Discussion Paper No. 2006-060 of *Sonderforschungsbereich 649 Economic Risk* (12 pp)

On maximal inequalities for some jump processes*

Pavel V. Gapeev

We present a solution to the considered in [5] and [22] optimal stopping problem for some jump processes. The method of proof is based on reducing the initial problem to an integro-differential free-boundary problem where the normal reflection and smooth fit may break down and the latter then be replaced by the continuous fit. The derived result is applied for determining the best constants in maximal inequalities for a compound Poisson process with linear drift and exponential jumps.

1. Introduction

The main aim of this paper is to present a solution to the optimal stopping problem (2.3) for the maximum associated with the process X that solves the stochastic differential equation (2.1) driven by a compound Poisson process with exponentially distributed jumps. The problem (2.3) was earlier considered for some particular classes of stochastic processes. In the articles [12] and [5], solutions of the given problem were found for a reflected Brownian motion and for Bessel processes, respectively, and then the derived results were applied for determining the best constants in the related maximal inequalities. The case of linear diffusion processes was considered in the papers [9]-[10]. A complete solution of the problem (2.3) for diffusion processes was obtained in the article [22] using the established *maximality principle* being equivalent to the superharmonic characterization of the value function. The case of Poisson process and a constant cost function was treated in the paper [17]. We also note that an explicit solution of a discounted variant of the problem (2.3) with the zero cost function (the Russian option problem) was derived in the articles [26]-[27].

In the papers mentioned above the solutions were obtained by reducing the initial problem to a free-boundary problem for a differential operator and solving the latter by means of the smooth-fit and normal-reflection conditions. By means of the same methodology, in this paper we derive a solution of the optimal stopping problem (2.3) for the defined in (2.1)-(2.2) jump process (X, S). We also remark that under some relationships on the parameters of the model the normal reflection and smooth fit may break down and the latter then be replaced by the

* This research was supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

Mathematics Subject Classification 2000: Primary 60G40, 34K10, 60E15. Secondary 60J60, 60J75.

Key words and phrases: Jump process, stochastic differential equation, maximum process, optimal stopping problem, compound Poisson process, Itô's formula, integro-differential free-boundary problem, normal reflection, continuous and smooth fit, maximality principle, maximal inequalities.

continuous fit. The breakdown of the smooth-fit principle and its replacement by the principle of continuous fit was earlier observed in optimal stopping problems for jump processes considered in the articles [23]-[24] (see also [1] for necessary and sufficient conditions for the occurrence of smooth-fit condition and references to the related literature and [25] for an extensive overview). Some other optimal stopping problems for jump processes related to financial mathematics were earlier considered in the articles [8], [19]-[20], [14]-[16], [2]-[3], and [7].

The paper is organized as follows. In Section 2, for the initial problem (2.3) we formulate the corresponding integro-differential free-boundary problem for the infinitesimal operator of the process (X, S). In Section 3, we present a solution to the free-boundary problem and derive (first-order) nonlinear ordinary differential equations for the optimal stopping boundary under different relationships on the parameters of the model. In Section 4, we verify that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem. In Section 5, the obtained result is applied for determining the best constants in some maximal inequalities for a compound Poisson process with linear drift and exponential jumps. The main result of the paper is stated in Theorem 4.1.

2. Formulation of the problem

2.1. For a precise probabilistic formulation of the problem let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a jump process $J = (J_t)_{t\geq 0}$ defined by $J_t = \sum_{i=1}^{N_t} Y_i$, where $N = (N_t)_{t\geq 0}$ is a Poisson process of the intensity λ, and $(Y_i)_{i\in\mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter 1 (N and $(Y_i)_{i\in\mathbb{N}}$ are supposed to be independent). It is assumed that there exists a process $X = (X_t)_{t\geq 0}$ solving the stochastic differential equation:

$$dX_t = \eta(X_t)\, dt + \theta\, dJ_t \quad (X_0 = x) \tag{2.1}$$

with a Lipschitz function $\eta(x) \neq 0$ on $\mathbb{R}$ and a constant $\theta \neq 0$, where $x \in \mathbb{R}$ is given and fixed. The processes of such type were considered e.g. in [4]. For simplicity of exposition throughout the paper we will assume that the state space of the process X is $\mathbb{R}$. With the process X let us associate the *maximum process* $S = (S_t)_{t\geq 0}$ defined by:

$$S_t = \Big(\max_{0\leq u\leq t} X_u \Big) \vee s \tag{2.2}$$

for an arbitrary $s \geq x$. The main purpose of the present paper is to give a solution to the optimal stopping problem for the time-homogeneous (strong) Markov process $(X, S) = (X_t, S_t)_{t\geq 0}$ given by:

$$V_*(x, s) = \sup_\tau E_{x,s} \left[S_\tau - \int_0^\tau c(X_t)\, dt \right], \tag{2.3}$$

where $P_{x,s}$ is a probability measure under which the process (X, S) starts at some $(x, s) \in E$, and the supremum is taken over all stopping times τ of the process X (i.e. stopping times with respect to $(\mathcal{F}_t^X)_{t\geq 0}$ denoting the natural filtration of X defined by $\mathcal{F}_t^X = \sigma\{X_u \,|\, 0 \leq u \leq t\}$, $t \geq 0$) satisfying the condition:

$$E_{x,s} \left[\int_0^\tau c(X_t)\, dt \right] < \infty \tag{2.4}$$

with some continuous cost function $c(x) > 0$ on $\mathbb{R}$. Here by $E = \{(x,s) \in \mathbb{R}^2 \,|\, x \le s\}$ we denote the state space of the process (X,S). By means of the same arguments as in [5] and [22] it can be shown that the optimal stopping time in the problem (2.3) should be given by:

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \le g_*(S_t)\} \tag{2.5}$$

for some function $g_*(s)$ such that $g_*(s) < s$ for all $s \in \mathbb{R}$. In this connection the function $g_*(s)$ is called an *optimal stopping boundary*. Note that $g_*(s)$ is the largest number x from $\mathbb{R}$ such that $V_*(x,s) = s$ for each $s \in \mathbb{R}$ fixed.

2.2. By means of standard arguments it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X,S) acts on a function $F \in C^{1,1}(E)$ according to the rule:

$$(\mathbb{L}F)(x,s) = \eta(x)\,\frac{\partial F}{\partial x}(x,s) + \int_0^\infty \Big(F(x+\theta y, (x+\theta y) \vee s) - F(x,s)\Big)\,\lambda e^{-y}\,dy \tag{2.6}$$

for all $x < s$. In order to find explicit expressions for the unknown value function $V_*(x,s)$ from (2.3) and the optimal stopping boundary $g_*(s)$ from (2.5), using the results of the general theory of optimal stopping problems for Markov processes (see, e.g., [11] and [28; Chapter III, Section 8] or [25]), we can formulate the following *integro-differential free-boundary problem*:

$$(\mathbb{L}V)(x,s) = c(x) \quad \text{for} \quad g(s) < x < s, \tag{2.7}$$

$$V(x,s)\big|_{x=g(s)+} = s \quad (\textit{continuous fit}), \tag{2.8}$$

$$V(x,s) = s \quad \text{for} \quad x < g(s), \tag{2.9}$$

$$V(x,s) > s \quad \text{for} \quad g(s) < x \le s \tag{2.10}$$

for each $s \in \mathbb{R}$. Note that by virtue of the superharmonic characterization of the value function (see [6] and [28]) it follows that $V_*(x,s)$ is the smallest function satisfying the conditions (2.7)-(2.10). Moreover, under some relations on the parameters of the model which are specified below, the following conditions can be satisfied or break down:

$$\frac{\partial V}{\partial x}(x,s)\Big|_{x=g(s)+} = 0 \quad (\textit{smooth fit}), \tag{2.11}$$

$$\frac{\partial V}{\partial s}(x,s)\Big|_{x=s-} = 0 \quad (\textit{normal reflection}) \tag{2.12}$$

for each $s \in \mathbb{R}$.

3. Solution of the free-boundary problem

3.1. Let us first assume that $\theta > 0$ and $\eta(x) < 0$ for all $x \in \mathbb{R}$. In this case, by means of straightforward calculations we get that the equation (2.7) takes the form:

$$\eta(x)\frac{\partial V}{\partial x}(x,s)e^{-\alpha x} + \int_x^\infty V(z, z \vee s)\,\lambda\alpha e^{-\alpha z}\,dz - V(x,s)\,\lambda e^{-\alpha x} = c(x)e^{-\alpha x} \tag{3.1}$$

with $\alpha = 1/\theta > 0$. Then, using the fact that by the integration-by-parts formula implies:

$$\int_x^s V(z,s)\,\alpha e^{-\alpha z}\,dz = \int_x^s \frac{\partial V}{\partial x}(z,s)e^{-\alpha z}\,dz - V(s,s)e^{-\alpha s} + V(x,s)e^{-\alpha x}, \tag{3.2}$$

we may conclude that the equation (3.1) is equivalent to the following (first order) ordinary differential equation:

$$-\eta(x)\frac{\partial G}{\partial x}(x,s) + \lambda G(x,s) = c(x)e^{-\alpha x}, \tag{3.3}$$

where we set:

$$G(x,s) = \int_x^s \frac{\partial V}{\partial x}(z,s)\,e^{-\alpha z}\,dz + \int_s^\infty V(z,z)\,\alpha e^{-\alpha z}\,dz - V(s,s)\,e^{-\alpha s} \tag{3.4}$$

for all $g(s) < x < s$. By virtue of the fact that in this case, leaving the continuation region $g_*(s) < x \leq s$ the process X can pass through the boundary $g_*(S)$ for the first time only continuously, let us further assume that the smooth-fit condition (2.11) holds. Solving the equation (3.3), we obtain that the function $G(x,s)$ admits the representation:

$$G(x,s) = \int_x^s \frac{c(y)e^{-\alpha y}}{\eta(y)} \exp\left(-\int_x^y \frac{\lambda dz}{\eta(z)}\right) dy + D(s)\exp\left(-\int_x^s \frac{\lambda dz}{\eta(z)}\right) \tag{3.5}$$

for $g(s) < x \leq s$, and since from (3.4) it follows that:

$$\frac{\partial V}{\partial x}(x,s) = -\frac{\partial G}{\partial x}(x,s)\,e^{\alpha x}, \tag{3.6}$$

from where, by means of the condition (2.11), we find that the function $D(s)$ from (3.5) takes the expression:

$$D(s) = \frac{c(g(s))}{\lambda e^{\alpha g(s)}} \exp\left(\int_{g(s)}^s \frac{\lambda dz}{\eta(z)}\right) - \int_{g(s)}^s \frac{c(y)e^{-\alpha y}}{\eta(y)} \exp\left(\int_y^s \frac{\lambda dz}{\eta(z)}\right) dy, \tag{3.7}$$

then, integrating the expression (3.6) and using the representation (3.5), we may conclude that the solution of the system (2.7)-(2.9) takes the form:

$$\begin{aligned} V(x,s;g(s)) = s &- \frac{c(g(s))}{e^{\alpha g(s)}} \int_{g(s)}^x \frac{\lambda e^{\alpha y}}{\eta(y)} \exp\left(\int_{g(s)}^y \frac{\lambda dz}{\eta(z)}\right) dy \\ &+ \int_{g(s)}^x \left(\frac{c(y)}{\eta(y)} + \frac{\lambda e^{\alpha y}}{\eta(y)} \int_{g(s)}^y \frac{c(z)e^{-\alpha z}}{\eta(z)} \exp\left(\int_z^y \frac{\lambda du}{\eta(u)}\right) dz\right) dy \end{aligned} \tag{3.8}$$

for all $g(s) < x \leq s$ and each $s \in \mathbb{R}$ with $\alpha = 1/\theta > 0$. In order to determine the optimal stopping boundary $g_*(s)$, we observe that setting $x = s$ into (3.4)-(3.5), it follows that for the function $D(s)$ we have the expression:

$$D(s) = \int_s^\infty V(z,z)\,\alpha e^{-\alpha z}\,dz - V(s,s)\,e^{-\alpha s} \tag{3.9}$$

for $s \in \mathbb{R}$. Then, substituting the expressions (3.7) for $D(s)$ and (3.8) for $V(s,s)$ into (3.9) and assuming that the functions $c(x)$ and $g(s)$ are continuously differentiable, differentiating both sides of the expression (3.9), after some transformations we obtain the equality:

$$\left(\frac{d}{ds} \frac{c(g(s))}{e^{\alpha g(s)}} \right) \left(\int_{g(s)}^{s} \frac{e^{\alpha y}}{\eta(y)} \exp\left(\int_{g(s)}^{y} \frac{\lambda dz}{\eta(z)} \right) dy - \lambda e^{\alpha s} \exp\left(\int_{g(s)}^{s} \frac{\lambda dz}{\eta(z)} \right) \right) = 1 \tag{3.10}$$

for each $s \in \mathbb{R}$ with $\alpha = 1/\theta > 0$.

3.2. Let us now assume that $\theta < 0$ and $\eta(x) > 0$ for all $x \in \mathbb{R}$. In this case, using the condition (2.9), by means of straightforward calculations we obtain that the equation (2.7) takes the form:

$$\eta(x) \frac{\partial V}{\partial x}(x,s)\, e^{-\alpha x} - \int_{g(s)}^{x} V(z,s)\, \lambda \alpha e^{-\alpha z}\, dz + s\, \lambda e^{-\alpha g(s)} - V(x,s)\, \lambda e^{-\alpha x} = c(x)\, e^{-\alpha x} \tag{3.11}$$

with $\alpha = 1/\theta < 0$. Then, using the fact that the integration-by-parts formula implies:

$$\int_{g(s)}^{x} V(z,s)\, \alpha e^{-\alpha z}\, dz = \int_{g(s)}^{x} \frac{\partial V}{\partial x}(z,s)\, e^{-\alpha z}\, dz - V(x,s)\, e^{-\alpha x} + V(g(s),s)\, e^{-\alpha g(s)} \tag{3.12}$$

and by virtue of the fact that the condition (2.8) yields $V(g(s),s)$, we may conclude that the equation (3.11) is equivalent to the following (first order) ordinary differential equation:

$$-\eta(x) \frac{\partial H}{\partial x}(x,s) + \lambda H(x,s) = c(x)\, e^{-\alpha x}, \tag{3.13}$$

where we set:

$$H(x,s) = - \int_{g(s)}^{x} \frac{\partial V}{\partial x}(z,s)\, e^{-\alpha z}\, dz \tag{3.14}$$

for all $g(s) < x < s$. Solving the equation (3.13), we obtain that the function $H(x,s)$ admits the representation:

$$H(x,s) = - \int_{g(s)}^{x} \frac{c(y) e^{-\alpha y}}{\eta(y)} \exp\left(\int_{y}^{x} \frac{\lambda dz}{\eta(z)} \right) dy \tag{3.15}$$

for $g(s) < x \le s$, and since from (3.14) it follows that:

$$\frac{\partial V}{\partial x}(x,s) = - \frac{\partial H}{\partial x}(x,s)\, e^{\alpha x}, \tag{3.16}$$

then integrating the expression (3.16) and using the representation (3.15), we may conclude that the solution of the system (2.7)-(2.9) takes the form:

$$V(x,s;g(s)) = s + \int_{g(s)}^{x} \left(\frac{c(y)}{\eta(y)} + \frac{\lambda e^{\alpha y}}{\eta(y)} \int_{g(s)}^{y} \frac{c(z) e^{-\alpha z}}{\eta(z)} \exp\left(\int_{z}^{y} \frac{\lambda du}{\eta(u)} \right) dz \right) dy \tag{3.17}$$

for all $g(s) < x \le s$ and each $s \in \mathbb{R}$ with $\alpha = 1/\theta < 0$. By virtue of the fact that in this case the process X can hit the diagonal in $\mathbb{R}^2$ only continuously, in order to determine the optimal stopping boundary $g_*(S)$, let us further assume that the normal-reflection condition (2.12) holds. Then, assuming that the function $g(s)$ is continuously differentiable, differentiating both sides of the expression (3.17) and setting $x = s$, after some transformations we obtain the equality:

$$g'(s)\frac{c(g(s))}{\eta(g(s))}\left(1+\int_{g(s)}^{s}\frac{\lambda e^{\alpha(y-g(s))}}{\eta(y)}\exp\left(\int_{g(s)}^{y}\frac{\lambda dz}{\eta(z)}\right)dy\right)=1 \tag{3.18}$$

for each $s \in \mathbb{R}$ with $\alpha = 1/\theta < 0$.

We will further assume that there exist maximal solutions $g_*(s)$ of the (first order) ordinary differential equations (3.10) and (3.18), staying strictly below the diagonal in $\mathbb{R}^2$, and show that these solutions turn out to be optimal stopping boundaries in (2.5).

4. Main result and proof

Taking into account the facts proved above let us now formulate the main assertion of the paper, which extends the results of the articles [5] and [22] to the case of some jump processes.

Theorem 4.1. *Suppose that the process (X, S) is defined in (2.1)-(2.2), under $\theta > 0$ and $\eta(x) < 0$ there exists a maximal solution $g_*(s)$ of the equation (3.10), and under $\theta < 0$ and $\eta(x) > 0$ there exists a maximal solution $g_*(s)$ of the equation (3.18), where in both cases $g_*(s) < s$ for all $s \in \mathbb{R}$. Then the stopping time τ_* defined in (2.5) is optimal in the problem (2.3) whenever it satisfies the condition (2.4), and the value function is finite and takes the expression:*

$$V_*(x,s)=\begin{cases}V(x,s;g_*(s)), & g_*(s)<x\le s,\\ s, & x\le g_*(s),\end{cases} \tag{4.1}$$

where under $\theta > 0$ and $\eta(x) < 0$ the function $V(x, s; g(s))$ is given by (3.8), and under $\theta < 0$ and $\eta(x) > 0$ the function $V(x, s; g(s))$ is given by (3.17).

Proof. Let us show that the function (4.1) coincides with the value function (2.3) and the maximal solutions $g_*(s)$ of the equations (3.10) and (3.18), staying strictly below the diagonal in $\mathbb{R}^2$, are the optimal stopping boundaries in (2.5). For this let us introduce the function:

$$V_g(x,s)=\begin{cases}V(x,s;g(s)), & g(s)<x\le s,\\ s, & x\le g(s),\end{cases} \tag{4.2}$$

where under $\theta > 0$ and $\eta(x) < 0$ the function $V(x, s; g(s))$ is given by (3.8) and the function $g(s)$ solves the equation (3.10), and under $\theta < 0$ and $\eta(x) > 0$ the function $V(x, s; g(s))$ is given by (3.17) and the function $g(s)$ solves the equation (3.18). In this case by straightforward calculations and the assumptions above it follows that the function $V(x, s)$ satisfies the system (2.7)-(2.9) as well as the condition (2.11) under $\theta > 0$ and $\eta(x) < 0$, and the condition (2.12) under $\theta < 0$ and $\eta(x) > 0$. Then, applying Itô's formula for semimartingales (see e.g. [13; Chapter I, Theorem 4.57] or [18; Chapter II, Theorem 6.1]) to $V_g(X_t, S_t)$, we obtain:

$$\begin{aligned}V_g(X_t,S_t)&=V_g(x,s)+\int_0^t(\mathbb{L}V_g)(X_u,S_u)I(X_u\neq g(S_u),X_u<S_u)\,du+M_t\\&+\int_0^t\frac{\partial V_g}{\partial s}(X_{u-},S_{u-})\,dS_u-\sum_{0<u\le t}\frac{\partial V_g}{\partial s}(X_{u-},S_{u-})\,\Delta S_u,\end{aligned} \tag{4.3}$$

where the process $(M_t)_{t \geq 0}$ defined by:

$$M_t = \int_0^t \int_0^\infty \Big(V_g\big(X_{u-} + \theta y, (X_{u-} + \theta y) \vee S_{u-}\big) - V_g(X_{u-}, S_{u-}) \Big) \, (\mu(du, dy) - \nu(du, dy)), \tag{4.4}$$

is a local martingale under the measure $P_{x,s}$ with respect to $(\mathcal{F}_t^X)_{t \geq 0}$, and $\mu(du, dy)$ is the measure of jumps of the process J having the compensator $\nu(du, dy) = \lambda du I(y > 0) e^{-y} dy$. Observe that when $\theta > 0$ and $\eta(x) < 0$ the time spent by the process X at the diagonal in $\mathbb{R}^2$ is of Lebesgue measure zero that permits to extend the function $(\mathbb{L}V_g)(x, s)$ arbitrarily to $x = s$, as well as by virtue of the fact that in this case we have $dS_u = \Delta S_u$, the integral with respect to dS_u in (4.3) is compensated by the sum with respect to ΔS_u. On the other hand, when $\theta < 0$ and $\eta(x) > 0$ the time spent by X at the boundary $g(S)$ is of Lebesgue measure zero that permits to extend $(\mathbb{L}V_g)(x, s)$ arbitrarily to $x = g(s)$, as well as the sum with respect to ΔS_u in (4.3) is equal to zero and the same is the integral with respect to dS_u, since in the latter case the process S can increase only at the diagonal in $\mathbb{R}^2$, where we assume that the condition (2.12) is satisfied.

By virtue of the arguments above we may conclude that $(\mathbb{L}V_g)(x, s) \leq c(x)$ for all $x < s$. Moreover, by means of straightforward calculations, it can be shown that the property (2.10) also holds, that together with the condition (2.9) implies $V_g(x, s) \geq s$ for all $x \leq s$. From the expression (4.3) it therefore follows that the inequalities:

$$S_\tau - \int_0^\tau c(X_u) \, du \leq V_g(X_\tau, S_\tau) - \int_0^\tau c(X_u) \, du \leq V_g(x, s) + M_\tau \tag{4.5}$$

hold for any stopping time τ of the process X.

Let $(\sigma_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(M_t)_{t \geq 0}$. Then taking in (4.5) expectation with respect to the measure $P_{x,s}$, by means of the optional sampling theorem (see e.g. [13; Chapter I, Theorem 1.39]) we get:

$$\begin{aligned} E_{x,s} \left[S_{\tau \wedge \sigma_n} - \int_0^{\tau \wedge \sigma_n} c(X_u) \, du \right] &\leq E_{x,s} \left[V_g(X_{\tau \wedge \sigma_n}, S_{\tau \wedge \sigma_n}) - \int_0^{\tau \wedge \sigma_n} c(X_u) \, du \right] \\ &\leq V_g(x, s) + E_{x,s}[M_{\tau \wedge \sigma_n}] = V_g(x, s) \end{aligned} \tag{4.6}$$

for all $x \leq s$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that for any stopping time τ satisfying the condition (2.4), the inequalities:

$$E_{x,s} \left[S_\tau - \int_0^\tau c(X_u) \, du \right] \leq E_{x,s} \left[V_g(X_\tau, S_\tau) - \int_0^\tau c(X_u) \, du \right] \leq V_g(x, s) \tag{4.7}$$

hold for all $x \leq s$. Taking in (4.7) the supremum over all stopping times τ satisfying the condition (2.4), and then infimum over all boundaries g, by virtue of the obvious fact that the function $g \mapsto V_g(x, s)$ is (strictly) decreasing, we may therefore conclude that:

$$V_*(x, s) \leq \inf_g V_g(x, s) = V_{g_*}(x, s) \tag{4.8}$$

for all $x \leq s$, from where it is seen that one should take maximal solutions of the equations (3.10) and (3.18) as candidates for the optimal stopping boundary in (2.5).

In order to show that the equalities in (4.7)-(4.8) are attained under τ_* from (2.5), let us use the fact that the function $V_{g_*}(x,s)$ from (4.2) together with the boundary $g_*(s)$ satisfy the system (2.7)-(2.9). In this case by the structure of the stopping time τ_* in (2.5) and the expression (4.3) it follows that the equality:

$$V_{g_*}(X_{\tau_*\wedge\sigma_n}, S_{\tau_*\wedge\sigma_n}) - \int_0^{\tau_*\wedge\sigma_n} c(X_u)\,du = V_{g_*}(x,s) + M_{\tau_*\wedge\sigma_n} \tag{4.9}$$

is satisfied, and by virtue of the expression (4.5), we may conclude that the inequalities:

$$-\int_0^{\tau_*\wedge\sigma_n} c(X_u)\,du \leq V_{g_*}(x,s) + M_{\tau_*\wedge\sigma_n} \leq V_{g_*}(X_{\tau_*\wedge\sigma_n}, S_{\tau_*\wedge\sigma_n}) - \int_0^{\tau_*\wedge\sigma_n} c(X_u)\,du \tag{4.10}$$

hold for all $x \leq s$, where $(\sigma_n)_{n\in\mathbb{N}}$ is a localizing sequence for $(M_t)_{t\geq 0}$. Hence, letting n go to infinity in the expression (4.9) and taking into account the equalities (4.7)-(4.8) as well as the property $V_{g_*}(X_{\tau_*}, S_{\tau_*}) = S_{\tau_*}$ also satisfied, by means of the Lebesgue bounded convergence theorem we obtain the equality:

$$E_{x,s}\left[S_{\tau_*} - \int_0^{\tau_*} c(X_u)\,du\right] = V_{g_*}(x,s) \tag{4.11}$$

for all $x \leq s$, from where the desired assertion follows. □

Remark 4.1. It can be easily verified that in case when $\theta > 0$ and $\eta(x) < 0$, for the function $V_*(x,s)$ from (4.1) the normal-reflection condition (2.12) breaks down, and at the same time the smooth-fit condition (2.11) at the boundary $g_*(s)$ is satisfied. This can be explained by the fact that in the given case the process X can hit the diagonal in $\mathbb{R}^2$ only by jumping, while it can leave the continuation region $g_*(s) < x \leq s$ only continuously.

Remark 4.2. On the other hand, by means of straightforward calculations, it can be shown that in case when $\theta < 0$ and $\eta(x) > 0$ for the function $V_*(x,s)$ from (4.1) the smooth-fit condition (2.11) at the boundary $g_*(s)$ breaks down, that can be explained by the fact that in the given case, leaving the continuation region $g_*(s) < x \leq s$ the process X can pass through the boundary $g_*(S)$ for the first time only by jumping. Such an effect was earlier observed and explained in [23]-[24] by solving some other optimal stopping problems for jump processes. According to the results in [1] we may conclude that this property appears because of finite intensity of jumps and exponential distribution of jump sizes of the compound Poisson process J.

Remark 4.3. Note that, at the same time, in case when $\theta < 0$ and $\eta(x) > 0$, for the function $V_*(x,s)$ from (4.1) the normal-reflection condition (2.12) is satisfied, that can be explained by the fact that the process X can hit the diagonal in $\mathbb{R}^2$ only continuously. This condition was earlier observed and explained in [5] and then in [22].

5. Maximal inequalities

Let us now consider the application of the results derived above for determining the best constants in some maximal inequalities for a compound Poisson process with linear drift and exponential jumps. For this in the course of all the section we assume that the functions $\eta(x)$ in (2.1) and $c(x)$ in (2.3)-(2.4) are constant, from where, in particular, it follows that $X = (X_t)_{t\geq 0}$ is a stationary process with independent increments (a Lévy process). In this case, if there exist maximal solutions of the equations (3.10) and (3.18), staying strictly below the diagonal in $\mathbb{R}^2$, then they get the form $g_*(s) = s - h_*$, so that, the optimal stopping time (2.5) has the structure:

$$\tau_* = \inf\{t \geq 0 \,|\, S_t - X_t \geq h_*\} \tag{5.1}$$

with some constant $h_* > 0$. Taking into account these arguments let us formulate the assertions, which straightforwardly follow from Theorem 3.1.

Corollary 5.1. Suppose that in (2.1) we have $\theta = 1$ and $\eta(x) = \eta < 0$ for all $x \in \mathbb{R}$. Then in case when $\eta < -1/\lambda^2$ and $0 < c < 1/\lambda^2$ as well as when $-1/\lambda^2 < \eta < 0$ and $\eta + 1/\lambda^2 < c < 1/\lambda^2$ the expression (3.8) takes the form:

$$V(x, s; g(s)) = s + \frac{c\lambda^2}{\lambda^2\eta + 1}\big(x - g(s)\big) - \frac{c\lambda^3\eta}{(\lambda^2\eta + 1)^2}\Big(e^{\alpha(x-g(s))} - 1\Big) \tag{5.2}$$

with $\alpha = 1/(\lambda\eta) + \lambda$ and for h_* in (5.1) we get the representation:

$$h_* = \frac{\lambda\eta}{\lambda^2\eta + 1} \log\left(\frac{\lambda^2(\eta - c) + 1}{\lambda^4\eta c}\right), \tag{5.3}$$

and in case when $\eta = -1/\lambda^2$ and $0 < c < 1/\lambda^2$ (3.8) has the form:

$$V(x, s; g(s)) = s + \frac{c\lambda^3}{2}\big(x - g(s)\big)^2 \tag{5.4}$$

and for h_* in (5.1) we have:

$$h_* = \frac{1 - c\lambda^2}{c\lambda^3}. \tag{5.5}$$

Corollary 5.2. Suppose that in (2.1) we have $\theta = -1$ and $\eta(x) = \eta > 0$ for all $x \in \mathbb{R}$. Then in case when $0 < \eta < 1/\lambda^2$ and $\eta < c$ as well as when $1/\lambda^2 < \eta$ and $\eta - 1/\lambda^2 < c < \eta$ the expression (3.17) takes the form:

$$V(x, s; g(s)) = s + \frac{c\lambda^2}{\lambda^2\eta - 1}\big(x - g(s)\big) + \frac{c\lambda}{(\lambda^2\eta - 1)^2}\Big(e^{\beta(x-g(s))} - 1\Big) \tag{5.6}$$

with $\beta = 1/(\lambda\eta) - \lambda$ and for h_* in (5.1) we get the representation:

$$h_* = -\frac{\lambda\eta}{\lambda^2\eta - 1} \log\left(\frac{\lambda^2\eta(c - \eta) + \eta}{c}\right), \tag{5.7}$$

and in case when $\eta = 1/\lambda^2$ and $0 < c < 1/\lambda^2$ (3.17) has the form:

$$V(x, s; g(s)) = s + c\lambda^2\big(x - g(s)\big) + \frac{c\lambda^3}{2}\big(x - g(s)\big)^2 \tag{5.8}$$

and for h_* in (5.1) we have (5.5).

Finally, setting $x = s = 0$ in (2.1)-(2.2) and underlying the dependence of the value function from the parameter c, we observe that under the assumptions above the expression (2.3) takes the form:

$$V_*(0,0;c) = \sup_\tau E\Big[\max_{0\le u\le\tau} X_u - c\tau\Big], \tag{5.9}$$

from where we obtain that for any arbitrary stopping time τ of the process X the following inequality is satisfied:

$$E\Big[\max_{0\le u\le\tau} X_u\Big] \le V_*(0,0;c) + cE[\tau]. \tag{5.10}$$

In this case the following assertions hold.

Example 5.1. Let the process $X = (X_t)_{t\ge0}$ be of the form $X_t = J_t - t/\lambda^2$ for all $t \ge 0$. Then from Corollary 4.1 and the inequality (5.10) it follows that for any stopping time τ of the process X we have the expression:

$$E\Big[\max_{0\le u\le\tau} X_u\Big] \le \inf_{0<c<1/\lambda^2}\left(\frac{(1-c\lambda^2)^2}{2c\lambda^3} + cE[\tau]\right), \tag{5.11}$$

where the infimum is attained at $c = 1/\sqrt{\lambda^4 + 2\lambda^3 E[\tau]}$. From (5.11) we may therefore conclude that for any stopping time τ the following inequality holds:

$$E\Big[\max_{0\le u\le\tau} X_u\Big] \le \frac{\sqrt{1+2E[\tau]/\lambda} - 1}{\lambda}. \tag{5.12}$$

Example 5.2. Let the process $X = (X_t)_{t\ge0}$ be of the form $X_t = t/\lambda^2 - J_t$ for all $t \ge 0$. Then from Corollary 4.2 and the inequality (5.10) it follows that for any stopping time τ of the process X such that $E[\tau] > \lambda$ we have the expression:

$$E\Big[\max_{0\le u\le\tau} X_u\Big] \le \inf_{0<c<1/\lambda^2}\left(\frac{1-c^2\lambda^4}{2c\lambda^3} + cE[\tau]\right), \tag{5.13}$$

where the infimum is attained at $c = 1/\sqrt{2\lambda^3 E[\tau] - \lambda^4}$. From (5.13) we may therefore conclude that for any stopping time τ such that $E[\tau] > \lambda$ the following inequality holds:

$$E\Big[\max_{0\le u\le\tau} X_u\Big] \le \frac{\sqrt{2E[\tau]/\lambda - 1}}{\lambda}. \tag{5.14}$$

Acknowledgments. The author thanks Goran Peskir for many useful discussions of optimal stopping problems for maxima processes.

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Pavel V. Gapeev
Weierstraß Institute
for Applied Analysis and Stochastics (WIAS)
Mohrenstr. 39, D-10117 Berlin, Germany
e-mail: gapeev@wias-berlin.de

(Russian Academy of Sciences
Institute of Control Sciences
Profsoyuznaya Str. 65
117997 Moscow, Russia)