sigma q_y]/q)_y(y,v) = 0$ for all $y \in \mathbb{R}$ and $v \in \mathbb{R}$, then $f(y)$ has the form (3.11) with an arbitrary $\gamma_0 \neq 0$. If $q_y(y) \neq 0$, but the latter part of (3.18) is satisfied for all $y \in \mathbb{R}$ and $v \in \mathbb{R}$, then $f(y)$ has the form (3.10) with $\gamma_0 = 0$ and $\gamma_1 = -[\sigma q_y]_y(y)/q_y(y)$.

Now we summarize the assertions above into the following sought-after reducibility criterion for the jump-diffusion processes.

THEOREM 3.1. *Under regularity conditions above the following assertions hold:*

(i) if conditions (3.15), and (3.18) or (3.19), are satisfied with $p(y)$ and $q(y)$ given by (3.14) and (3.17), respectively, then the homogeneous variant of the equation (3.1) is reducible to (2.7) with (2.8), which is solvable in the explicit form (2.13);

(ii) if conditions (3.18) or (3.19) are satisfied with $q(y)$ from (3.17), and after taking $f(y)$ as in (3.10) with $\gamma_0 = 0$, we receive a Lipschitz function $\beta(x)$, $x \in \mathbb{R}$, then the equation (3.1) is reducible to (2.7) with (2.14), which is reducible to the ordinary differential equation (2.17).

We finish the section by giving some examples of diffusions and related transformations.

EXAMPLE 3.3. (Cox-Ingersoll-Ross process I [6]) Suppose that in (3.1) we have $\eta(y) = \eta_0 + \eta_1 y$, $\sigma(y) = \sigma_0\sqrt{y}$, $\eta_0 \geq \sigma_0^2/2$, and $\theta(y,v) = 0$ for all $y > 0$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(2\sqrt{y})$, $y > 0$ (with the inverse $g(x) = (\log x/2)^2$, $x > 1$) reduces (3.1) to the equation (2.7) with (2.14), where $\beta(x) = x[2\eta_0 \log^2 x + \eta_1 \log^4 x/2 + 2\sigma_0^2(\log x - 1)]/\log^3 x$, $\gamma_1 = \sigma_0$ and $\delta_1(v) = 0$ for all $x > 1$ and $v \in \mathbb{R}$.

EXAMPLE 3.4. (Cox-Ingersoll-Ross process II [6]) Suppose that in (3.1) we have $\eta(y) = 0$, $\sigma(y) = \sigma_0\sqrt{y^3}$ and $\theta(y, v) = 0$ for all $y > 0$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(-2/\sqrt{y})$, $y > 0$ (with the inverse $g(x) = (-2/\log x)^2$, $x \in \langle 0, 1\rangle$) reduces (3.1) to the equation (2.7) with (2.14), where $\beta(x) = \sigma_0^2 x(1 + 3/\log x)/2$, $\gamma_1 = \sigma_0$ and $\delta_1(v) = 0$ for all $x \in \langle 0, 1\rangle$ and $v \in \mathbb{R}$.

EXAMPLE 3.5. (A nonlinear filter process [10; Chapter IX]) Suppose that in (3.1) we have $\eta(y) = \eta_0(1 - y)$, $\sigma(y) = \sigma_0 y(1 - y)$ and $\theta(y, v) = 0$ for all $y \in \langle 0, 1\rangle$ and $v \in \mathbb{R}$. Then the function $f(y) = y/(1 - y)$, $y \in \langle 0, 1\rangle$ (with the inverse $g(x) = x/(1 + x)$, $x > 0$) reduces (3.1) to the equation (2.7) with (2.14), where $\beta(x) = \eta_0(1+x)+\sigma_0^2 x^2/(1+x)$, $\gamma_1 = \sigma_0$ and $\delta_1(v) = 0$ for all $x > 0$ and $v \in \mathbb{R}$.

EXAMPLE 3.6. (Jacobi diffusion [9; p. 335]) Suppose that in (3.1) we have $\eta(y) = \sigma_0^2[\eta_0(1-y) - \eta_1 y]/2$, $\sigma(y) = \sigma_0\sqrt{y(1 - y)}$, $\eta_0 \geq 1$, $\eta_1 \geq 1$, and $\theta(y, v) = 0$ for all $y \in \langle 0, 1\rangle$ and $v \in \mathbb{R}$. Then the function $f(y) = \exp(2 \arcsin\sqrt{y})$, $y \in \langle 0, 1\rangle$ (with the inverse $g(x) = \sin^2(\log\sqrt{x})$, $x \in \langle 1, e^\pi\rangle$) reduces (3.1) to the equation (2.7) with (2.14), where $\beta(x) = \sigma_0^2 x[\eta_0 \cos^2(\log\sqrt{x}) - \eta_1 \sin^2(\log\sqrt{x}) + (\sin(2\log\sqrt{x}) - \cos(2\log\sqrt{x}))/2]/\sin(2\log\sqrt{x})$, $\gamma_1 = \sigma_0$ and $\delta_1(v) = 0$ for all $x \in \langle 1, e^\pi\rangle$ and $v \in \mathbb{R}$.

4 Constructing jump analogues of some diffusions

In this section we will suppose that on the initial stochastic base there exist a Wiener process $W' = (W'_t)_{t\geq 0}$ and a Poisson random measure $\mu'(dt, dv)$ with the same compensator $\nu(dt, dv) = dt \otimes F(dv)$, where W' and μ' are assumed to be independent also from W and μ.

4.1. Let $Y = (Y_t)_{t\geq 0}$ be a continuous process solving the stochastic differential equation (3.1) with $\theta(t, y, v) = 0$, $t \geq 0$, $y \in \mathbb{R}$, $v \in \mathbb{R}$. Suppose that there exists an invertible transformation $f(t, y) \in C^{1,2}(\mathbb{R}_+, \mathbb{R})$ satisfying (3.7) - (3.9) and such that the process $X = (X_t)_{t\geq 0}$, $X_t = f(t, Y_t)$, solves the equation (2.7) with $\delta_i(t, v) = 0$, $i = 0, 1$, $t \geq 0$, $v \in \mathbb{R}$. Let us take some continuous functions $\delta'_i(t, v)$, $i = 0, 1$, satisfying conditions (2.5) - (2.6) with $\delta_i(t, v)$ replaced by $\delta'_i(t, v)$ and the rule

$$\delta'_i(t, v) \neq 0 \quad \text{if and only if} \quad \gamma_i(t) \neq 0 \tag{4.1}$$

for $i = 0, 1$ and all $t \geq 0$, $v \in \mathbb{R}$, and consider the stochastic differential equation

$$dX'_t = \beta(t, X'_t)dt + (\gamma_0(t) + \gamma_1(t)X'_t)dW'_t + \int(\delta'_0(t, v) + \delta'_1(t, v)X'_{t-})\mu'(dt, dv), \quad X'_0 = x, \tag{4.2}$$

where $\beta(t, x)$ satisfies (2.8) or the condition

$$\gamma_0(t) = \delta'_0(t, v) = 0 \tag{4.3}$$

holds for all $t \geq 0$ and $v \in \mathbb{R}$. Then according to the arguments in Section 2, we conclude that equation (4.2) is solvable in a closed form, and applying to the solution $X' = (X'_t)_{t\geq 0}$ the inverse transformation $g(t, x)$, $t \geq 0$, $x \in \mathbb{R}$, we obtain that the process $Y'_t = g(t, X'_t)$ solves the equation

$$dY'_t = \eta(t, Y'_t)dt + \sigma(t, Y'_t)dW'_t + \int \theta'(t, Y'_{t-}, v)\mu'(dt, dv), \quad Y'_0 = y, \tag{4.4}$$

with

$$\theta'(t,y,v) = g(t, \delta_0'(t,v) + (1+\delta_1'(t,v))f(t,y)) - g(t, f(t,y)). \tag{4.5}$$

We will call such process $Y' = (Y_t')_{t\geq 0}$ a *jump analogue* of the diffusion process $Y = (Y_t)_{t\geq 0}$.

In order to prove that the method for construction of jump analogues presented above is correct, we show that the supports of marginal distributions of the constructed jump-diffusion process and of the initial continuous process happen to coincide.

THEOREM 4.1. *Suppose that in the assumptions of this section $\beta(t,x)$ satisfies (2.8) or the condition (4.3) holds for all $t \geq 0$ and $v \in \mathbb{R}$. Then* $\mathrm{Supp}\{\mathcal{L}(Y_t')\} = \mathrm{Supp}\{\mathcal{L}(Y_t)\} \subseteq \overline{\mathbb{R}}$ *for all* $t \geq 0$.

PROOF. Since the transformations $f(t,y)$ and $g(t,x)$ are supposed to be mutually invertible, it remains us to prove that if (4.1) is satisfied with (2.8) or (4.3), then $\mathrm{Supp}\{\mathcal{L}(X_t')\} = \mathrm{Supp}\{\mathcal{L}(X_t)\}$ for all $t \geq 0$. For this, let us introduce the process $Z' = (Z_t')_{t\geq 0}$ defined as

$$Z_t' = \exp\left(\int_0^t \frac{\gamma_1^2(s)}{2} ds - \int_0^t \gamma_1(s) dW_s' - \int_0^t \int \log(1+\delta_1'(s,v))\mu'(ds,dv)\right), \tag{4.6}$$

and observe that if in the equation (4.2) the function $\beta(t,x)$ satisfies (2.8), then by means of the arguments in (2.11) - (2.13) we get that the solution $X' = (X_t')_{t\geq 0}$ admits the representation

$$\begin{aligned} X_t' =& Z_t'^{-1}\left(e^{\int_0^t \beta_1(s)ds} x + \int_0^t e^{\int_s^t \beta_1(u)du} Z_s'[\beta_0(s) - \gamma_0(s)\gamma_1(s)]ds\right. \\ &\left. + \int_0^t e^{\int_s^t \beta_1(u)du} Z_s'\gamma_0(s) dW_s' + \int_0^t e^{\int_s^t \beta_1(u)du} Z_{s-}' \int \delta_0'(s,v)\mu'(ds,dv)\right), \end{aligned} \tag{4.7}$$

and if in (4.2) we have (4.3), then applying Itô's formula to the process $G' = (G_t')_{t\geq 0}$ given by

$$G_t' = Z_t' X_t', \tag{4.8}$$

by analogy with the arguments in (2.15) - (2.17), we obtain that for all $\omega \in \Omega$ the process $G'(\omega) = (G_t'(\omega))_{t\geq 0}$ is a unique solution of the ordinary differential equation

$$dG_t'(\omega) = Z_t'(\omega)\beta(t, Z_t'^{-1}(\omega)G_t'(\omega))dt, \quad G_0'(\omega) = x. \tag{4.9}$$

Note that since $\nu([0,t]\times A) < \infty$ for all $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R})$, the Poisson random measure $\mu'(dt,dv)$ as well as the processes Z', X' and Y' have a finite number of jumps on each finite time interval. Also we recall that conditions (2.5) - (2.6) are satisfied with $\delta_i'(t,v)$ instead of $\delta_i(t,v)$, and in the assumptions of this section $\delta_i(t,v) = 0$, $i = 0,1$, $t \geq 0$, $v \in \mathbb{R}$.

Observe that if in (4.2) we have (2.8) and for given and fixed $t > 0$ there exists a set $B \subseteq [0,t]$ with $\lambda(B) > 0$ (by λ we denote the Lebesgue measure on $\mathbb{R}_+$) such that $\gamma_0(s) \neq 0$ and $\delta_0'(s,v) \neq 0$ for all $s \in B$ and $v \in \mathbb{R}$, then the structure of the solutions X from (2.13) and X' from (4.7) implies that the random variables X_t and X_t' have absolutely continuous distributions on $\mathbb{R}$ (with respect to the Lebesgue measure), and hence $\mathrm{Supp}\{\mathcal{L}(X_t')\} = \mathrm{Supp}\{\mathcal{L}(X_t)\} = \overline{\mathbb{R}}$.

Now we suppose that in (4.2) the property (4.3) holds for λ-a.e. $t \geq 0$ and fix some point $a \in \text{Supp}\{\mathcal{L}(X'_t)\}$, i.e., such that for an arbitrary $\rho > 0$ we have $P[a-\rho < X'_t < a+\rho] > 0$. Let us introduce the sets $\Sigma(D) := \{|\log Z'_s| \leq D, |\log Z_s| \leq D, |X'_s| \leq D, |X_s| \leq D, s \in [0,t]\} \subseteq \Omega$ and $\Phi(\varepsilon) := \{|Z'_t - Z_t| < \varepsilon, \int_0^t |Z'_s - Z_s| ds < \varepsilon\} \subseteq \Omega$. Then from the sample path properties of the Wiener process and independence of W', μ' and W, for sufficiently large $D > 0$ and arbitrarily small $\varepsilon > 0$ we have $P[\{a - \rho < X'_t < a + \rho\} \cap \Sigma(D) \cap \Phi(\varepsilon)] > 0$.

Observe that from (2.15) + (2.17) and (4.8) - (4.9) it follows that for all $\omega \in \Omega$

$$X_t(\omega) = Z_t^{-1}(\omega)x + \int_0^t Z_s(\omega)\beta(s, X_s(\omega))ds \tag{4.10}$$

and

$$X'_t(\omega) = Z_t'^{-1}(\omega)x + \int_0^t Z'_s(\omega)\beta(s, X'_s(\omega))ds. \tag{4.11}$$

Then for $\omega \in \Sigma(D) \cap \Phi(\varepsilon)$ we have

$$\begin{aligned} |X'_t(\omega) - X_t(\omega)| \leq & |Z_t'^{-1}(\omega) - Z_t^{-1}(\omega)||x| \\ & + \left| Z_t'^{-1}(\omega) \int_0^t Z'_s(\omega)\beta(s, X'_s(\omega))ds - Z_t^{-1}(\omega) \int_0^t Z_s(\omega)\beta(s, X_s(\omega))ds \right| \\ \leq & |Z_t'^{-1}(\omega) - Z_t^{-1}(\omega)| \left(|x| + \int_0^t |Z'_s(\omega)||\beta(s, X'_s(\omega))|ds \right) \\ & + |Z_t^{-1}(\omega)| \int_0^t |Z'_s(\omega) - Z_s(\omega)||\beta(s, X'_s(\omega))|ds \\ & + |Z_t^{-1}(\omega)| \int_0^t |Z_s(\omega)||\beta(s, X'_s(\omega)) - \beta(s, X_s(\omega))|ds, \end{aligned} \tag{4.12}$$

where by means of Lipschitz and linear growth conditions (2.3) - (2.4) we get that

$$|X'_t(\omega) - X_t(\omega)| \leq \varepsilon[(1 + D)(1 + Ce^{2D}t)e^D + De^{2D}] + Ce^{2D} \int_0^t |X'_s(\omega) - X_s(\omega)|ds, \tag{4.13}$$

and applying the Gronwall inequality (see, e.g., [11; Chapter V, Example 5.17]) we obtain that

$$|X'_t(\omega) - X_t(\omega)| \leq \varepsilon[(1 + D)(1 + Ce^{2D}t)e^D + De^{2D}] \exp(Ce^{2D}t) \tag{4.14}$$

for all $\omega \in \Sigma(D) \cap \Phi(\varepsilon)$. Taking $\varepsilon := \rho[(1+D)(1+Ce^{2D}t)e^D + De^{2D}]^{-1} \exp(-Ce^{2D}t)$, we receive that $\{a - \rho < X'_t < a + \rho\} \cap \Sigma(D) \cap \Phi(\varepsilon) \subseteq \{|X'_t - X_t| \leq \rho\} \cap \{a - \rho < X'_t < a + \rho\} \cap \Sigma(D) \subseteq \{a - 2\rho < X_t < a + 2\rho\}$, and thus, we obtain that the property $P[a - \rho < X'_t < a + \rho] > 0$ implies $P[a - 2\rho < X_t < a + 2\rho] > 0$ for an arbitrary $\rho > 0$, i.e., $a \in \text{Supp}\{\mathcal{L}(X_t)\}$. Therefore, we get that $\text{Supp}\{\mathcal{L}(X'_t)\} \subseteq \text{Supp}\{\mathcal{L}(X_t)\}$, $t \geq 0$, and since the inverse inclusion $\text{Supp}\{\mathcal{L}(X_t)\} \subseteq \text{Supp}\{\mathcal{L}(X'_t)\}$ easily follows from the fact that the process X' may have no jumps on the interval $[0, t]$ (with a positive probability), we conclude that $\text{Supp}\{\mathcal{L}(X_t)\} = \text{Supp}\{\mathcal{L}(X'_t)\}$, and hence, the invertibility of transformations $f(t, y)$ and $g(t, x)$ implies that $\text{Supp}\{\mathcal{L}(Y_t)\} = \text{Supp}\{\mathcal{L}(Y'_t)\}$ for all $t \geq 0$.

REMARK 4.1. After constructing the jump analogue $Y' = (Y'_t)_{t\geq 0}$ of the given diffusion process $Y = (Y_t)_{t\geq 0}$ we can also define a *pure jump analogue* $Y'' = (Y''_t)_{t\geq 0}$ of $Y = (Y_t)_{t\geq 0}$ by setting $\sigma(t,y) = 0$ in (4.4) for all $t \geq 0$ and $y \in \mathbb{R}$, i.e., as a process solving the equation

$$dY''_t = \eta(t, Y''_t)dt + \int \theta'(t, Y''_{t-}, v)\mu(dt, dv), \quad Y''_0 = y, \tag{4.15}$$

with $\theta'(t,y,v)$ given by (4.5). In this case, from the arguments of the proof of Theorem 4.1 we easily deduce that $\mathrm{Supp}\{\mathcal{L}(Y''_t)\} \subseteq \mathrm{Supp}\{\mathcal{L}(Y'_t)\} = \mathrm{Supp}\{\mathcal{L}(Y_t)\} \subseteq \overline{\mathbb{R}}$ for all $t \geq 0$.

4.2. In the rest of the section we give some examples of jump analogues of diffusion processes cited in Section 3. Actually, first of them were considered in Examples 2.1 - 2.3 above.

EXAMPLE 4.1. Suppose that in (4.4) we have the same $\eta(t,y)$ and $\sigma(t,y)$ as in Example 3.1. Then for a jump analogue in (4.5) we can take $\delta'_1(t,v) = 0$, and thus $\theta'(t,y,v) = y[\exp(\delta'_0(t,v)) - 1]$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$.

EXAMPLE 4.2. Suppose that in (4.4) we have the same $\eta(t,y)$ and $\sigma(t,y)$ as in Example 3.2. Then for a jump analogue in (4.5) we can take $\delta'_0(t,v) = 0$, and thus $\theta'(t,y,v) = -y[\delta'_1(t,v)/(1 + \delta'_1(t,v))]$ for all $t \geq 0$, $y > 0$ and $v \in \mathbb{R}$.

EXAMPLE 4.3. Suppose that in (4.4) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.3. Then for a jump analogue in (4.5) we can take $\theta'(y,v) = \sqrt{y}\log(1 + \delta'_1(v)) + \log^2(1 + \delta'_1(v))/4$ for all $y > 0$ and $v \in \mathbb{R}$.

EXAMPLE 4.4. Suppose that in (4.4) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.4. Then for a jump analogue in (4.5) we can take $\theta'(y,v) = y\sqrt{y}\log\sqrt{1 + \delta'_1(v)}(2 - \sqrt{y}\log\sqrt{1 + \delta'_1(v)})/(\sqrt{y}\log\sqrt{1 + \delta'_1(v)} - 1)^2$ for all $y > 0$ and $v \in \mathbb{R}$.

EXAMPLE 4.5. Suppose that in (4.4) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.5. Then for a jump analogue in (4.5) we can take $\theta'(y,v) = y(1-y)\delta'_1(v)/(1 + y\delta'_1(v))$ for all $y \in \langle 0, 1 \rangle$ and $v \in \mathbb{R}$ (see [10; Chapter XIX]).

EXAMPLE 4.6. Suppose that in (4.4) we have the same $\eta(y)$ and $\sigma(y)$ as in Example 3.6. Then for a jump analogue in (4.5) we can take $\theta'(y,v) = \sin[2\arcsin\sqrt{y} + \log\sqrt{1 + \delta'_1(v)}]\sin[\log\sqrt{1 + \delta'_1(v)}]$ for all $y \in \langle 0, 1 \rangle$ and $v \in \mathbb{R}$.

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