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# The integral option in a model with jumps

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We present a closed form solution to the considered in [15] optimal stopping problem for the case of geometric compound Poisson process with exponential jumps. The method of proof is based on reducing the initial problem to an integro-differential free-boundary problem and solving the latter by using continuous and smooth fit. The result can be interpreted as pricing perpetual integral options in a model with jumps.

## 1. Introduction

The main aim of this paper is to present a closed form solution to the optimal stopping problem (3) for the process $S$ defined in (1)-(2). This problem is related to the option pricing theory in mathematical insurance, where the process $S$ can describe the risk process of a company (see, e.g., [10] and [25; Chapter I, Section 3c]). In that case, the value (3) can be interpreted as a *fair price* of a *perpetual integral* option in a jump market model.

It is known that the change-of-measure theorem allows to reduce the dimension of optimal stopping problems for continuous time Markov processes. By means of introducing the so-called *dual* martingale measure, Shepp and Shiryaev [23] reduced the Russian option problem to an optimal stopping problem for a one-dimensional Markov reflected diffusion process. By using similar arguments, Kramkov and Mordecki [15] reduced the perpetual integral option

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problem to an optimal stopping problem for Shiryaev's diffusion process playing the central role in the 'disorder' problems (see, e.g., [24; Chapter IV, Section 4]). These arguments were recently applied by Peskir and Uys [21] for solving the early exercise Asian option problem in the classical Black-Merton-Scholes model, and in [8] for solving the perpetual average option problem in a diffusion-type model with delay. Following the same methodology, in the present paper we solve the problem (3) being a discounted optimal stopping problem for an integral of a jump process $S$ defined in (1)-(2) under some relationships on the parameters of the model. In order to be able to derive a closed form solution we let the jumps of the driving compound Poisson process be exponentially distributed. Some other optimal stopping problems in such a model were solved, for example, in [9], [16]-[17], [13]-[14], [3]-[4] and [7]. The key point in solving optimal stopping problems for jump processes established in [18]-[19] is that the smooth fit at the optimal boundary may break down and then be replaced by the continuous fit (see also [2] for necessary and sufficient conditions for the occurrence of smooth-fit condition and references to the related literature, and [20] for an extensive overview).

The paper is organized as follows. In Section 2, by using change-of-measure arguments, for the initial problem (3) we construct the equivalent optimal stopping problem (10), where the process $X$ defined in (7) can be considered as an analogue of Shiryaev's process for the jump model (1)-(2). By analyzing the sample-path behavior of the process $X$, we give explicit estimations for the optimal stopping boundary under several relationships on the parameters of the model. In Section 3, we formulate the corresponding integro-differential free-boundary problem for the infinitesimal operator of the process $X$ and derive a solution, which can be expressed by means of Gauss' and Kummer's hypergeometric functions and thus admits a representation in a closed form. In Section 4, we verify that the solution of the free-boundary problem turns out to be a solution of the initial optimal stopping problem and comment the structure of the solution under different relationships on the parameters of the model. The main result of the paper is stated in Theorem 4.1.

## 2. Formulation of the problem

In this section we introduce the setting and notation of the optimal stopping problem which is related to pricing integral option.

2.1. For a precise formulation of the problem let us consider a probability space $(\Omega, \mathcal{F}, P)$

with a jump process $J = (J_t)_{t \ge 0}$ defined by $J_t = \sum_{i=1}^{N_t} Y_i$, where $N = (N_t)_{t \ge 0}$ is a Poisson process of the intensity $\lambda > 0$, and $(Y_i)_{i \in \mathbb{N}}$ is a sequence of independent random variables exponentially distributed with parameter 1 ($N$ and $(Y_i)_{i \in \mathbb{N}}$ are supposed to be independent). Assume that there exists a process $S = (S_t)_{t \ge 0}$ given by:

$$S_t = s \, \exp\Big( \big(r - \delta - \lambda\theta/(1-\theta)\big)\, t + \theta \, J_t \Big) \tag{1}$$

where $0 \le \delta < r$ and $\theta < 1$, $\theta \neq 0$. It follows that the process $S$ solves the stochastic differential equation:

$$dS_t = (r-\delta) S_{t-} \, dt + S_{t-} \int_0^\infty \Big( e^{\theta y} - 1 \Big) \big(\mu(dt,dy) - \nu(dt,dy)\big) \quad (S_0 = s) \tag{2}$$

where $s > 0$ is given and fixed. It can be assumed that the process $S$ describes the risk process of an insurance company, where $r$ is the riskless interest rate and the rate payed by the company is $\delta$. Here $\mu(dt,dy)$ is the measure of jumps of the process $J$ with the compensator $\nu(dt,dy) = \lambda dt I(y>0) e^{-y} dy$, which means that we work directly under a martingale measure for $S$ (see, e.g., [25; Chapter VII, Section 3g]). Note that the assumption $\theta < 1$ guarantees that the jumps of $S$ are integrable under the martingale measure, which is no restriction. The main purpose of the present paper is to derive a solution to the optimal stopping problem:

$$V_* = \sup_\tau E \left[ e^{-r\tau} \Big( \int_0^\tau S_u \, du + x \Big) \right] \tag{3}$$

where the supremum is taken over all finite stopping times $\tau$ with respect to the natural filtration $(\mathcal{F}_t)_{t \ge 0}$ of $S$. The value (3) coincides with an *arbitrage-free price* of a perpetual integral option. This can be argued according to the concept of *neutral derivative pricing*, which was adapted in [12] to the case of American-type options and can be applied for the considered incomplete jump market model. From the structure of the payoff in (3) it follows that without loss of generality we can further assume that $s = 1$. For the case when $S$ was a geometric Brownian motion the problem (3) was formulated and explicitly solved in [15].

2.2. Following the arguments from [23; Section 2] and [15; Section 1], we introduce the probability measures $\widetilde{P}^t$ by:

$$\frac{d\widetilde{P}^t}{dP^t} = \exp\Big( \theta \, J_t - \big(\lambda\theta/(1-\theta)\big)\, t \Big) \tag{4}$$

where $P^t = P \,|\, \mathcal{F}_t$ for all $t \ge 0$. Then, by means of the result of [22; Chapter VIII, Proposition 1.13] we may conclude that there exists a probability measure $\widetilde{P}$ such that $\widetilde{P} \,|\, \mathcal{F}_t = \widetilde{P}^t$, and $\widetilde{P}$ is locally equivalent to $P$ on the filtration $(\mathcal{F}_t)_{t \ge 0}$ with the density process (4). Hence, by virtue of Girsanov's theorem for semimartingales (see, e.g., [11; Chapter III, Theorem 5.34]) we may conclude that the process $J$ has the compensator $\widetilde{\nu}(dt, dy) = \lambda dt I(y > 0) e^{-(1-\theta)y} dy$ under $\widetilde{P}$ playing the role of *dual* martingale measure. Observe that, by using the explicit expression (1) as well as the assumption $s = 1$, from (4) we obtain that the expression:

$$\frac{d\widetilde{P} \,|\, \mathcal{F}_\tau}{dP \,|\, \mathcal{F}_\tau} = e^{-(r-\delta)\tau} \, S_\tau \tag{5}$$

holds for all finite stopping times $\tau$. It therefore follows that the value (3) takes the form:

$$V_* = \sup_\tau \widetilde{E}\big[e^{-\delta\tau} \, X_\tau\big] \tag{6}$$

where the process $X = (X_t)_{t \ge 0}$ is given by:

$$X_t = \frac{1}{S_t} \Big( \int_0^t S_u \, du + x \Big). \tag{7}$$

By using Itô's formula for semimartingales (see, e.g., [11; Chapter I, Theorem 4.57]), in this case it is shown that the process $X$ solves the stochastic differential equation:

$$dX_t = (1 - (r - \delta) X_{t-}) \, dt - X_{t-} \int_0^\infty \Big(1 - e^{-\theta y}\Big) \, (\mu(dt, dy) - \widetilde{\nu}(dt, dy)) \quad (X_0 = x). \tag{8}$$

It can be easily verified that $X$ is a time-homogeneous (strong) Markov process under $\widetilde{P}$ with respect to its natural filtration, which clearly coincides with $(\mathcal{F}_t)_{t \ge 0}$. Therefore, the supremum in (6) can be equivalently taken over all finite stopping times of the process $X$ playing the role of sufficient statistic in the given optimal stopping problem. We also note that if, in addition, $0 < \lambda\theta/(1-\theta) < r - \delta$ holds, then:

$$\widehat{B} = 1 \Big/ \Big( r - \delta - \frac{\lambda\theta}{1-\theta} \Big) \tag{9}$$

turns out to be a *singularity point* of equation (8) in the sense that the drift rate of the continuous part of the process $X$ is positive on the interval $[0, \widehat{B})$, negative on $(\widehat{B}, \infty)$, and equal to zero at the point $\widehat{B}$.

2.3. In order to compute the value (6), let us consider the following optimal stopping problem for the Markov process $X$ given by:

$$V_*(x) = \sup_\tau \widetilde{E}_x\big[e^{-\delta\tau} \, X_\tau\big] \tag{10}$$

where the supremum in (10) is taken over all finite stopping times $\tau$ with respect to $(\mathcal{F}_t)_{t\ge 0}$, and $\widetilde{E}_x$ denotes the expectation under the assumption that the process $X$ defined in (7)-(8) starts at $x \ge 0$. Taking into account the structure of the payoff function in the problem (10), we will search for an optimal stopping time in the form:

$$\tau_* = \inf\{t \ge 0 \,|\, X_t \ge B_*\} \tag{11}$$

for some number $B_*$ to be determined. Observe that, by applying Itô's formula to $e^{-\delta t} X_t$ and by using the equation (8), it follows that:

$$e^{-\delta t} X_t = x + \int_0^t e^{-\delta u} \, (1 - rX_u) \, du + \int_0^t \int_0^\infty e^{-\delta u} \, X_{u-} \Big( e^{-\theta y} - 1 \Big) \left( \mu(dt, dy) - \widetilde{\nu}(dt, dy) \right) \tag{12}$$

where the last term is a martingale under the measure $\widetilde{P}_x$. Hence, taking in (12) expectation with respect to the measure $\widetilde{P}_x$, by means of the optional sampling theorem (see, e.g., [11; Chapter I, Theorem 1.39]), from (12) we obtain:

$$\widetilde{E}_x \big[ e^{-\delta \sigma} \, X_\sigma \big] = x + \widetilde{E}_x \left[ \int_0^\sigma e^{-\delta u} \, (1 - rX_u) \, du \right] \tag{13}$$

for any stopping time $\sigma$ being the exit time of the process $X$ from a sufficiently small ball. Therefore, it is easily seen from (13) that one should not stop the process $X$ in the interval $[0, \overline{B})$ with $\overline{B} = 1/r$ being a lower estimation for the optimal stopping boundary $B_*$ in the sense that $0 < \overline{B} \le B_*$.

2.4. By using the schema of arguments from [19] and by analyzing the sample path behavior of the process $X$, let us now make some conclusions on the optimal stopping boundary $B_*$ under several relationships on the parameters of the model.

**Remark 2.1.** Observe that if $\theta < 0$ then the process $X$ can have only positive jumps, it can leave $[0, \widehat{B})$ only by jumping and fluctuating in $(\widehat{B}, \infty)$ cannot enter $[0, \widehat{B})$. If $X$ gets into $\widehat{B}$, then it is trapped there until the next jump of $J$ occurs. Moreover, if $X$ is located in $[0, \widehat{B})$ or in $(\widehat{B}, \infty)$, then under the absence of jumps of $J$ the process $X$ will never reach $\widehat{B}$, because while it approaches to $\widehat{B}$ its local drift decreases to zero at the same time with linear order. Hence, if $0 < -\lambda\theta/(1-\theta) \le \delta$ also holds, then we have $\overline{B} \le \widehat{B}$. Recalling that the process $X$ is monotone increasing on $[0, \widehat{B})$, from the representation (13) we may therefore conclude that one should not stop $X$ on $[0, \overline{B})$, but one should stop it immediately after passing through $\overline{B}$, because after leaving $[0, \overline{B})$ the process $X$ never returns back. In other words, in this case for the optimal stopping boundary we have $B_* = \overline{B} \equiv 1/r$.

**Remark 2.2.** Note that if $0 < \theta < 1$ then the process $X$ can have only negative jumps. If, in addition, $r - \delta - \lambda\theta/(1-\theta) > 0$ holds, then $X$ is monotone decreasing on $(\widehat{B}, \infty)$, and by virtue of the structure of the value function (10), it follows that one should not stop $X$ on $(\widehat{B}, \infty)$. From the expression (13) it therefore follows that for the boundary $B_*$ we should have $\overline{B} \le B_* < \widehat{B}$, because otherwise it would not be optimal.

## 3. Solution of the free-boundary problem

In this section we derive a solution of the free-boundary problem associated with the initial optimal stopping problem.

3.1. By means of standard arguments it can be shown that the infinitesimal operator $\mathbb{L}$ of the process $X = (X_t)_{t \ge 0}$ acts on an arbitrary function $F(x)$ from the class $C^1$ on $(0, \infty)$ according to the rule:

$$(\mathbb{L}F)(x) = (1 - (r - \delta + \zeta)x)F'(x) + \int_0^\infty \Big(F\big(xe^{-\theta y}\big) - F(x)\Big)\, \lambda e^{-(1-\theta)y}\, dy \tag{14}$$

for all $x > 0$, where we denote $\zeta = -\lambda\theta/(1-\theta)$. In order to find explicit expressions for the unknown value function $V_*(x)$ from (10) and the boundary $B_*$ from (11), let us use the results of general theory of optimal stopping problems for Markov processes (see, e.g., [24; Chapter III, Section 8] and [20; Chapter IV, Section 8]). We can reduce the optimal stopping problem (10) to the free-boundary problem:

$$(\mathbb{L}V)(x) = \delta V(x) \quad \text{for} \quad 0 < x < B \tag{15}$$

$$V(B-) = B \quad (\textit{continuous fit}) \tag{16}$$

$$V(x) = x \quad \text{for} \quad x > B \tag{17}$$

$$V(x) > x \quad \text{for} \quad 0 \le x < B \tag{18}$$

for some $\overline{B} \le B [\le \widehat{B}]$, where (16) plays the role of instantaneous-stopping condition. Note that by virtue of the superharmonic characterization of the value function (see [6], [24] and [20; Chapter IV, Section 9]) it follows that $V_*(x)$ is the smallest function satisfying the conditions (15)-(18). Moreover, we further assume that the condition:

$$V'(B-) = 1 \quad (\textit{smooth fit}) \quad \text{if} \quad 0 < \theta < 1 \tag{19}$$

is satisfied for some $\overline{B} \le B [\le \widehat{B}]$. The latter can be explained by the fact that according to Remark 2.2, in this case, leaving the continuation region $[0, B_*)$ the process $X$ can pass through the boundary $B_*$ continuously. This property was earlier observed in [18; Section 2] and [19] by solving some other optimal stopping problems for jump processes.

3.2. By means of straightforward calculations we reduce the equation (15) to the form:

$$(1-(r-\delta+\zeta)x)V'(x)+(1-\alpha)\lambda x^{\alpha}G(x)=\Big(\delta-\frac{\lambda(1-\alpha)}{\alpha}\Big)\,V(x) \tag{20}$$

with $\alpha=1-1/\theta$ and $\zeta=-\lambda\theta/(1-\theta)$, where taking into account conditions (16)-(17) we set:

$$G(x)=-\int_x^B V(z)\,\frac{dz}{z^{\alpha+1}}+\frac{B^{1-\alpha}}{1-\alpha}\quad \text{if}\quad \alpha=1-1/\theta>1 \tag{21}$$

$$G(x)=\int_0^x V(z)\,\frac{dz}{z^{\alpha+1}}\quad \text{if}\quad \alpha=1-1/\theta<0 \tag{22}$$

for all $0<x<B$. Then, from (20) and (21)-(22) it follows that the function $G(x)$ solves the following (second-order) ordinary differential equation:

$$\begin{aligned} &x(1-(r-\delta+\zeta)x)G''(x) \\ &+\Big[(\alpha+1)(1-(r-\delta+\zeta)x)-\Big(\delta-\frac{\lambda(1-\alpha)}{\alpha}\Big)\,x\Big]\,G'(x)+(1-\alpha)\lambda G(x)=0 \end{aligned} \tag{23}$$

for $0<x<B$. Observe that equation (20) as well as (23) has the singularity point $\widehat{B}\equiv 1/(r-\delta+\zeta)$ whenever $r-\delta+\zeta>0$.

3.3. Let us now assume that $r-\delta+\zeta\neq 0$ holds with $\zeta=-\lambda\theta/(1-\theta)$. In this case, (23) is a Gauss' hypergeometric equation, which has the general solution:

$$G(x)=C_1\,A_1(x)+C_2\,x^{-\alpha}\,A_2(x) \tag{24}$$

where $C_1$ and $C_2$ are some arbitrary constants and the functions $A_1(x)$ and $A_2(x)$ are defined by:

$$A_1(x)=F\Big(\gamma_1,\gamma_2;\alpha+1;(r-\delta+\zeta)x\Big),\quad A_2(x)=F\Big(\gamma_1-\alpha,\gamma_2-\alpha;1-\alpha;(r-\delta+\zeta)x\Big) \tag{25}$$

for $0\le x[<\widehat{B}]$, and $\gamma_i$ for $i=1,2$ are explicitly given by:

$$\gamma_i=\Big(\frac{\alpha(\delta+\lambda)-1}{2\alpha(r-\delta+\zeta)}+\frac{\alpha}{2}\Big)+(-1)^i\sqrt{\Big(\frac{\alpha(\delta+\lambda)-1}{2\alpha(r-\delta+\zeta)}+\frac{\alpha}{2}\Big)^2+\frac{\lambda(1-\alpha)}{r-\delta+\zeta}} \tag{26}$$

with $\alpha=1-1/\theta$ and $\zeta=-\lambda\theta/(1-\theta)$. Here $F(a,b;c;x)$ denotes Gauss' hypergeometric function, which admits the integral representation:

$$F(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt \tag{27}$$

for $c > b > 0$ and the series expansion:

$$F(a,b;c;x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \tag{28}$$

for $c \neq 0, -1, -2, \ldots$ and $(c)_k = c(c+1)\cdots(c+k-1)$, $k \in \mathbb{N}$, where $\Gamma$ denotes Euler's Gamma function and the series converges under all $|x| < 1$ (see, e.g., [1; Chapter XV] and [5; Chapter II]). Therefore, differentiating both sides of the formulas (21)-(22), by using (24) we obtain that in this case the integro-differential equation (20) has the general solution:

$$V(x) = C_1\, x^{\alpha+1}\, A_1'(x) + C_2\, [x\, A_2'(x) - \alpha\, A_2(x)] \tag{29}$$

for $0 \le x [< \widehat{B}]$. Hence, applying conditions (21), (16) and (19) to the functions (24) and (29), respectively, we get that the following equalities:

$$C_1\, B^{\alpha}\, A_1(B) + C_2\, A_2(B) = \frac{B}{1-\alpha} \tag{30}$$

$$C_1\, B^{\alpha+1}\, A_1'(B) + C_2\, [B\, A_2'(B) - \alpha\, A_2(B)] = B \tag{31}$$

$$C_1\, B^{\alpha}\, [B\, A_1''(B) + (\alpha+1)\, A_1'(B)] + C_2\, [B\, A_2''(B) + (1-\alpha)\, A_2'(B)] = 1 \tag{32}$$

hold for some $\overline{B} \le B [\le \widehat{B}]$, where condition (32) is satisfied when $\alpha = 1 - 1/\theta < 0$.

Note that if, in addition, $\alpha = 1 - 1/\theta > 1$ and $0 < -\lambda\theta/(1-\theta) \le \delta$ holds, then by Remark 2.1 we may conclude that for the optimal stopping boundary we have $B_* = \overline{B} \equiv 1/r$. Hence, solving the system (30)-(31), by means of straightforward calculations we obtain that the solution of the system (15)-(17) is given by:

$$\begin{aligned} V(x; B_*) = & \frac{B_*^{2-\alpha} A_2'(B_*) - B_*^{1-\alpha} A_2(B_*)}{(1-\alpha) D(B_*)}\, x^{1+\alpha}\, A_1'(x) \\ & + \frac{(1-\alpha) B_* A_1(B_*) - B_*^2 A_1'(B_*)}{(1-\alpha) D(B_*)}\, [x\, A_2'(x) - \alpha\, A_2(x)] \end{aligned} \tag{33}$$

where the function $D(x)$ is defined by:

$$D(x) = x\, A_1(x)\, A_2'(x) - x\, A_1'(x)\, A_2(x) - \alpha\, A_1(x)\, A_2(x) \tag{34}$$

for all $0 \le x < B_* < \widehat{B}$, and under $B_* = \widehat{B}$ in (33) we may set $V(x; B_*) = V(x; B_*-)$. Here the functions $A_1'(x)$ and $A_2'(x)$ are given by:

$$A_1'(x) = \frac{\gamma_1 \gamma_2 (r - \delta + \zeta)}{\alpha + 1}\, F\Big(\gamma_1 + 1, \gamma_2 + 1; \alpha + 2; (r - \delta + \zeta) x\Big) \tag{35}$$

$$A_2'(x) = \frac{(\gamma_1 - \alpha)(\gamma_2 - \alpha)(r - \delta + \zeta)}{1 - \alpha}\, F\Big(\gamma_1 - \alpha + 1, \gamma_2 - \alpha + 1; 2 - \alpha; (r - \delta + \zeta) x\Big) \tag{36}$$

for $0 \le x < \widehat{B}$.

Observe that if, in addition, $\alpha = 1 - 1/\theta < 0$ holds, then we have $C_1 = 0$ in (24) and (29), since otherwise, from expression (20) it would follow that $V'(x) \to \pm\infty$ under $x \downarrow 0$ that should be excluded by virtue of the easily proved fact that the value function $V_*(x)$ from (10) is convex and increasing on the interval $[0, \infty)$. The latter fact follows from the known result that the value function of an optimal stopping problem with a convex reward is convex. Thus, solving the system (31)-(32) with $C_1 = 0$, by using straightforward calculations we obtain that the solution of the system (15)-(17)+(19) is given by:

$$V(x; B_*) = B_* \, \frac{x A_2'(x) - \alpha A_2(x)}{B_* A_2'(B_*) - \alpha A_2(B_*)} \tag{37}$$

for all $0 \le x < B_* [< \widehat{B}]$, where the boundary $B_*$ satisfies the equation:

$$B \, \frac{B A_2''(B) + (1-\alpha) A_2'(B)}{B A_2'(B) - \alpha A_2(B)} = 1. \tag{38}$$

Here the function $A_2''(x)$ is given by:

$$A_2''(x) = \frac{(\gamma_1 - \alpha)(\gamma_1 - \alpha + 1)(\gamma_2 - \alpha)(\gamma_2 - \alpha + 1)(r - \delta + \zeta)^2}{(1-\alpha)(2-\alpha)} \times F\Big(\gamma_1 - \alpha + 2, \gamma_2 - \alpha + 2; 3 - \alpha; (r - \delta + \zeta)x\Big) \tag{39}$$

for $0 \le x [< \widehat{B}]$.

3.4. Let us finally assume that $\alpha = 1 - 1/\theta < 0$ and $r - \delta + \zeta = 0$ with $\zeta = -\lambda\theta/(1-\theta)$ holds. In this case, equation (23) turns out to be a confluent hypergeometric equation, which has the general solution:

$$G(x) = C_1 \, H_1(x) + C_2 \, H_2(x) \tag{40}$$

where $C_1$ and $C_2$ are some arbitrary constants and the functions $H_1(x)$ and $H_2(x)$ are defined by:

$$H_1(x) = U\Big(-\lambda(1-\alpha)/\eta, \alpha + 1; \eta x\Big), \quad H_2(x) = M\Big(\lambda(1-\alpha)/\eta, -\alpha - 1; \eta x\Big) \tag{41}$$

for $x \ge 0$ with $\alpha = 1 - 1/\theta$ and $\eta = \delta + \lambda + \lambda\theta/(1-\theta)$. Here $U(a, b; x)$ is the confluent hypergeometric function, which admits the integral representation:

$$U(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} \, t^{a-1} (1+t)^{b-a-1} \, dt \tag{42}$$

for $a > 0$, and $M(a, b; x)$ is Kummer's confluent hypergeometric function, which admits the integral representation:

$$M(a, b; x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xt} \, t^{a-1} (1-t)^{b-a-1} \, dt \tag{43}$$

for $b > a > 0$ and has the series expansion:

$$M(a,b;x) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!} \tag{44}$$

for $b \neq 0, -1, -2, \ldots$ and $(b)_k = b(b+1)\cdots(b+k-1)$, $k \in \mathbb{N}$, where the series converges under all $x > 0$ (see, e.g., [1; Chapter XIII] and [5; Chapter VI] with a different parametrization). Therefore, differentiating both sides of the formula (22), by using (40) we get that in this case $\widehat{B} = \infty$ and the integro-differential equation (20) has the general solution:

$$V(x) = C_1\, x^{\alpha+1}\, H_1'(x) + C_2\, x^{\alpha+1}\, H_2'(x) \tag{45}$$

for $x \geq 0$. Hence, applying conditions (16) and (19) to the function (45), we get that the following equalities:

$$C_1\, B^{\alpha+1}\, H_1'(B) + C_2\, B^{\alpha+1}\, H_2'(B) = B \tag{46}$$

$$C_1\, B^{\alpha}\, [(\alpha+1)H_1'(B) + B\, H_1''(B)] + C_2\, B^{\alpha}\, [(\alpha+1)H_2'(B) + B\, H_2''(B)] = 1 \tag{47}$$

hold for some $B \geq \overline{B}$.

It thus follows that in (40) and (45) we have $C_1 = 0$, since otherwise $V(x) \to \pm\infty$ as $x \downarrow 0$, which should be excluded due to the obvious fact that the value function $V_*(x)$ from (10) is bounded under $x \downarrow 0$. Therefore, solving the system (46)-(47) with $C_1 = 0$, by using straightforward calculations we obtain that in this case the solution of the system (15)-(17)+(19) is given by:

$$V(x; B_*) = B_*\, \frac{x^{\alpha+1} H_2'(x)}{B_*^{\alpha+1} H_2'(B_*)} \tag{48}$$

for all $0 \leq x < B_*$, where the boundary $B_*$ satisfies the equation:

$$B\, \frac{H_2''(B)}{H_2'(B)} = -\alpha. \tag{49}$$

Here the functions $H_2'(x)$ and $H_2''(x)$ are given by:

$$H_2'(x) = -\frac{\lambda(1-\alpha)}{\alpha+1}\, M\Big(1 - \lambda(1-\alpha)/\eta, -\alpha; \eta x\Big) \tag{50}$$

$$H_2''(x) = \frac{\lambda(1-\alpha)(\eta+\lambda-\lambda\alpha)}{\alpha(\alpha+1)}\, M\Big(2 - \lambda(1-\alpha)/\eta, 1-\alpha; \eta x\Big) \tag{51}$$

for $x \geq 0$.

3.5. Since it is difficult to give a direct proof of uniqueness of solutions of equations (38) and (49), let us clarify this point by means of the following arguments. We first note that when $\alpha = 1 - 1/\theta < 0$ the two curves $V(x; B')$ and $V(x; B'')$ do not intersect on the interval $(\overline{B}, B']$ as solutions of the integro-differential equation (20) started at two different points $B'$ and $B''$ according to the condition (16) whenever $\overline{B} < B' < B''[< \widehat{B}]$ (see Remark 4.3 below). Hence, by virtue of the properties of the function $F(a, b; c; x)$ defined in (27)-(28) and the function $M(a, b; x)$ defined in (43)-(44), by applying straightforward calculations we obtain that $0 < V'(\overline{B}; B) < 1$ for all $\overline{B} < B[< \widehat{B}]$ as well as $\lim_{B \uparrow \infty[\widehat{B}]} V'(B-; B) > 1$. Thus, by using the fact that the function $V(x; B)$ is convex on $[0, B]$ for each $\overline{B} < B[< \widehat{B}]$ fixed, we may conclude that there exists a unique point $\overline{B} < B_*[< \widehat{B}]$, at which the curve $V(x; B_*)$ hits the line $x$ smoothly implying that equations (38) and (49) have unique solutions.

## 4. Main result and proof

Taking into account the facts proved above, let us now formulate the main assertion of the paper, which extends the result of the article [15] to the case of some jump processes.

**Theorem 4.1.** *Let the process $S$ be given by (1)-(2) with $\delta \geq -\lambda\theta/(1-\theta)$, and thus the process $X$ is given by (7)-(8). Then the value function of the problem (10) takes the expression:*

$$V_*(x) = \begin{cases} V(x; B_*), & \text{if} \quad 0 \leq x < B_* \\ x, & \text{if} \quad x \geq B_* \end{cases} \tag{52}$$

*and the optimal stopping time $\tau_*$ has the structure (11), where the function $V(x; B_*)$ and the optimal stopping boundary $B_*$ are specified as follows:*

*(i): if $\theta < 0$ and $\delta > -\lambda\theta/(1-\theta)$ then the function $V(x; B_*)$ is given by (33), and $B_* = \overline{B} \equiv 1/r$;*

*(ii): if $\theta < 0$ and $\delta = -\lambda\theta/(1-\theta)$ then the function $V(x; B_*) = V(x; B_*-)$ is also given by (33), and $B_* = \widehat{B} = \overline{B} \equiv 1/r$;*

*(iii): if $0 < \theta < 1$ and $r - \delta \neq \lambda\theta/(1-\theta)$ then $V(x; B_*)$ is given by (37) and $B_*$ is uniquely determined from the equation (38);*

*(iv): if $0 < \theta < 1$ and $r - \delta = \lambda\theta/(1-\theta)$ then $V(x; B_*)$ is given by (48) and $B_*$ is uniquely determined from the equation (49).*

**Proof.** (i)+(ii) Observe that in this case we have $\overline{B} \leq \widehat{B}$. Hence, by Remark 2.1 we get that $B_*$ coincides with $\overline{B} \equiv 1/r$, and by means of the existence and uniqueness theorem for hypergeometric equations we may conclude that under the assumptions above the value function (10) admits the unique representation (52) with $V(x; B_*)$ given by (33).

(iii)+(iv) Let us show that the function (52) coincides with the value function (3)+(10) and that the stopping time $\tau_*$ from (11) with the boundary $B_*$ specified above is optimal. For this, let us denote by $V(x)$ the right-hand side of the expression (52). In this case, by means of straightforward calculations and by construction from the previous section it follows that the function $V(x)$ solves the system (15)-(17) as well as the smooth-fit condition (19) is satisfied. Then, by applying Itô's formula to $e^{-\delta t} V(X_t)$ we obtain:

$$e^{-\delta t}\, V(X_t) = V(x) + \int_0^t e^{-\delta u}\, (\mathbb{L}V - \delta V)(X_u)\, du + \widetilde{M}_t \tag{53}$$

where the process $(\widetilde{M}_t)_{t \geq 0}$ defined by:

$$\widetilde{M}_t = \int_0^t \int_0^\infty e^{-\delta u} \Big( V\big(X_{u-} e^{-\theta y}\big) - V(X_{u-}) \Big) \left( \mu(du, dy) - \widetilde{\nu}(du, dy) \right) \tag{54}$$

is a local martingale with respect to the measure $\widetilde{P}_x$.

By using the facts proved in the previous section, following the arguments from [8; Section 3] it can be shown that $(d/dx) \log V(x; B_*) < 1/x$ being equivalent to $V'(x; B_*) < V(x; B_*)/x$ for $0 < x < B_*$. This is done by checking that the left-hand side of the latter inequality is increasing to one and the right-hand side is decreasing from infinity to one on the interval $(0, B_*)$ under the considered relationships on the parameters of the model. Hence, it is shown that the inequality (18) also holds, that together with (16)-(17) yields $V(x) \geq x$ for all $x \geq 0$. Moreover, by using the arguments similar to [8; Section 4] it is shown that $(\mathbb{L}V - \delta V)(B_*-) \leq (\mathbb{L}V - \delta V)(B_*+) = 0$ as well as the function $(\mathbb{L}V - \delta V)(x)$ is increasing (since its derivative is a positive function) and thus negative on $(0, B_*)$, that together with (15) yields $(\mathbb{L}V - \delta V)(x) \leq 0$ for all $x > 0$. From the expression (53) it therefore follows that the inequalities:

$$e^{-\delta \tau}\, X_\tau \leq e^{-\delta \tau}\, V(X_\tau) \leq V(x) + \widetilde{M}_\tau \tag{55}$$

hold for any finite stopping time $\tau$ of the process $X$ started at $x \geq 0$.

Let $(\sigma_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the process $(\widetilde{M}_t)_{t \geq 0}$. Then, taking in (55) expectation with respect to the measure $\widetilde{P}_x$, by means of the optional sampling theorem we get:

$$\widetilde{E}_x \big[ e^{-\delta (\tau \wedge \sigma_n)}\, X_{\tau \wedge \sigma_n} \big] \leq V(x) + \widetilde{E}_x \big[ \widetilde{M}_{\tau \wedge \sigma_n} \big] = V(x) \tag{56}$$

for all $x \geq 0$. Hence, letting $n$ go to infinity and using Fatou's lemma, we obtain that for any finite stopping time $\tau$ the inequalities:

$$\widetilde{E}_x\big[e^{-\delta\tau}\, X_\tau\big] \leq \widetilde{E}_x\big[e^{-\delta\tau}\, V(X_\tau)\big] \leq V(x) \tag{57}$$

are satisfied for all $x \geq 0$.

In order to show that the equality in (57) is attained at $\tau_*$ from (11), let us first prove that the property $\widetilde{P}_x[\tau_* < \infty] = 1$ holds. For this, we observe that from (8) it follows that in the case $r - \delta + \zeta \neq 0$ the continuous part of the process $X$ is given by $\widehat{B} - \widehat{B}\exp(-t/\widehat{B})$, and in the case $r - \delta + \zeta = 0$ it is equal to $t$ for all $t \geq 0$. Then, under the absence of jumps, when $r - \delta + \zeta > 0$ the process $X$ started at $x < \widehat{B}$ will reach the boundary $\widehat{B} - \varepsilon$ by the time not greater than $-\widehat{B}\log(\varepsilon/\widehat{B})$, when $r - \delta + \zeta \leq 0$ the process $X$ started at $x \geq 0$ will reach the boundary $1/\varepsilon$ by the time not greater than $-\widehat{B}\log(1 - 1/(\varepsilon\widehat{B}))$ whenever $r - \delta + \zeta < 0$, and by the time not greater than $1/\varepsilon$ whenever $r - \delta + \zeta = 0$, for any sufficiently small $\varepsilon > 0$ given and fixed. Since from the sample path properties of Poisson processes, by applying the Borel-Cantelli lemma it follows that the $\widetilde{P}_x$-probability of the event that the time between any two jumps of the process $N$ (and thus of $J$) will never exceed $\rho(\varepsilon)$ is equal to zero, we may thus conclude that $\widetilde{P}_x[\tau_* < \infty] = 1$.

By virtue of the fact that the function $V(x)$ together with the boundary $B_*$ satisfy the system (15)-(18), by the structure of the stopping time $\tau_*$ in (11) and by expression (53) it follows that the equality:

$$e^{-\delta(\tau_* \wedge \sigma_n)}\, X_{\tau_* \wedge \sigma_n} = e^{-\delta(\tau_* \wedge \sigma_n)}\, V(X_{\tau_* \wedge \sigma_n}) = V(x) + \widetilde{M}_{\tau_* \wedge \sigma_n} \tag{58}$$

holds for all $x \geq 0$ and any localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of $(\widetilde{M}_t)_{t\geq 0}$. Observe that when $0 < \theta < 1$ the jumps of the process $X$ are negative, and according to Remark 2.2 it is decreasing on $(\widehat{B}, \infty)$ whenever $r - \delta + \zeta > 0$ and increasing on $(0, \infty)$ between the jumps by bounded drift whenever $r - \delta + \zeta \leq 0$. Then, by applying standard arguments to the expression (12) we may conclude that the property:

$$\widetilde{E}_x\Big[\sup_{t\geq 0} e^{-\delta(\tau_* \wedge t)}\, X_{\tau_* \wedge t}\Big] < \infty \tag{59}$$

holds for all $x \geq 0$ and the variable $e^{-\delta\tau_*} X_{\tau_*}$ is bounded on the set $\{\tau_* = \infty\}$. Hence, letting $n$ go to infinity in the expression (58) and using the conditions (16)-(17) as well as the proved above fact that $\widetilde{P}_x[\tau_* < \infty] = 1$, by means of the Lebesgue dominated convergence theorem we obtain that the equality:

$$\widetilde{E}_x\big[e^{-\delta\tau_*}\, X_{\tau_*}\big] = V(x) \tag{60}$$

holds for all $x \geq 0$, that together with (57) directly implies the desired assertion. $\square$

**Remark 4.1.** By means of straightforward calculations it can be verified that in the conditions of the case (i) of Theorem 4.1 for the function $V(x; B_*)$ from (33) we have the equality $V'(B_*-; B_*) = 1$, and by proving the assertions in the cases (iii)-(iv) we have used the equalities (38) and (49), that means that the smooth-fit condition (19) is satisfied. As in [18]-[19] (see also [2] and [20]), this property can be explained by the fact that in the given cases leaving the continuation region $[0, B_*)$ the process $X$ can pass through the boundary $B_*$ continuously.

**Remark 4.2.** On the other hand, in the conditions of the case (ii) of Theorem 4.1 it can be shown that for the function $V(x; B_*)$ from (33) the inequality $V'(B_*-; B_*) < 1$ holds, so that the smooth-fit condition (19) breaks down. As in [18; Section 2] and [19], this property can be explained by the fact that in the given case, by leaving the continuation region $[0, B_*)$ the process $X$ may pass through $B_*$ only by jumping. According to the results in [2] we may conclude that this property appears because of finite intensity of jumps and exponential distribution of jump sizes of the compound Poisson process $J$.

**Remark 4.3.** Assume that the curves $V(x; B')$ and $V(x; B'')$ do intersect at some point $C$ such that $C < B' < B''$, and let $\alpha' x + \beta'$ be the tangent of $V(x; B')$ at the point $C$ with some $\alpha', \beta' > 0$. Following the arguments from [18; Remark 2.2] let us consider the optimal stopping problem (10) with the gain function $(\alpha' x + \beta') \vee x$ instead of $x$ and denote by $W_*(x)$ its value function. Let us also consider the function $W(x)$ defined by $W(x) = V(x; B')$ for $0 \leq x < B'$ and $W(x) = x$ for $x \geq B'$. Since $W(x)$ is non-smooth only at one point $B'$, by means of the arguments of the proof of Theorem 4.1 it can be shown that $W(x) \geq W_*(x)$ for all $x \geq 0$. On the other hand, for the stopping time $\sigma_* = \inf\{t \geq 0 \,|\, X_t \notin (C, B'')\}$ we have $V(x; B'') = \widetilde{E}_x \big[ e^{-\delta \sigma_*} X_{\sigma_*} \big]$ and thus $V(x; B'') > W_*(x)$ for $C \leq x \leq B''$, that leads to a contradiction. We therefore conclude that the curves $V(x; B')$ and $V(x; B'')$ do not intersect.

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