arXiv:1010.3430v1 [math.ST] 17 Oct 2010

Non-additive disorder problems for some diffusion processes

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We study the Bayesian problem of detecting a change in the drift rate of an observable diffusion process with certain non-additive detection delay penalty criterions. We express the Bayesian risk function through the current state of a multi-dimensional diffusion process playing the role of a Markovian sufficient statistic. In the case of exponential delay penalty costs, the optimal time of alarm is found as the first time at which the weighted likelihood ratio hits a stochastic boundary depending on the current observations. The proof is based on an embedding of the initial problem into an appropriate multi-dimensional optimal stopping problem and the analysis of the associated parabolic-type free-boundary problem. We provide closed form estimates for the value function and the boundary for that case, under certain nontrivial relations between the coefficients of the observable diffusion.

Mathematics Subject Classification 2000: Primary 60G40, 62M20, 34K10. Secondary 62C10, 62L15, 60J60.

Key words and phrases: Disorder detection, diffusion process, multi-dimensional optimal stopping, stochastic boundary, parabolic-type free-boundary problem, a change-of-variable formula with local time on surfaces.

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Date: October 19, 2010

1. Introduction

The problem of quickest disorder detection for an observable diffusion process seeks to determine a stopping time of alarm τ which is as close as possible to the unknown time of *disorder* (or change-point) θ at which the local drift rate of the process changes from $\mu_0(x)$ to $\mu_1(x)$. In the classical Bayesian formulation, it is assumed that the random time θ takes the value 0 with probability π and is exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$. Shiryaev [22]-[23] proposed an optimality criterion for the time of alarm to minimize a linear combination of the false alarm probability and the average time delay in detecting of the disorder correctly, for sequences of i.i.d. observations. An explicit solution of the problem of detecting a change in the constant drift rate of an observable Wiener process with the same optimality criterion was derived in Shiryaev [25]-[26]. The appropriate optimal stopping problem for the posterior probability of the occurrence of disorder was reduced to the associated free-boundary problem for an ordinary differential operator (see also [27; Chapter IV, Section 4] or [17; Chapter VI, Section 22]). A finite time horizon version of the Wiener disorder problem was studied in Gapeev and Peskir [9].

The idea of replacing the initial average time delay by a certain non-additive detection delay penalty criterion was originally introduced by Shiryaev [24]. The resulting Bayesian risk function was expressed through the current state of a multi-dimensional Markovian sufficient statistic having state space components which are different from the posterior probability. Such a process contained all the necessary information to determine the structure of the optimal time of alarm (see also more recent works [29], [30] and [6]). In the case of exponential penalty costs for a delay, it was observed by Poor [18] that the weighted likelihood ratio process turns out to be a one-dimensional Markovian sufficient statistic, for sequences of i.i.d. observations. This idea was taken further by Beibel [4], who solved the corresponding problem of detecting a change in the drift rate of an observable Wiener process as a generalized parking problem. Bayraktar and Dayanik [1] recognized the same property from the structure of the ordinary differential-difference equation in the free-boundary problem associated with the Bayesian problem of detecting a change in the constant intensity rate of an observable Poisson process. Some other formulations of the problem for the case of detecting a change in the arrival rate of a Poisson process, leading to the appearance of essentially multi-dimensional Markovian sufficient statistics, were recently studied by Bayraktar, Dayanik and Karatzas [2]-[3]. Extensive overviews of these and other related quickest sequential change-point detection methods were provided in Shiryaev [28] and Poor and Hadjiliadis [19] among others.

In the present paper, we derive stochastic differential equations for Markovian sufficient statistics in the Bayesian disorder problem for observable diffusions with certain nonlinear functions expressing the penalty costs for a delay in the detection of the change correctly. For the case of exponential delay penalty functions, we make an embedding of the initial problem into an extended optimal stopping problem for a three-dimensional Markov diffusion process, having the posterior probability, weighted likelihood ratio, and the observations as its state space components. We show that the optimal stopping time is expressed as the first time at which the weighted likelihood ratio process hits a stochastic boundary depending on the current state of the observation process only. We verify that the value function and the optimal stopping boundary are characterised by means of the associated free-boundary problem for a second order partial differential operator. The latter turns out to be of parabolic type, since the observation process is a one-dimensional diffusion. We also derive closed form estimates for the value function and the boundary for a special nontrivial subclass of observable diffusions. The Bayesian sequential testing problem for such processes was recently solved in [10]. Another related problem of transient signal detection and identification of two-sided changes in the drift rates of observable diffusion processes was considered in Pospisil, Vecer and Hadjiliadis [20].

The paper is organized as follows. In Section 2, after formulating the Bayesian disorder detection problem for observable diffusion processes, we obtain a finite-dimensional Markovian sufficient statistic for the case in which the delay penalty cost function has the same form as in [24] and [29]. In Section 3, for the disorder problem with exponential penalty costs for a delay, we construct a three-dimensional optimal stopping problem and formulate the associated free-boundary problem. We reduce the resulting parabolic-type partial differential operator to the normal form which is amenable for further considerations. Applying the change-of-variable formula with local time on surfaces obtained by Peskir [16], we verify that the solution of the free-boundary problem, which satisfies certain additional conditions, provides the solution of the initial optimal stopping problem. In Section 4, we derive closed form estimates for the value function and the boundary, which are uniquely determined as solutions of ordinary differential equations, under certain nontrivial relations between the coefficients of the observable diffusion. The main result of the paper is stated in Theorem 3.3.

2. Preliminaries

In this section, we give the Bayesian formulation of the problem (see [27; Chapter IV, Section 4] or [17; Chapter VI, Section 22] for the case of Wiener processes) in which it is assumed that one observes a sample path of the diffusion process $X = (X_t)_{t \ge 0}$ with the drift rate changing from $\mu_0(x)$ to $\mu_1(x)$ at some random time θ taking the value 0 with probability π and being exponentially distributed with parameter $\lambda > 0$ under $\theta > 0$.

2.1. (Formulation of the problem.) For a precise probabilistic formulation of the Bayesian disorder detection problem, suppose that all the considerations take place on a probability space $(\Omega, \mathcal{F}, P_\pi)$ where the probability measure P_π has the structure:

$$P_\pi = \pi P^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda s} P^s \, ds \tag{2.1}$$

for any $\pi \in [0,1]$. Let θ be a nonnegative random variable satisfying $P_\pi(\theta = 0) = \pi$ and $P_\pi(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ for all $t \ge 0$ and some $\lambda > 0$, and let $W = (W_t)_{t \ge 0}$ be a standard Wiener process started at zero under P_π. It is assumed that θ and W are independent.

Suppose that we observe a continuous process $X = (X_t)_{t \ge 0}$ solving the stochastic differential equation:

$$dX_t = \big(\mu_0(X_t) + I(\theta \le t)(\mu_1(X_t) - \mu_0(X_t)) \big) \, dt + \sigma(X_t) \, dW_t \quad (X_0 = x) \tag{2.2}$$

where $\mu_i(x)$, $i = 0, 1$, and $\sigma(x) > 0$ are some continuously differentiable functions on $(0, \infty)$. For simplicity of exposition, we assume the state space of the process X to be the positive half line $(0, \infty)$, since that is the case in the examples considered below. It thus follows from [14; Theorem 4.6] that the equation in (2.2) admits a unique strong solution under $\theta = s$, and hence, $P_\pi(X \in \cdot \,|\, \theta = s) = P^s(X \in \cdot\,)$ is the distribution law of a time-homogeneous diffusion process started at some $x > 0$, with diffusion coefficient $\sigma^2(x)$ and the drift rate changing from $\mu_0(x)$ to $\mu_1(x)$ at time $s \in [0, \infty]$. It is assumed that the time of *disorder* θ is unknown, that is, it cannot be observed directly.

Being based upon the continuous observation of the process X, our task is to find among the stopping times τ of X (i.e., stopping times with respect to the natural filtration $\mathcal{F}_t = \sigma(X_s \,|\, 0 \le s \le t)$ of the process X, for $t \ge 0$) an optimal time τ_* at which an *alarm* should be sounded *as close as possible* to the unobservable time θ. More precisely, the problem consists of computing the Bayesian risk function:

$$V_*(\pi) = \inf_{\tau} \Big(P_\pi(\tau < \theta) + E_\pi\big[F(\tau - \theta) I(\tau \ge \theta)\big] \Big) \tag{2.3}$$

and finding the optimal stopping time τ_*, called the π-Bayes time, at which the infimum is attained in (2.3). Here, $P_\pi(\tau < \theta)$ is the probability of a *false alarm* and $E_\pi[F(\tau - \theta) I(\tau \ge \theta)]$ is the *average cost of delay* in detecting of the disorder correctly (i.e. when $\tau \ge \theta$), where the *delay penalty costs* function $F(t)$ satisfies the conditions $F(t) \ge 0$ for $t \ge 0$ and $F(t) = 0$ for $t \le 0$.

2.2. (Markovian sufficient statistics.) By means of the standard arguments from [24] (see also [29]), it is shown that the expectation in (2.3) admits the representation:

$$E_\pi\big[F(\tau - \theta) I(\tau \ge \theta)\big] = E_\pi\left[F(\tau)\, P_\pi(\theta = 0 \,|\, \mathcal{F}_\tau) + \int_0^\tau F(\tau - u)\, \frac{\partial P_\pi(\theta \le u \,|\, \mathcal{F}_\tau)}{\partial u}\, du \right] \tag{2.4}$$

under the assumption of continuous differentiability of the conditional probability in the integrand above. Using the fact that the probability measure P^s is absolutely continuous with respect to P_π on $\mathcal{F}_t$, for any $s \in [0, \infty]$, by means of Bayes' formula (see, e.g. [14; Theorem 7.23]), we get that:

$$P_\pi(\theta \le u \,|\, \mathcal{F}_t) = \pi\, \frac{d(P^0 \,|\, \mathcal{F}_t)}{d(P_\pi \,|\, \mathcal{F}_t)} + (1 - \pi) \int_0^u \frac{d(P^s \,|\, \mathcal{F}_t)}{d(P_\pi \,|\, \mathcal{F}_t)}\, \lambda e^{-\lambda s}\, ds \tag{2.5}$$

holds for all $0 \le u \le t$. Moreover, we see that:

$$P_\pi(\theta > t \,|\, \mathcal{F}_t) = (1 - \pi) \int_t^\infty \frac{d(P^s \,|\, \mathcal{F}_t)}{d(P_\pi \,|\, \mathcal{F}_t)}\, \lambda e^{-\lambda s}\, ds = (1 - \pi)\, e^{-\lambda t}\, \frac{d(P^t \,|\, \mathcal{F}_t)}{d(P_\pi \,|\, \mathcal{F}_t)} \tag{2.6}$$

is satisfied, where the last equality holds, since the probability measure P^s coincides with P^t on $\mathcal{F}_t$, for all $0 \le t \le s$. Using the fact that, for the underlying model of (2.2), the properties:

$$\frac{d(P^s \,|\, \mathcal{F}_t)}{d(P_\pi \,|\, \mathcal{F}_t)}\, \frac{d(P_\pi \,|\, \mathcal{F}_t)}{d(P^t \,|\, \mathcal{F}_t)} = \frac{d(P^s \,|\, \mathcal{F}_t)}{d(P^0 \,|\, \mathcal{F}_t)}\, \frac{d(P^0 \,|\, \mathcal{F}_t)}{d(P^t \,|\, \mathcal{F}_t)} = \frac{d(P^s \,|\, \mathcal{F}_s)}{d(P^0 \,|\, \mathcal{F}_s)}\, \frac{d(P^0 \,|\, \mathcal{F}_t)}{d(P^t \,|\, \mathcal{F}_t)} \tag{2.7}$$

hold for $0 \le s \le t$, we obtain from (2.5) and (2.6) that the representation:

$$\frac{P_\pi(\theta \le u \,|\, \mathcal{F}_t)}{1 - \Pi_t} = Z_t \left(\frac{\pi}{1 - \pi} + \lambda \int_0^u \frac{ds}{Z_s} \right) \tag{2.8}$$

is admitted. Here, the *posterior probability* process $\Pi = (\Pi_t)_{t \ge 0}$ is defined by $\Pi_t = P_\pi(\theta \le t \,|\, \mathcal{F}_t)$ and the process $Z = (Z_t)_{t \ge 0}$ is given by:

$$Z_t = e^{\lambda t} \, \frac{d(P^0 \,|\, \mathcal{F}_t)}{d(P^t \,|\, \mathcal{F}_t)} \equiv e^{\lambda t} \, \frac{d(P^0 \,|\, \mathcal{F}_t)}{d(P^\infty \,|\, \mathcal{F}_t)} \tag{2.9}$$

for all $t \ge 0$. It thus follows from (2.8) that the conditional probability in (2.5) is continuously differentiable with respect to u, so that the representation of (2.4) for the Bayesian risk function in (2.3) takes the form:

$$V_*(\pi) = \inf_\tau E_\pi \left[(1 - \Pi_\tau) \left(1 + F(\tau) \, Z_\tau \, \frac{\pi}{1 - \pi} + \lambda \int_0^\tau F(\tau - u) \, \frac{Z_\tau}{Z_u} \, du \right) \right] \tag{2.10}$$

for each $\pi \in [0, 1)$ fixed.

Following the schema of arguments from [24] and [29], we further assume that the delay penalty costs function is given by:

$$F(t) = \sum_{k=1}^{n} \sum_{l=1}^{m(k)} a_{k,l} \, e^{\beta_k t} \, t^{l-1} \tag{2.11}$$

for all $t \ge 0$, some $a_{k,l}, \beta_k \in \mathbb{R}$ for $l = 1, \ldots, m(k)$, $k = 1, \ldots, n$, and $m, n \in \mathbb{N}$ fixed. For this case, let us define the processes $\Psi^{k,l} = (\Psi^{k,l}_t)_{t \ge 0}$ by:

$$\Psi^{k,l}_t = \lambda \int_0^t e^{\beta_k (t-u)} \, \frac{(t-u)^{l-1}}{(l-1)!} \, \frac{Z_t}{Z_u} \, du \tag{2.12}$$

for every $l = 1, \ldots, m(k)$ and $k = 1, \ldots, n$. In this notation, we therefore conclude that the value function in (2.10) admits the representation:

$$V_*(\pi) = \inf_\tau E_\pi \left[(1 - \Pi_\tau) \left(1 + F(\tau) \, Z_\tau \, \frac{\pi}{1 - \pi} + \sum_{k=1}^{n} \sum_{l=1}^{m(k)} c_{k,l} \, \Psi^{k,l}_\tau \right) \right] \tag{2.13}$$

with $c_{k,l} = a_{k,l}(l-1)!/\lambda$, for every $l = 1, \ldots, m(k)$, $k = 1, \ldots, n$, and each $\pi \in [0, 1)$.

2.3. (Stochastic differential equations.) By means of Girsanov's theorem for diffusion-type processes (see [14; Theorem 7.19]), it follows that the process Z in (2.9) admits the representation:

$$Z_t = \exp \left(\lambda t + \int_0^t \frac{\mu_1(X_u) - \mu_0(X_u)}{\sigma^2(X_u)} \, dX_u - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_u) - \mu_0^2(X_u)}{\sigma^2(X_u)} \, du \right) \tag{2.14}$$

for all $t \ge 0$. Then, applying Itô's formula (see, e.g. [14; Chapter IV, Theorem 4.4] or [21; Chapter IV, Theorem 3.3]), we get that Z solves the stochastic differential equation:

$$dZ_t = \lambda \, dt + Z_t \, \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)} \, (dX_t - \mu_0(X_t) \, dt) \tag{2.15}$$

and thus, the processes $\Psi^{k,l}$ from (2.12) admit the representations:

$$d\Psi_t^{k,1} = \big(\lambda + (\lambda + \beta_k)\Psi_t^{k,1}\big)\, dt + \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)}\, \Psi_t^{k,1}\, (dX_t - \mu_0(X_t)\, dt) \tag{2.16}$$

$$d\Psi_t^{k,l} = \big(\Psi_t^{k,l-1} + (\lambda + \beta_k)\Psi_t^{k,l}\big)\, dt + \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)}\, \Psi_t^{k,l}\, (dX_t - \mu_0(X_t)\, dt) \tag{2.17}$$

for every $l = 2, \ldots, m(k)$ and $k = 1, \ldots, n$.

Let us now define the *likelihood ratio* process $\widetilde{\Phi} = (\widetilde{\Phi}_t)_{t \geq 0}$ by:

$$\widetilde{\Phi}_t = \frac{\Pi_t}{1 - \Pi_t} \equiv Z_t \left(\frac{\pi}{1 - \pi} + \lambda \int_0^t \frac{ds}{Z_s} \right) \tag{2.18}$$

for all $t \geq 0$. Then, using Itô's formula again, we get that the processes $\widetilde{\Phi}$ and Π solve the stochastic differential equations:

$$d\widetilde{\Phi}_t = \left(\lambda(1 + \widetilde{\Phi}_t) + \left(\frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \right)^2 \frac{\widetilde{\Phi}_t^2}{1 + \widetilde{\Phi}_t} \right) dt + \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)}\, \widetilde{\Phi}_t\, d\overline{W}_t \quad (\widetilde{\Phi}_0 = \phi) \tag{2.19}$$

with $\phi \equiv \pi/(1 - \pi)$, and

$$d\Pi_t = \lambda(1 - \Pi_t)\, dt + \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)}\, \Pi_t(1 - \Pi_t)\, d\overline{W}_t \quad (\Pi_0 = \pi) \tag{2.20}$$

where the *innovation* process $\overline{W} = (\overline{W}_t)_{t \geq 0}$ defined by:

$$\overline{W}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \left(\frac{\mu_0(X_s)}{\sigma(X_s)} + \Pi_s\, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \right) ds \tag{2.21}$$

is a standard Wiener process under the measure P_π with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$, according to P. Lévy's characterisation theorem (see, e.g. [14; Theorem 4.1] or [21; Chapter IV, Theorem 3.6]). It therefore follows from (2.21) that the process X admits the representation:

$$dX_t = \big(\mu_0(X_t) + \Pi_t\, (\mu_1(X_t) - \mu_0(X_t))\big)\, dt + \sigma(X_t)\, d\overline{W}_t \quad (X_0 = x). \tag{2.22}$$

By means of Remark to [14; Chapter IV, Theorem 4.6] (see also [15; Chapter V, Theorem 5.2.1]), we thus conclude that the multi-dimensional process having Π, Z, $\Psi^{k,l}$, $l = 1, \ldots, m(k)$, $k = 1, \ldots, n$, and X as its state space components turns out to be a unique strong solution of the system of stochastic differential equations in (2.15), (2.16)-(2.17), (2.20) and (2.22). Hence, according to [15; Chapter VII, Theorem 7.2.4], such a process turns out to be a (time-homogeneous strong) Markov process with respect to its natural filtration, which obviously coincides with $(\mathcal{F}_t)_{t\geq 0}$. We may therefore conclude that the infimum in (2.13) is taken over all stopping times of this multi-dimensional process which plays the role of a *Markovian sufficient statistic* in the problem (see [27; Chapter II, Section 15] for a discussion of this notion).

Summarizing the facts proved above, we now formulate the following assertion.

Lemma 2.1. *Suppose that* $\mu_i(x)$*,* $i = 0, 1$*, and* $\sigma(x) > 0$ *are continuously differentiable functions on* $(0, \infty)$ *in (2.2). Assume that the delay penalty cost function* $F(t)$ *is given by (2.11). Then, in the disorder detection problem of (2.3), the Bayesian risk function admits the representation in (2.13), where the processes* Π*,* Z *and* $\Psi^{k,l}$*,* $l = 1, \ldots, m(k)$*,* $k = 1, \ldots, n$*, and* X *given by (2.2), (2.12), (2.14) and (2.18), form a multi-dimensional Markovian sufficient statistic.*

Example 2.2. Assume that we have $F(t) = ct^{\alpha}$ in (2.3), for all $t \geq 0$ and some $\alpha > 0$ and $c > 0$ fixed. In this case, we see that if $\alpha \in \mathbb{N}$, then $F(t)$ is of the type (2.11). Hence, by means of the arguments above, it is shown that the Bayesian risk function in (2.13) admits the representation:

$$V_*(\pi) = \inf_{\tau} E_{\pi}\bigg[(1 - \Pi_{\tau})\bigg(1 + c\, Z_{\tau}\, \tau^{\alpha}\, \frac{\pi}{1 - \pi} + c\, \Gamma(\alpha + 1) \sum_{l=1}^{\alpha+1} \Psi_{\tau}^{1,l}\bigg)\bigg] \tag{2.23}$$

where Γ denotes the Euler Gamma function. Otherwise, if $\alpha > 0$ but $\alpha \notin \mathbb{N}$, then the delay penalty costs function $F(t)$ cannot be expressed in the form of (2.11) with a finite number of summands.

Example 2.3. Assume that we have $F(t) = c(e^{\alpha t} - 1)$ for all $t \geq 0$ and some $c, \alpha > 0$ fixed (see [24; Example 4], [18], [4] and [1]). Then, it is shown by means of standard arguments from [1] that the Bayesian risk function in (2.3) admits the representation:

$$V_*(\pi) = \inf_{\tau} E_{\pi}\left[1 - \Pi_{\tau} + \int_0^{\tau} (1 - \Pi_t)\, c\alpha \Phi_t\, dt\right] \tag{2.24}$$

where the infimum is taken over all $(\mathcal{F}_t)_{t\geq 0}$-stopping times such that the integral above has a finite expectation. Here, the *weighted likelihood ratio* process $\Phi = (\Phi_t)_{t\geq 0}$ is defined by:

$$\Phi_t = \frac{Z_t}{e^{-\alpha t}} \left(\frac{\pi}{1 - \pi} + \int_0^t \frac{\lambda e^{-\alpha u}}{Z_u}\, du\right) \tag{2.25}$$

for all $t \geq 0$, and, according to Itô's formula and the expressions in (2.15) and (2.22), solves the stochastic differential equation:

$$d\Phi_t = \left(\lambda + (\lambda + \alpha)\, \Phi_t + \left(\frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \right)^2 \Pi_t \, \Phi_t \right) dt + \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma(X_t)} \, \Phi_t \, d\overline{W}_t \quad (\Phi_0 = \phi) \tag{2.26}$$

with $\phi \equiv \pi/(1-\pi)$. In this case, the process (Π, Φ, X) turns out to be a three-dimensional Markovian sufficient statistic.

Example 2.4. Assume now that we have $F(t) = ct$ for all $t \geq 0$ and some $c > 0$ fixed (see [25], [26], [27; Chapter IV] and [17; Chapter VI, Section 22]). Then, it is shown by means of standard arguments from [27; Chapter IV, Section 3] that the Bayesian risk function in (2.3) admits the representation:

$$V_*(\pi) = \inf_\tau E_\pi \left[1 - \Pi_\tau + \int_0^\tau c \Pi_t \, dt \right] \equiv \inf_\tau E_\pi \left[\frac{1}{1 + \widetilde{\Phi}_\tau} + \int_0^\tau \frac{c \widetilde{\Phi}_t}{1 + \widetilde{\Phi}_t} \, dt \right] \tag{2.27}$$

where the infimums are taken over all stopping times of the processes Π or $\widetilde{\Phi}$ such that the integrals above have finite expectations. In this case, by virtue of the one-to-one correspondence in (2.18), the processes (Π, X) and $(\widetilde{\Phi}, X)$ turn out to be equivalent two-dimensional Markovian sufficient statistics.

3. The case of exponential delay penalty costs

In this section, we formulate and prove the main assertions of the paper related to the disorder detection problem with the exponential delay penalty costs indicated in Example 2.3.

3.1. For the Bayesian disorder detection problem of (2.24), let us consider the following extended optimal stopping problem:

$$V_*(\pi, \phi, x) = \inf_\tau E_{\pi,\phi,x} \left[1 - \Pi_\tau + \int_0^\tau (1 - \Pi_t) \, c\alpha \Phi_t \, dt \right] \tag{3.1}$$

where $P_{\pi,\phi,x}$ is a measure of the diffusion process (Π, Φ, X) started at some $(\pi, \phi, x) \in [0,1] \times [0, \infty) \times (0, \infty)$ and solving the three-dimensional system of equations in (2.20), (2.26) and (2.22). The infimum in (3.1) is therefore taken over all stopping times of (Π, Φ, X) such that the integral there has a finite expectation. By means of the results of general theory of optimal stopping (see, e.g. [27; Chapter III] or [17; Chapter I, Section 2.1]), it follows from the structure of the reward functional in (3.1) that the optimal stopping time is given by:

$$\tau_* = \inf\{t \geq 0 \,|\, V_*(\Pi_t, \Phi_t, X_t) = 1 - \Pi_t\} \tag{3.2}$$

whenever the integral above has a finite expectation, so that the continuation region has the form:

$$C_* = \{(\pi, \phi, x) \in [0,1] \times [0,\infty) \times (0,\infty) \,|\, V_*(\pi, \phi, x) < 1 - \pi\}. \tag{3.3}$$

3.2. In order to specify the structure of the stopping time in (3.2), let us fix some (π, ϕ, x) from the continuation region C_* in (3.3) and denote by $\tau_* = \tau_*(\pi, \phi, x)$ the optimal stopping time in the problem of (3.1). Then, by means of the general optimal stopping theory for Markov processes (see, e.g. [27; Chapter III] or [17; Chapter I, Section 2.2]), we conclude that:

$$V_*(\pi, \phi, x) - (1 - \pi) = E_{\pi,\phi,x}\left[1 - \Pi_{\tau_*} + \int_0^{\tau_*} (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] - (1 - \pi) < 0 \tag{3.4}$$

holds. Hence, taking any ϕ' such that $\phi' < \phi$ and using the explicit expression for the process Φ through its starting point $\phi \equiv \pi/(1-\pi)$ in (2.25), we obtain from (3.1) that the inequalities:

$$\begin{aligned} V_*(\pi, \phi', x) - (1 - \pi) &\leq E_{\pi,\phi',x}\left[1 - \Pi_{\tau_*} + \int_0^{\tau_*} (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] - (1 - \pi) \\ &\leq E_{\pi,\phi,x}\left[1 - \Pi_{\tau_*} + \int_0^{\tau_*} (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] - (1 - \pi) \end{aligned} \tag{3.5}$$

are satisfied. Thus, by virtue of the inequality in (3.4), we see that $(\pi, \phi', x) \in C_*$. Taking into account the multiplicative structure of the integrand in (3.1), we can therefore extend the approach used in [18], [4] and [1], and further assume that there exists a function $g_*(x)$ such that $0 < g_*(x) < 1$ for $x > 0$, and the continuation region in (3.3) for the optimal stopping problem of (3.1) takes the form:

$$C_* = \{(\pi, \phi, x) \in [0,1] \times [0,\infty) \times (0,\infty) \,|\, \phi < g_*(x)\} \tag{3.6}$$

and thus, the corresponding stopping region is the closure of the set:

$$D_* = \{(\pi, \phi, x) \in [0,1] \times [0,\infty) \times (0,\infty) \,|\, \phi > g_*(x)\}. \tag{3.7}$$

3.3. In order to describe some properties of the boundary $g_*(x)$ in (3.6)-(3.7), we follow the arguments from [9; Subsection 2.5] and use Itô's formula to get:

$$1 - \Pi_t = 1 - \pi - \lambda \int_0^t (1 - \Pi_s)\, ds + N_t \tag{3.8}$$

where the process $N = (N_t)_{t \ge 0}$ defined by:

$$N_t = -\int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \, \Pi_s(1-\Pi_s) \, d\overline{W}_s \tag{3.9}$$

is a continuous local martingale under P_π. Then, assuming that the process $(N_{\tau_* \wedge t})_{t \ge 0}$ is a uniformly integrable martingale (as it turns out to be under the assumptions of Lemma 3.2 below) and applying Doob's optional sampling theorem (see, e.g. [14; Theorem 3.6] or [21; Chapter II, Theorem 3.2]), we get from the expression in (3.8) that:

$$E_{\pi,\phi,x}\left[1 - \Pi_{\tau_*} + \int_0^{\tau_*} (1-\Pi_t) \, c\alpha\Phi_t \, dt\right] = 1 - \pi + E_{\pi,\phi,x} \int_0^{\tau_*} (1-\Pi_t) \, (c\alpha\Phi_t - \lambda) \, dt \tag{3.10}$$

holds for all $(\pi, \phi, x) \in [0,1] \times [0,\infty) \times (0,\infty)$. Taking into account the structure of the reward in the right-hand side of (3.10), it is seen from (3.1) that it is never optimal to stop when $\Phi_t < \lambda/(c\alpha)$ for any $t \ge 0$. This shows that all points (π, ϕ, x) with $\phi < \lambda/(c\alpha)$ belong to the continuation region C_* in (3.3) and (3.6), so that $0 < \lambda/(c\alpha) \le g_*(x)$ holds for all $x > 0$.

3.4. By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(\Pi,\Phi,X)}$ of the process (Π, Φ, X) from (2.20), (2.26) and (2.22) has the structure:

$$\begin{aligned} \mathbb{L}_{(\Pi,\Phi,X)} &= \lambda(1-\pi) \frac{\partial}{\partial \pi} + \left(\lambda + (\lambda+\alpha)\,\phi + \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}\right)^2 \pi\,\phi\right) \frac{\partial}{\partial \phi} \\ &+ \big(\mu_0(x) + (\mu_1(x) - \mu_0(x))\,\pi\big) \frac{\partial}{\partial x} + (\mu_1(x) - \mu_0(x)) \left(\pi(1-\pi) \frac{\partial^2}{\partial \pi \partial x} + \phi \frac{\partial^2}{\partial \phi \partial x}\right) \\ &+ \frac{1}{2} \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}\right)^2 \left(\pi^2(1-\pi)^2 \frac{\partial^2}{\partial \pi^2} + 2\pi(1-\pi)\,\phi \frac{\partial^2}{\partial \pi \partial \phi} + \phi^2 \frac{\partial^2}{\partial \phi^2}\right) + \frac{1}{2}\,\sigma^2(x) \frac{\partial^2}{\partial x^2} \end{aligned} \tag{3.11}$$

for all $(\pi, \phi, x) \in [0,1] \times [0,\infty) \times (0,\infty)$.

In order to find analytic expressions for the unknown value function $V_*(\pi, \phi, x)$ from (3.4) and the boundary $g_*(x)$ from (3.6)-(3.7), we use the results of the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [11], [27; Chapter III, Section 8] and [17; Chapter IV, Section 8]) to formulate the associated *free-boundary problem*:

$$(\mathbb{L}_{(\Pi,\Phi,X)}V)(\pi, \phi, x) = -(1-\pi)\,c\alpha\phi \quad \text{for} \quad (\pi, \phi, x) \in C \tag{3.12}$$

$$V(\pi, \phi, x)\big|_{\phi = g(x)-} = 1 - \pi \quad (\textit{instantaneous stopping}) \tag{3.13}$$

$$V(\pi, \phi, x) = 1 - \pi \quad \text{for} \quad (\pi, \phi, x) \in D \tag{3.14}$$

$$V(\pi, \phi, x) < 1 - \pi \quad \text{for} \quad (\pi, \phi, x) \in C \tag{3.15}$$

where C and D are defined as C_* and D_* in (3.6) and (3.7) with $g(x)$ instead of $g_*(x)$, and the *instantaneous stopping* condition in (3.13) is satisfied for all $\pi \in [0,1]$ and $x > 0$.

Note that the superharmonic characterisation of the value function (see [7], [27; Chapter III, Section 8] and [17; Chapter IV, Section 9]) implies that $V_*(\pi, \phi, x)$ from (3.1) is the largest function satisfying (3.12)-(3.15) with the boundary $g_*(x)$.

Remark 3.1. Observe that, since the system in (3.12)-(3.15) admits multiple solutions, we need to find some additional conditions which would specify the appropriate solution providing the value function and the optimal stopping boundary for the initial problem of (3.1). In order to derive such conditions, we shall reduce the operator in (3.11) to the normal form. We also note that the fact that the stochastic differential equations for the posterior probability, the weighted likelihood ratio and the observation process in (2.20), (2.26) and (2.22), respectively, are driven by the same (one-dimensional) innovation Wiener process yields the property that the infinitesimal operator in (3.11) turns out to be of parabolic type.

3.5. In order to find the normal form of the operator in (3.11) and formulate the associated optimal stopping and free-boundary problem, we use the one-to-one correspondence transformation of processes proposed by A.N. Kolmogorov in [12]. For this, let us define the process $Y = (Y_t)_{t \geq 0}$ by:

$$Y_t = \log \Phi_t - \int_z^{X_t} \frac{\mu_1(w) - \mu_0(w)}{\sigma^2(w)} \, dw \tag{3.16}$$

for all $t \geq 0$, and any $z > 0$ fixed. Then, taking into account the assumption that the functions $\mu_i(x)$, $i = 0, 1$, and $\sigma(x)$ are continuously differentiable on $(0, \infty)$, by means of Itô's formula, we get that the process Y admits the representation:

$$dY_t = \left(\frac{\lambda}{\Phi_t} + \lambda + \alpha - \frac{\sigma^2(X_t)}{2} \left[\frac{\mu_1^2(X_t) - \mu_0^2(X_t)}{\sigma^4(X_t)} + \frac{\partial}{\partial x} \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma^2(x)} \right) \bigg|_{x = X_t} \right] \right) dt \quad (Y_0 = y) \tag{3.17}$$

with

$$y = \log \phi - \int_z^{x} \frac{\mu_1(w) - \mu_0(w)}{\sigma^2(w)} \, dw \tag{3.18}$$

for any $z > 0$ fixed. It is seen from the equation in (3.17) that the process Y started at $y \in \mathbb{R}$ is of bounded variation. We further assume that:

$$\text{either} \quad \mu_0(x) < \mu_1(x) \quad \text{or} \quad \mu_0(x) > \mu_1(x) \quad \text{holds for all} \quad x > 0. \tag{3.19}$$

In this case, it follows from the relation in (3.16) that there exists a one-to-one correspondence between the processes (Π, Φ, X) and (Π, Φ, Y). Hence, for any $z > 0$ fixed, the value function $V_*(\pi, \phi, x)$ from (3.1) is equal to the one of the optimal stopping problem:

$$U_*(\pi, \phi, y) = \inf_{\tau} E_{\pi,\phi,y} \bigg[1 - \Pi_\tau + \int_0^\tau (1 - \Pi_t) \, c\alpha\Phi_t \, dt \bigg] \tag{3.20}$$

where the supremum is taken over all stopping times with respect to the natural filtration of (Π, Φ, Y), which clearly coincides with $(\mathcal{F}_t)_{t \ge 0}$. Here, $E_{\pi,\phi,y}$ denotes the expectation under the assumption that the three-dimensional Markov process (Π, Φ, Y) from (2.18), (2.25) and (3.16) starts at some $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$. It thus follows from (3.6)-(3.7) that there exists a function $h_*(y)$ such that $0 < h_*(y) < 1$ for $y \in \mathbb{R}$, and the optimal stopping time in the problem of (3.20) has the structure:

$$\tau_* = \inf\{ t \ge 0 \,|\, \Phi_t \ge h_*(Y_t) \} \tag{3.21}$$

whenever the integral above has a finite expectation.

3.6. Standard arguments then show that the infinitesimal operator $\mathbb{L}_{(\Pi,\Phi,Y)}$ of the process (Π, Φ, Y) from (2.20), (2.26) and (3.17) has the structure:

$$\begin{aligned} \mathbb{L}_{(\Pi,\Phi,Y)} = {} & \lambda(1-\pi) \, \frac{\partial}{\partial \pi} + \left(\lambda + (\lambda + \alpha) \, \phi + \left(\frac{\mu_1(x(\phi,y)) - \mu_0(x(\phi,y))}{\sigma(x(\phi,y))} \right)^2 \pi \, \phi \right) \frac{\partial}{\partial \phi} \\ & + \frac{1}{2} \left(\frac{\mu_1(x(\phi,y)) - \mu_0(x(\phi,y))}{\sigma(x(\phi,y))} \right)^2 \left(\pi^2 (1-\pi)^2 \frac{\partial^2}{\partial \pi^2} + 2\pi(1-\pi) \, \phi \, \frac{\partial^2}{\partial \pi \partial \phi} + \phi^2 \, \frac{\partial^2}{\partial \phi^2} \right) \\ & + \left(\frac{\lambda}{\phi} + \lambda + \alpha - \frac{\sigma^2(x(\phi,y))}{2} \left[\frac{\mu_1^2(x(\phi,y)) - \mu_0^2(x(\phi,y))}{\sigma^4(x(\phi,y))} + \frac{\partial}{\partial x} \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma^2(x)} \right) \bigg|_{x = x(\phi,y)} \right] \right) \frac{\partial}{\partial y} \end{aligned} \tag{3.22}$$

for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$. Here, by virtue of the assumption in (3.19), the expression for $x(\phi, y) \equiv x(\phi, y; z)$ is uniquely determined by the relation in (3.18), for any $z > 0$.

We are now ready to formulate the associated free-boundary problem for the unknown value function $U_*(\pi, \phi, y) \equiv U_*(\pi, \phi, y; z)$ from (3.20) and the boundary $h_*(y) \equiv h_*(y; z)$ from (3.21):

$$(\mathbb{L}_{(\Pi,\Phi,Y)} U)(\pi, \phi, y) = -(1-\pi) \, c\alpha\phi \quad \text{for} \quad \phi < h(y) \tag{3.23}$$

$$U(\pi, \phi, y) \big|_{\phi = h(y)-} = 1 - \pi \tag{3.24}$$

$$U(\pi, \phi, y) = 1 - \pi \quad \text{for} \quad \phi > h(y) \tag{3.25}$$

$$U(\pi, \phi, y) < 1 - \pi \quad \text{for} \quad \phi < h(y) \tag{3.26}$$

where the instantaneous stopping condition in (3.24) is satisfied for all $\pi \in (0,1)$ and $y \in \mathbb{R}$. Moreover, we assume that the following conditions hold:

$$\frac{\partial U}{\partial \phi}(\pi, \phi, y)\Big|_{\phi = h(y)-} = 0 \quad (\textit{smooth fit}) \tag{3.27}$$

$$\frac{\partial U}{\partial \phi}(\pi, \phi, y)\Big|_{\phi = 0+} \quad \text{is finite} \tag{3.28}$$

and the one-sided derivative:

$$\frac{\partial U}{\partial y}(\pi, \phi, y)\Big|_{\phi = h(y)-} \qquad \text{exists} \tag{3.29}$$

for all $\pi \in (0,1)$, $y \in \mathbb{R}$, and any $z > 0$ fixed.

We further search for solutions of the parabolic-type free-boundary problem in (3.23)-(3.26) satisfying the conditions in (3.27)-(3.29) and such that the resulting boundaries are continuous and of bounded variation. Since such free-boundary problems cannot, in general, be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the related variational inequalities and their connection with the optimal stopping problems have been extensively studied in the literature (see, e.g. [8], [5], [13] or [15]).

3.7. We continue with the following verification assertion related to the free-boundary problem in (3.23)-(3.29).

Lemma 3.2. *Suppose that $\mu_i(x)$, $i = 0, 1$, and $\sigma(x) > 0$ are continuously differentiable functions on $(0, \infty)$ in (2.2). Assume that the function $U(\pi, \phi, y; h_*(y)) \equiv (1-\pi) H(\phi, y; h_*(y))$ and the continuous boundary of bounded variation $h_*(y)$ form a unique solution of the free-boundary problem in (3.23)-(3.26) satisfying the conditions of (3.27)-(3.29). Then, the value function of the optimal stopping problem in (3.20) takes the form:*

$$U_*(\pi, \phi, y) = \begin{cases} (1-\pi) H(\phi, y; h_*(y)), & \text{if } 0 \le \phi < h_*(y) \\ 1 - \pi, & \text{if } \phi \ge h_*(y) \end{cases} \tag{3.30}$$

and $h_*(y)$ provides the optimal stopping boundary for (3.21) whenever the corresponding integral integral has a finite expectation, for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$.

Proof. Let us denote by $U(\pi, \phi, y)$ the right-hand side of the expression in (3.30). Hence, applying the change-of-variable formula with local time on surfaces from [16] to $U(\pi, \phi, y)$ and

$h_*(y)$, and taking into account the smooth-fit condition in (3.27), we obtain:

$$U(\Pi_t, \Phi_t, Y_t) = U(\pi, \phi, y) + \int_0^t (\mathbb{L}_{(\Pi,\Phi,Y)} U)(\Pi_s, \Phi_s, Y_s) \, I(\Phi_s \neq h_*(Y_s)) \, ds + M_t \tag{3.31}$$

where the process $M = (M_t)_{t \geq 0}$ defined by:

$$\begin{aligned} M_t = & \int_0^t \frac{\partial U}{\partial \pi}(\Pi_s, \Phi_s, Y_s) \, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \, \Pi_s(1 - \Pi_s) \, d\overline{W}_s \\ & + \int_0^t \frac{\partial U}{\partial \phi}(\Pi_s, \Phi_s, Y_s) \, \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma(X_s)} \, \Phi_s \, d\overline{W}_s \end{aligned} \tag{3.32}$$

is a continuous local martingale under $P_{\pi,\phi,y}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$.

It follows from the equation in (3.23) and the conditions of (3.25)-(3.26) that the inequality $(\mathbb{L}_{(\Pi,\Phi,Y)} U)(\pi, \phi, y) \geq -(1 - \pi) c \alpha \phi$ holds for any $(\pi, \phi, y) \in [0, 1] \times (0, \infty) \times \mathbb{R}$ such that $\phi \neq h_*(y)$, as well as $U(\pi, \phi, y) \leq 1 - \pi$ is satisfied for all $(\pi, \phi, y) \in [0, 1] \times (0, \infty) \times \mathbb{R}$. Recall the assumption that the boundary $h_*(y)$ is continuous and of bounded variation and the fact that the process Y from (3.16) is of bounded variation too. We thus conclude from the assumption of continuous differentiability of the functions $\mu_i(x)$, $i = 0, 1$, and $\sigma(x)$ that the time spent by the process Φ at the boundary $h_*(Y)$ is of Lebesgue measure zero, so that the indicator which appears in (3.31) can be ignored. Hence, the expression in (3.31) yields that the inequalities:

$$\begin{aligned} & 1 - \Pi_\tau + \int_0^\tau (1 - \Pi_t) \, c \alpha \Phi_t \, dt \\ & \geq U(\Pi_\tau, \Phi_\tau, Y_\tau) + \int_0^\tau (1 - \Pi_t) \, c \alpha \Phi_t \, dt \geq U(\pi, \phi, y) + M_\tau \end{aligned} \tag{3.33}$$

hold for any stopping time τ of the process (Π, Φ, Y) started at $(\pi, \phi, y) \in [0, 1] \times (0, \infty) \times \mathbb{R}$.

Let $(\tau_n)_{n \in \mathbb{N}}$ be an arbitrary localizing sequence of stopping times for the processes M. Taking the expectations with respect to the probability measure $P_{\pi,\phi,y}$ in (3.33), by means of Doob's optional sampling theorem, we get that the inequalities:

$$\begin{aligned} & E_{\pi,\phi,y} \left[1 - \Pi_{\tau \wedge \tau_n} + \int_0^{\tau \wedge \tau_n} (1 - \Pi_t) \, c \alpha \Phi_t \, dt \right] \\ & \geq E_{\pi,\phi,y} \left[U(\Pi_{\tau \wedge \tau_n}, \Phi_{\tau \wedge \tau_n}, Y_{\tau \wedge \tau_n}) + \int_0^{\tau \wedge \tau_n} (1 - \Pi_t) \, c \alpha \Phi_t \, dt \right] \\ & \geq U(\pi, \phi, y) + E_{\pi,\phi,y} \big[M_{\tau \wedge \tau_n} \big] = U(\pi, \phi, y) \end{aligned} \tag{3.34}$$

hold for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$. Hence, letting n go to infinity and using Fatou's lemma, we obtain:

$$\begin{aligned} & E_{\pi,\phi,y}\left[1 - \Pi_\tau + \int_0^\tau (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] \\ & \quad \geq E_{\pi,\phi,y}\left[U(\Pi_\tau, \Phi_\tau, Y_\tau) + \int_0^\tau (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] \geq U(\pi, \phi, y) \end{aligned} \tag{3.35}$$

for any stopping time τ and all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$. By virtue of the structure of the stopping time in (3.21), it is readily seen that the inequalities in (3.35) hold with τ_* instead of τ when $\phi \geq h_*(y)$.

It remains to show that the equalities are attained in (3.35) when τ_* replaces τ, for $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$ such that $0 \leq \phi < h_*(y)$. By virtue of the fact that the function $U(\pi, \phi, y)$ and the bounded boundary $h_*(y)$ satisfy the conditions in (3.23) and (3.24), it follows from the expression in (3.31) and the structure of the stopping time in (3.21) that the equalities:

$$U(\Pi_{\tau_* \wedge \tau_n}, \Phi_{\tau_* \wedge \tau_n}, Y_{\tau_* \wedge \tau_n}) + \int_0^{\tau_* \wedge \tau_n} (1 - \Pi_t)\, c\alpha \Phi_t \, dt = U(\pi, \phi, y) + M_{\tau_* \wedge \tau_n} \tag{3.36}$$

hold for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of M. Hence, taking into account the assumption that the integral in (3.20) taken up to the optimal stopping time τ_* has a finite expectation and using the fact that $0 \leq U(\pi, \phi, y) \leq 1$ holds, we conclude from the expression in (3.36) that the process $(M_{\tau_* \wedge t})_{t \geq 0}$ is a uniformly integrable martingale. Therefore, taking the expectations in (3.36) and letting n go to infinity, we apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$\begin{aligned} & E_{\pi,\phi,y}\left[1 - \Pi_{\tau_*} + \int_0^{\tau_*} (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] \\ & \quad = E_{\pi,\phi,y}\left[U(\Pi_{\tau_*}, \Phi_{\tau_*}, Y_{\tau_*}) + \int_0^{\tau_*} (1 - \Pi_t)\, c\alpha \Phi_t \, dt\right] = U(\pi, \phi, y) \end{aligned} \tag{3.37}$$

for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$, which together with the inequalities in (3.35) directly imply the desired assertion. $\square$

3.8. We are now in a position to formulate the main assertion of the paper, which follows from a straightforward combination of Lemma 3.2 above and the standard change-of-variable arguments. More precisely, after obtaining the solution $U_*(\pi, \phi, y) \equiv (1 - \pi) H_*(\phi, y; z)$ with

$h_*(y) \equiv h_*(y; z)$ of the free-boundary problem in (3.23)-(3.26), which satisfies the conditions in (3.27)-(3.29), we put $y = y(\pi, x; z)$ and $z = x$, in order to get the solution of the initial Bayesian disorder problem with exponential penalty costs for a detection delay stated in (2.24).

Theorem 3.3. *Suppose that the assumptions of Lemma 3.2 as well as the property in (3.19) hold. Then, in the disorder detection problem of (2.24) and (3.1) for the observation process* X *from (2.2), the Bayesian risk function takes the form* $V_*(\pi, \phi, x) = U_*(\pi, \phi, y(\phi, x)) \equiv (1-\pi) H_*(\phi, y(\phi, x; x); x)$ *and the optimal stopping time has the form:*

$$\tau_* = \inf\{t \geq 0 \,|\, \Phi_t \geq g_*(X_t)\} \tag{3.38}$$

whenever the corresponding integral has a finite expectation, where the boundary $0 < \lambda/(c\alpha) \leq g_*(x)$ *is uniquely determined by the equation* $g(x) = h_*(y(g(x), x)) \equiv h_*(y(g(x), x; x); x)$*, for each* $x > 0$ *fixed. Here, the function* $U_*(\pi, \phi, y) \equiv (1-\pi) H_*(\phi, y; z)$ *and the continuous boundary of bounded variation* $h_*(y) \equiv h_*(y; z)$ *form a unique solution of the free-boundary problem in (3.23)-(3.29), and the expression for* $y(\phi, x) \equiv y(\phi, x; z)$ *is explicitly determined by the relation in (3.18), for all* $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$ *and any* $z > 0$ *fixed.*

Remark 3.4. Observe that the optimal stopping time in the problem of (3.1) does not depend on the dynamics of the process Π, so that the two-dimensional process (Φ, X) turns out to be a *sufficient statistic*. This fact is recognized as a consequence of the structure of the partial differential equation in (3.11)-(3.12). However, the process (Φ, X) is not Markovian itself under the probability measure P_π, and, in order to solve the optimal stopping problem of (3.1), we need to add the component Π to be able to operate with a Markov process (Π, Φ, X).

Remark 3.5. Note that in the disorder detection problem with the linear delay penalty costs indicated in Example 2.4, it follows from (3.8) that, under the assumption that the process $(N_{\tau_* \wedge t})_{t \geq 0}$ in (3.9) is a uniformly integrable martingale, the equality:

$$E_{\phi,x}\left[\frac{1}{1+\widetilde{\Phi}_\tau} + \int_0^\tau \frac{c\widetilde{\Phi}_t}{1+\widetilde{\Phi}_t}\, dt\right] = \frac{1}{1+\phi} + E_{\phi,x} \int_0^{\tau_*} \frac{c\widetilde{\Phi}_t - \lambda}{1+\widetilde{\Phi}_t}\, dt \tag{3.39}$$

holds. Here, $P_{\phi,x}$ is a measure of the diffusion process $(\widetilde{\Phi}, X)$ started at some $(\phi, x) \in [0, \infty) \times (0, \infty)$ and solving the two-dimensional system of equations in (2.19) and (2.22) with (2.18). Taking into account the structure of the reward in (2.27), it is also seen from (3.39) that it is never optimal to stop when $\widetilde{\Phi}_t < \lambda/c$ for any $t \geq 0$.

4. Conclusions

In this section, we provide closed form estimates for the value function and the stopping boundary, for some particular cases of observable diffusions.

4.1. In order to pick up some special cases in which the free-boundary problem in (3.23)-(3.29) can admit a simpler structure, for the rest of the paper, we assume that the property:

$$\mu_i(x) = \frac{\eta_i \sigma^2(x)}{x} \quad \text{for some} \quad \eta_i \in \mathbb{R},\, i=0,1, \quad \text{such that} \quad \eta_0 \neq \eta_1 \quad \text{and} \quad \eta_0 + \eta_1 = 1 \tag{4.1}$$

holds for all $x > 0$. Moreover, we assume that the diffusion coefficient $\sigma(x)$ satisfies:

$$\sigma(x) \sim A_0 \, x^{\alpha} \quad \text{as} \quad x \downarrow 0 \qquad \text{and} \qquad \sigma(x) \sim A_\infty \, x^{\beta} \quad \text{as} \quad x \uparrow \infty \tag{4.2}$$

with some $A_0, A_\infty > 0$ as well as $\alpha, \beta \in \mathbb{R}$ such that $(1-\alpha)\eta \le 0$ and $(1-\beta)\eta \ge 0$ holds, where we set $\eta = 1/(\eta_1 - \eta_0)$. Then, the process $Y = (Y_t)_{t \ge 0}$ takes the form:

$$Y_t = \log \Phi_t - \frac{1}{\eta} \log \frac{z}{X_t} \equiv \log \phi + \int_0^t \left(\frac{\lambda}{\Phi_s} + \lambda + \alpha \right) ds \tag{4.3}$$

for any $z > 0$ fixed. It is easily seen from the structure of the expression in (4.3) that the one-to-one correspondence between the processes (Π, Φ, X) and (Π, Φ, Y) remains true in this case. Getting the expression for X_t from (4.3) and substituting it into the equations of (2.20) and (2.26), we obtain:

$$d\Pi_t = \lambda(1-\Pi_t)\, dt + \frac{\sigma(z e^{-\eta Y_t} \Phi_t^{\eta})}{\eta z e^{-\eta Y_t} \Phi_t^{\eta}} \, \Pi_t (1-\Pi_t) \, d\overline{W}_t \quad (\Pi_0 = \pi) \tag{4.4}$$

and

$$d\Phi_t = \left(\lambda + (\lambda+\alpha)\Phi_t + \frac{\sigma^2(z e^{-\eta Y_t} \Phi_t^{\eta})}{\eta^2 z^2 e^{-2\eta Y_t} \Phi_t^{2\eta}} \, \Pi_t \, \Phi_t \right) dt + \frac{\sigma(z e^{-\eta Y_t} \Phi_t^{\eta})}{\eta z e^{-\eta Y_t} \Phi_t^{\eta}} \, \Phi_t \, d\overline{W}_t \quad (\Phi_0 = \phi) \tag{4.5}$$

for any $z > 0$ fixed. Applying Itô's formula to the expression in (4.3) and taking into account the representations in (2.20) and (2.22) as well as the assumption of (4.1), we get:

$$dY_t = \left(\frac{\lambda}{\Phi_t} + \lambda + \alpha \right) dt \quad (Y_0 = y) \tag{4.6}$$

for all $t \ge 0$. It thus follows that the infinitesimal operator $\mathbb{L}_{(\Pi,\Phi,Y)}$ from (3.22) takes the form:

$$\begin{aligned} \mathbb{L}_{(\Pi,\Phi,Y)} &= \lambda(1-\pi) \frac{\partial}{\partial \pi} + \left(\lambda + (\lambda+\alpha)\phi + \frac{\sigma^2(z e^{-\eta y} \phi^{\eta})}{\eta^2 z^2 e^{-2\eta y} \phi^{2\eta}} \, \pi \, \phi \right) \frac{\partial}{\partial \phi} \\ &+ \frac{1}{2} \frac{\sigma^2(z e^{-\eta y} \phi^{\eta})}{\eta^2 z^2 e^{-2\eta y} \phi^{2\eta}} \left(\pi^2 (1-\pi)^2 \frac{\partial^2}{\partial \pi^2} + 2\pi(1-\pi)\, \phi \, \frac{\partial^2}{\partial \pi \partial \phi} + \phi^2 \frac{\partial^2}{\partial \phi^2} \right) + \left(\frac{\lambda}{\phi} + \lambda + \alpha \right) \frac{\partial}{\partial y} \end{aligned} \tag{4.7}$$

for all $(\pi, \phi, y) \in [0,1] \times (0, \infty) \times \mathbb{R}$ and any $z > 0$ fixed.

4.2. Let us now introduce a function $\widehat{U}(\pi, \phi, y) \equiv (1-\pi)\widehat{H}(\phi, y)$ and a boundary $\widehat{h}(y)$ as a solution of the free-boundary problem consisting of the differential equation:

$$\Big(\big(\lambda + (\lambda + \alpha)\phi\big) \frac{\partial H}{\partial \phi} + \frac{1}{2} \frac{\sigma^2(ze^{-\eta y}\phi^{\eta})}{\eta^2 z^2 e^{-2\eta y} \phi^{2\eta}} \, \phi^2 \, \frac{\partial^2 H}{\partial \phi^2} - \lambda H \Big)(\phi, y) = -c\alpha\phi \quad \text{for} \quad \phi < h(y) \tag{4.8}$$

instead of that in (3.23), for each $y > 0$ fixed, and the conditions of (3.24)-(3.26) as well as (3.27)-(3.29). The general solution of the resulting second order *ordinary* differential equation in (4.8) takes the form:

$$H(\phi, y) = C_0(y) \, H_0(\phi, y) + C_\infty(y) \, H_\infty(\phi, y) - c(1 + \phi) \tag{4.9}$$

where $H_i(\phi, y)$, $i = 0, \infty$, form a system of fundamental positive solutions (i.e. nontrivial linearly independent particular solutions) of the corresponding *homogeneous* differential equation, and $C_i(y)$, $i = 0, \infty$, are some arbitrary continuously differentiable functions, so that the condition in (3.29) holds. By virtue of the assumptions of (4.2) with $(1-\alpha)\eta \leq 0$ and $(1-\beta)\eta \geq 0$, taking into account the arguments from [10; Section 4], we can identify by $H_0(\phi, y)$ a decreasing solution that has a singularity at zero and by $H_\infty(\phi, y)$ an increasing solution that has a singularity at infinity.

Observe that we should have $C_0(y) = 0$ in (4.9), since otherwise $U(\pi, \phi, y) \equiv (1 - \pi)H(\phi, y) \to \pm\infty$ as $\phi \downarrow 0$, that must be excluded by virtue of the obvious fact that the value function in (3.20) is bounded under $\phi \downarrow 0$, for any $y \in \mathbb{R}$ fixed. Then, applying the conditions of (3.24) and (3.27) to the function in (4.9) with $C_0(y) = 0$, we get that the equalities:

$$C_\infty(y) \, H_\infty(h(y), y) = c(1 + h(y)) + 1 \quad \text{and} \quad C_\infty(y) \, \frac{\partial H_\infty}{\partial \phi}(\phi, y) \Big|_{\phi = h(y)} = c \tag{4.10}$$

hold for $y \in \mathbb{R}$ fixed. Hence, solving the equations of (4.10), we get that the solution of the system of (4.8) with (3.24) and (3.27)-(3.28) is given by:

$$H(\phi, y; \widehat{h}(y)) = (c(1 + \widehat{h}(y)) + 1) \, \frac{H_\infty(\phi, y)}{H_\infty(\widehat{h}(y), y)} - c(1 + \phi) \tag{4.11}$$

for all $0 \leq \phi < \widehat{h}(y)$, where $\widehat{h}(y)$ satisfies the equation:

$$\frac{\partial H_\infty}{\partial \phi}(\phi, y) \Big|_{\phi = h(y)} = \frac{cH_\infty(h(y), y)}{c(1 + h(y)) + 1} \tag{4.12}$$

for any $y \in \mathbb{R}$ fixed.

Taking into account the facts proved above, let us formulate the following assertion.

Corollary 4.1. *Suppose that* $\mu_i(x)$*,* $i = 0, 1$*, and* $\sigma(x) > 0$ *are continuously differentiable functions on* $(0, \infty)$ *in (2.2) such that the conditions of (3.19) as well as (4.1)-(4.2) hold with* $(1-\alpha)\eta \le 0$ *and* $(1-\beta)\eta \ge 0$*, where* $\eta = 1/(\eta_1 - \eta_0)$*. Assume that the inequalities* $0 \le H(\phi, y; \widehat{h}(y)) \equiv H(\phi, y; z; \widehat{h}(y; z)) \le 1$ *hold for the function in (4.11), and the equation in (4.12) has a unique solution. Then, using the same arguments as in the proof of Lemma 3.2 above, it is shown that the function* $\widehat{U}(\pi, \phi, y) \equiv (1-\pi)\widehat{H}(\phi, y)$ *with:*

$$\widehat{H}(\phi, y) = \begin{cases} H(\phi, y; \widehat{h}(y)), & \textit{if } \ 0 \le \phi < \widehat{h}(y) \\ 1, & \textit{if } \ \phi \ge \widehat{h}(y) \end{cases} \tag{4.13}$$

coincides with the value function of the optimal stopping problem:

$$\begin{aligned} &\widehat{U}(\pi, \phi, y) && (4.14) \\ &= \inf_{\tau} E_{\pi,\phi,y}\bigg[1 - \Pi_\tau + \int_0^\tau (1-\Pi_t)\bigg(c\alpha\Phi_t - \bigg(\frac{\lambda}{\Phi_t} + \lambda + \alpha\bigg)\frac{\partial \widehat{H}}{\partial y}(\Phi_t, Y_t)\, I(\Phi_t < \widehat{h}(Y_t))\bigg)\, dt\bigg] \end{aligned}$$

which corresponds to the Bayesian risk function in (2.24). Moreover, $\widehat{h}(y) \equiv \widehat{h}(y; z)$ *determined by (4.12) provides a hitting boundary for the stopping time:*

$$\widehat{\tau} = \inf\{t \ge 0 \,|\, \Phi_t \ge \widehat{h}(Y_t)\} \tag{4.15}$$

which turns out to be optimal in (4.14) whenever the integral above has a finite expectation, for any $z > 0$ *fixed.*

Remark 4.2. Under the assumptions of Corollary 4.1, let us now define the function $\overline{U}(\pi, \phi, y)$ by:

$$\overline{U}(\pi, \phi, y) = E_{\pi,\phi,y}\bigg[1 - \Pi_{\widehat{\tau}} + \int_0^{\widehat{\tau}} (1-\Pi_t)\, c\alpha\Phi_t\, dt\bigg] \tag{4.16}$$

where the stopping time $\widehat{\tau}$ is given by (4.15). It follows by virtue of the strong Markov property of the process (Π, Φ, Y) that the value in (4.16) takes the form:

$$\overline{U}(\pi, \phi, y) = \begin{cases} U(\pi, \phi, y; \widehat{h}(y)), & \text{if } \ 0 \le \phi < \widehat{h}(y) \\ 1 - \pi, & \text{if } \ \phi \ge \widehat{h}(y) \end{cases} \tag{4.17}$$

where the function $U(\pi,\phi,y;\widehat{h}(y)) \equiv (1-\pi)H(\phi,y;\widehat{h}(y))$ satisfies the partial differential equation in (3.23) for $\phi < \widehat{h}(y)$. Moreover, since a particular stopping time is used in (4.16), while the infimum in (3.20) is taken over all stopping times such that the integral there has a finite expectation, the value $\overline{U}(\pi,\phi,y)$ provides an *upper estimate* for $U_*(\pi,\phi,y)$, for any $(\pi,\phi,y) \in (0,1)\times(0,\infty)\times\mathbb{R}$. Let us set $\overline{h}(y) = \sup\{\phi > 0 \,|\, H(\phi,y;\widehat{h}(y)) < 1\}$ for any $y \in \mathbb{R}$ fixed. Hence, standard comparison arguments imply that $\overline{h}(y) \vee \lambda/(c\alpha) \le h_*(y)$, where $\widehat{h}(y)$ is assumed to be a unique solution of the equation in (4.12), for each $y \in \mathbb{R}$.

4.3. Let us finally consider the case of linear delay penalty costs indicated in Example 2.4. In this case, following the schema of arguments above, we obtain the corresponding equation:

$$\left(\lambda(1+\phi)\,\frac{\partial U}{\partial \phi} + \frac{\sigma^2(ze^{-\eta y}\phi^\eta)}{\eta^2 z^2 e^{-2\eta y}\phi^{2\eta}}\left(\frac{\phi^2}{1+\phi}\,\frac{\partial U}{\partial \phi} + \frac{\phi^2}{2}\,\frac{\partial^2 U}{\partial \phi^2}\right)\right)(\phi,y) = -\frac{c\phi}{1+\phi} \quad \text{for} \quad \phi < h(y) \tag{4.18}$$

for any $h(y)$ and each $y \in \mathbb{R}$ fixed. The general solution of the resulting first order linear ordinary differential equation for $\phi \mapsto (\partial U/\partial\phi)(\phi,y)$ takes the form:

$$\begin{aligned}\frac{\partial U}{\partial \phi}(\phi,y) = {} & \frac{C(y)}{(1+\phi)^2}\,\exp\left(\int_\phi^w \frac{\lambda(1+u)}{u^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}u^{2\eta}}{\sigma^2(ze^{-\eta y}u^\eta)}\,du\right) \\ & - \int_0^\phi \frac{cu(1+u)}{(1+\phi)^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}u^{2\eta}}{\sigma^2(ze^{-\eta y}u^\eta)}\,\exp\left(-\int_u^\phi \frac{\lambda(1+v)}{v^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}v^{2\eta}}{\sigma^2(ze^{-\eta y}v^\eta)}\,dv\right)du\end{aligned} \tag{4.19}$$

where $C(y)$ is an arbitrary continuously differentiable function, for each $y \in \mathbb{R}$ and any $z, w > 0$ fixed. By virtue of the assumptions of (4.2) with $(1-\alpha)\eta \le 0$ and $(1-\beta)\eta \ge 0$, we see that the term in the first line of (4.19) above tends to infinity as $\phi \downarrow 0$, so that $(\partial U/\partial\phi)(\phi,y) \to \pm\infty$ as $C(y) \neq 0$, for any $y \in \mathbb{R}$ fixed. We should thus choose $C(y) = 0$ that is equivalent to the property in (3.28). Hence, integrating the equation in (4.19), we therefore obtain that the solution of the system of (4.18) with (3.24) and (3.27)-(3.28) is given by:

$$\begin{aligned}& U(\phi,y;\widetilde{h}(y)) = 1/(1+\phi) \\ & + \int_\phi^{\widetilde{h}(y)} \int_0^w \frac{cu(1+u)}{(1+w)^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}u^{2\eta}}{\sigma^2(ze^{-\eta y}u^\eta)}\,\exp\left(-\int_u^w \frac{\lambda(1+v)}{v^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}v^{2\eta}}{\sigma^2(ze^{-\eta y}v^\eta)}\,dv\right)du\,dw\end{aligned} \tag{4.20}$$

for all $0 \le \phi < \widetilde{h}(y)$, where $\widetilde{h}(y)$ satisfies the equation:

$$\int_0^{h(y)} cu(1+u)\,\frac{2\eta^2 z^2 e^{-2\eta y}u^{2\eta}}{\sigma^2(ze^{-\eta y}u^\eta)}\,\exp\left(-\int_u^{h(y)} \frac{\lambda(1+v)}{v^2}\,\frac{2\eta^2 z^2 e^{-2\eta y}v^{2\eta}}{\sigma^2(ze^{-\eta y}v^\eta)}\,dv\right)du = 1 \tag{4.21}$$

for each $y \in \mathbb{R}$ and any $z > 0$ fixed.

Summarising the facts proved above, let us formulate the following assertion.

Corollary 4.3. *Suppose that* $\mu_i(x)$*,* $i = 0, 1$*, and* $\sigma(x) > 0$ *are continuously differentiable functions on* $(0, \infty)$ *in (2.2) such that the conditions of (3.19) as well as (4.1)-(4.2) hold with* $(1-\alpha)\eta \leq 0$ *and* $(1-\beta)\eta \geq 0$*, where* $\eta = 1/(\eta_1 - \eta_0)$*. Assume that the inequalities* $0 \leq U(\phi, y; \widetilde{h}(y)) \equiv U(\phi, y; z; \widetilde{h}(y; z)) \leq 1/(1+\phi)$ *hold for the function in (4.20), and the equation in (4.21) has a unique solution. Then, using the same arguments as in the proof of Lemma 3.2 above, it is shown that the function:*

$$\widetilde{U}(\phi, y) = \begin{cases} U(\phi, y; \widetilde{h}(y)), & \text{if } 0 \leq \phi < \widetilde{h}(y) \\ 1/(1+\phi), & \text{if } \phi \geq \widetilde{h}(y) \end{cases} \tag{4.22}$$

coincides with the value function of the optimal stopping problem:

$$\widetilde{U}(\phi, y) = \inf_{\tau} E_{\phi,y} \left[\frac{1}{1+\widetilde{\Phi}_t} + \int_0^{\tau} \left(\frac{c\widetilde{\Phi}_t}{1+\widetilde{\Phi}_t} - \left(\frac{\lambda}{\widetilde{\Phi}_t} + \lambda \right) \frac{\partial \widetilde{U}}{\partial y}(\widetilde{\Phi}_t, Y_t)\, I(\widetilde{\Phi}_t < \widetilde{h}(Y_t)) \right) dt \right] \tag{4.23}$$

which corresponds to the Bayesian risk function in (2.27). Moreover, $\widetilde{h}(y) \equiv \widetilde{h}(y; z)$ *determined by (4.21) provides a hitting boundary for the stopping time:*

$$\widetilde{\tau} = \inf\{t \geq 0 \,|\, \widetilde{\Phi}_t \geq \widetilde{h}(Y_t)\} \tag{4.24}$$

which turns out to be optimal in (4.23) whenever the integral above has a finite expectation, for any $z > 0$ *fixed. Here,* $E_{\phi,y}$ *denotes the expectation under the assumption that the two-dimensional Markov process* $(\widetilde{\Phi}, Y)$ *from (2.18) and (3.16) with* $\widetilde{\Phi}_t$ *instead of* Φ_t *starts at some* $(\phi, y) \in (0, \infty) \times \mathbb{R}$*.*

Example 4.4. Suppose that we have $\sigma(x) = x$ in (4.1), for all $x > 0$, and some $\eta_i \in \mathbb{R}$, $i = 0, 1$, where the restriction $\eta_0 + \eta_1 = 1$ is omitted. In this case, the process X in (2.2) is a geometric Brownian motion under $\theta = i$, for every $i = 0, 1$. It is easily seen that the initial problem of (3.1) is then equivalent to the Bayesian disorder problem for the observable Wiener process $\log X$ with the changing drift rate from $\eta_0 - 1/2$ to $\eta_1 - 1/2$. The latter problem was reduced to (3.1) and solved as an optimal stopping problem for a one-dimensional Markov process Π in [25] (see also [26], [27; Chapter IV, Section 4] or [17; Chapter VI, Section 22]).

Remark 4.5. We finally note that the associated variational formulation of the problem can be considered following the structure of arguments similar to the one used in [9] (see also

[17; Chapter VI, Section 22]). Those arguments are based on the embedding of the latter problem into the associated Bayesian one, and then the specifying of the appropriate stopping time for the admissible probability of the false alarm given.

Acknowledgments. This research was partially supported by Deutsche Forschungsgemeinschaft through the SFB 649 Economic Risk.

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