We study the perpetual American call option pricing problem in a model of a financial market in which the firm issuing a risky asset can regulate the dividend rate by switching it between two constant values. The firm dividend policy is unknown for small investors who can only observe the prices available from the market. The asset price dynamics are described by a geometric Brownian motion with a random drift rate modeled by a continuous time Markov chain with two states. The optimal exercise time of the option for small investors is found as the first time at which the asset price hits a boundary depending on the current state of the filtering dividend rate estimate. The proof is based on an embedding of the initial problem into a two-dimensional optimal stopping problem and the analysis of the associated parabolic-type free-boundary problem. We also provide closed form estimates for the rational option price and the optimal exercise boundary.

1 Introduction

The problem of pricing of American options is an important problem of the mathematical theory of modern finance. This problem consists of finding a rational (no-arbitrage) price of the option as well as an optimal stopping time at which the option should be exercised. For
the case of perpetual American put options in the Black-Merton-Scholes model, this problem was first studied by McKean [22], who proved optimality of the first time at which the price of the underlying risky asset hits a constant boundary (see also Shiryaev [28; Chapter VIII; Section 2a] and Peskir and Shiryaev [25; Chapter VII; Section 25] for an extensive overview of other related results in the area). More recently, Guo [14]-[15] and Guo and Zhang [16] obtained closed form solutions of the perpetual American lookback and put options pricing problem in an extension of that model to the case in which the drift and volatility coefficients switch their values between two constant ones, according to the change of the state of a continuous time Markov chain. Jobert and Rogers [18] considered the perpetual American put option problem within an extension of that model to the case of several states for the Markov chain and solved the corresponding problem with finite expiry numerically. In the model with a two-state Markov chain and no diffusion part, Dalang and Hongler [6] presented a complete and essentially explicit solution to a similar problem, which exhibited a surprisingly rich structure. These results were further extended by Jiang and Pistorius [17], who studied the perpetual American put option problem within the framework of an exponential jump-diffusion model with observable dynamics of regime-switching behaving parameters. In the present paper, we study the problem of pricing of perpetual American call options on assets continuously paying dividends in a model with a hidden continuous time Markov chain describing the dynamics of the dividend rate. This stays within the general framework of information-based approach for derivative pricing introduced by Brody, Hughston and Macrina [5]. The efficient pricing of derivatives on assets with discretely paid dividends was recently studied by Vellekoop and Nieuwenhuis [30].

We consider a model of a financial market in which the firm issuing the asset can regulate the dividend rate paid to shareholders. To simplify the exposition, we assume that the firm policy allows only two possible states for the dividend rate. Roughly speaking, the firm can be either in a good or bad economic state, so that the dividend rate may take only two values, accordingly. We further assume that, being inaccessible to small investors trading in the market, the future dividend policy of the issuing firm is hidden into the dynamics of the asset prices, under the risk-neutral probability measure. From the point of view of such investors, the firm future dividend policy and thus the dynamics of the dividend rate can be described by a continuous Markov chain with two states.

Suppose that the dynamics of the underlying asset prices are described by a geometric Brownian motion with a random drift rate having the following structure with respect to the
risk-neutral probability measure. We assume that the drift switches its rate from one constant value to another, according to the change of the state of the continuous time Markov chain. Such a switching model for the description of the interest rate dynamics was proposed by Shiryaev [28; Chapter III, Section 4a]. More general hidden continuous time Markov chains were recently used by Elliott and Wilson [8] for modeling of the interest rate dynamics. A model for several asset prices with one-time switching random dividends, which reflects an influence of certain unobservable external events, was recently considered in Gapeev and Jeanblanc [10], where the rational prices of European contingent claims were computed. Some other models with random dividends were earlier considered in the literature (see, e.g. Geske [12]), where the possibility of significance of stochastic dividend effects on the rational values of contingent claims was emphasized.

The paper is organized as follows. In Section 2, we make an embedding of the initial perpetual American option pricing problem into an extended optimal stopping problem for a two-dimensional Markov diffusion process having the asset price and the filtering dividend rate estimate as its state space components. We show that the optimal time of exercise is expressed as the first time at which the asset price process hits a stochastic boundary depending on the current state of the dividend rate estimate. We also formulate an associated free-boundary problem for a parabolic-type second order partial differential operator. Moreover, we construct lower and upper bounds for the value function and the stopping boundary. In Section 3, we reduce the resulting parabolic-type partial differential operator to the normal form. We verify that the solution of the free-boundary problem, which satisfies certain additional conditions, provides the solution of the initial optimal stopping problem. For this, we apply the change-of-variable formula with local time on surfaces, obtained by Peskir [24]. In Section 4, we construct other related estimates, which coincide with the true value function and the exercise boundary, under certain relations between the parameters of the model. We also clarify the validity of the smooth-fit principle for the underlying two-dimensional optimal stopping problem. The main results of the paper are stated in Theorem 3.1 and Corollary 4.1.

Note that other optimal stopping problems for diffusions under partial information were earlier considered by Beibel and Lerche [3; Section 2.6], Rishel [26], and Shiryaev and Novikov [29] for underlying one-dimensional processes, and, more recently, in Gapeev and Shiryaev [11] for an essentially two-dimensional model.
2 Preliminaries

In this section, we introduce the setting and notation of the two-dimensional optimal stopping problem, which is related to the pricing of perpetual American call options under partial information, and formulate the associated free-boundary problem.

2.1 Formulation of the problem

Let us suppose that on a probability space \((\Omega, \mathcal{G}, P)\) there exist a standard Brownian motion \(B = (B_t)_{t \geq 0}\) and a continuous-time Markov chain \(\Theta = (\Theta_t)_{t \geq 0}\) with two states 0 and 1. Assume that \(\Theta\) has initial distribution \(\{1 - \pi, \pi\}\) for \(\pi \in [0, 1]\), transition probability matrix \(\{e^{-\lambda t}, 1 - e^{-\lambda t}; 1 - e^{-\lambda t}, e^{-\lambda t}\}\) for \(t \geq 0\), and intensity matrix \(\{-\lambda, \lambda; \lambda, -\lambda\}\) for some \(\lambda \geq 0\) fixed. Suppose that the processes \(B\) and \(\Theta\) are independent. Let us define the process \(S = (S_t)_{t \geq 0}\) started at some \(s > 0\) by:

\[
S_t = s \exp \left( \int_0^t \left( r - \frac{\sigma^2}{2} - \delta_0 - (\delta_1 - \delta_0) \Theta_u \right) du + \sigma B_u \right)
\]

(2.1)

which solves the stochastic differential equation:

\[
dS_t = (r - \delta_0 - (\delta_1 - \delta_0) \Theta_t) S_t dt + \sigma S_t dB_t \quad (S_0 = s)
\]

(2.2)

where \(\sigma > 0\) and \(0 < \delta_i < r\) are some given constants, for every \(i = 0, 1\).

Assume that the process \(S\) describes the risk-neutral dynamics of the price of a dividend paying asset changing its dividend rate at the switching times of the process \(\Theta\). Suppose that \(\Theta\) reflects the switching behavior of the economic state of the firm issuing the asset from 0 (the firm is in the so-called bad state) to 1 (the firm is in the so-called good state) and vice versa. In those cases, the asset pays dividends at the rate \(\delta_0\) if \(\Theta_t = 0\), and the dividend rate is \(\delta_1\) if \(\Theta_t = 1\), for all \(t \geq 0\). We let the time of each stay be exponentially distributed with parameter \(\lambda\). Here, \(r\) is the interest rate of a riskless bank account and \(\sigma\) is the volatility coefficient.

Suppose that the dividend rate regulation process \(\delta_0 + (\delta_1 - \delta_0) \Theta\) is unknown to small investors trading at the market, who can only observe the dynamics of the asset prices \(S\). The main purpose of the present paper is to derive a solution to the optimal stopping problem:

\[
V_s = \sup_\tau E \left[ e^{-r\tau} (S_\tau - K)^+ \right]
\]

(2.3)

where the supremum is taken over all stopping times \(\tau\) with respect to the natural filtration \(\mathcal{F}_t = \sigma(S_u)_{0 \leq u \leq t} \subseteq \mathcal{G}, t \geq 0\), of the asset price process \(S\). Since we work under
the martingale probability measure $P$ on the filtration $(\mathcal{F}_t)_{t \geq 0}$, the value of (2.3) represents a rational (no-arbitrage) price of a perpetual American call option with the strike price $K > 0$ in a model with partial information.

It is shown by means of standard arguments (see, e.g. [21; Chapter IX] or [7; Chapter VIII]) that the asset price process $S$ from (2.2) admits the representation:

$$dS_t = (r - \delta_0 - (\delta_1 - \delta_0) \Pi_t) S_t \, dt + \sigma S_t \, dB_t \quad (S_0 = s) \quad (2.4)$$

and the filtering estimate $\Pi = (\Pi_t)_{t \geq 0}$ defined by $\Pi_t = E[\Theta_t | \mathcal{F}_t] \equiv P(\Theta_t = 1 | \mathcal{F}_t)$ solves the stochastic differential equation:

$$d\Pi_t = \lambda (1 - 2\Pi_t) \, dt - \frac{\delta_1 - \delta_0}{\sigma} \Pi_t (1 - \Pi_t) \, dB_t \quad (\Pi_0 = \pi) \quad (2.5)$$

for some $(s, \pi) \in (0, \infty) \times [0, 1]$ fixed. Here, the innovation process $B = (B_t)_{t \geq 0}$ defined by:

$$B_t = \int_0^t \frac{dS_u}{\sigma S_u} - \frac{1}{\sigma} \int_0^t (r - \delta_0 - (\delta_1 - \delta_0) \Pi_u) \, du \quad (2.6)$$

is a standard Brownian motion, according to P. Lévy’s characterization theorem (see, e.g. [21; Theorem 4.1]). It can be verified that $(S, \Pi)$ is a (time-homogeneous strong) Markov process under $P$ with respect to its natural filtration $(\mathcal{F}_t)_{t \geq 0}$ as a unique strong solution of the system of stochastic differential equations in (2.4) and (2.5).

2.2 The optimal stopping problem

For the problem of (2.3), let us consider the following extended optimal stopping problem:

$$V^*_s(s, \pi) = \sup_{\tau} E_{s,\pi}[e^{-r\tau} (S_\tau - K)^+] \quad (2.7)$$

where $P_{s,\pi}$ is a measure of the diffusion process $(S, \Pi)$ started at some $(s, \pi) \in (0, \infty) \times [0, 1]$ and solving the two-dimensional system of equations in (2.4) and (2.5). The supremum in (2.7) is therefore taken over all stopping times $\tau$ of $(S, \Pi)$ being a Markovian sufficient statistic in the problem (see [27; Chapter II, Section 15] for a discussion of this notion). By means of the results of general theory of optimal stopping (see, e.g. [27; Chapter III] or [25; Chapter I, Section 2.1]), it follows from the structure of the reward function in (2.7) that the optimal stopping time is given by:

$$\tau^*_s = \inf\{t \geq 0 \mid V^*_s(S_t, \Pi_t) = (S_t - K)^+\} \quad (2.8)$$
so that the continuation region has the form:

\[ C_* = \{(s, \pi) \in (0, \infty) \times [0, 1] \mid V_\pi(s, \pi) > (s - K)^+ \}. \]

(2.9)

In order to specify the structure of the optimal stopping time in (2.8), we apply the change-of-variable formula from [24] to the function \( e^{-rt}(s - K)^+ \) to get:

\[
e^{-rt}(s - K)^+ = (s - K)^+ + M^K_t \\
+ \int_0^t e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^t e^{-ru} I(S_u = K) \, d\ell^K_u(S)
\]

(2.10)

where we set \( H(s, \pi) = rK - (\delta_0 + (\delta_1 - \delta_0)\pi) s \), and the process \( \ell^K(S) = (\ell^K_t(S))_{t \geq 0} \) is the local time of \( S \) at the point \( K \) given by:

\[
\ell^K_t(S) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t I(K - \varepsilon < S_u < K + \varepsilon) \, \sigma S_u^2 \, du
\]

(2.11)

as a limit in probability. Here, the process \( M^K = (M^K_t)_{t \geq 0} \) defined by:

\[
M^K_t = \int_0^t e^{-ru} I(S_u > K) \, \sigma S_u d\overline{B}_u
\]

(2.12)

is a continuous square integrable martingale under \( P_{s,\pi} \), and \( I(\cdot) \) denotes the indicator function. Therefore, applying Doob’s optional sampling theorem (see, e.g. [21; Theorem 3.6]) and using the expression in (2.10), we get that:

\[
E_{s,\pi}[e^{-r\tau}(S_\tau - K)^+] = (s - K)^+ \\
+ E_{s,\pi} \left[ \int_0^\tau e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^\tau e^{-ru} I(S_u = K) \, d\ell^K_u(S) \right]
\]

(2.13)

holds for any stopping time \( \tau \) and all \((s, \pi) \in (0, \infty) \times [0, 1]\). Choosing \( \tau \) to be the exit time of \( S \) from a small ball, it is seen from (2.13) that it is never optimal to stop when \( H(S_t, \Pi_t) \leq 0 \) and \( S_t > K \), for \( t \geq 0 \). In other words, this shows that all points \((s, \pi)\) such that \( K < s \leq b(\pi) \) with \( b(\pi) = rK/(\delta_0 + (\delta_1 - \delta_0)\pi) \) for \( \pi \in [0, 1] \) belong to the continuation region in (2.9), which clearly contains the rectangle \( \{(s, \pi) \in (0, K) \times [0, 1]\} \).

Let us fix some \((s, \pi)\) from the continuation region \( C_* \) in (2.9) and let \( \tau_* = \tau_*(s, \pi) \) denote the optimal stopping time in the problem of (2.7). Then, by means of general optimal stopping theory for Markov processes (see, e.g. [27; Chapter III] or [25; Chapter I, Section 2.2]), we conclude from (2.13) that:

\[
V_\pi(s, \pi) - (s - K)^+ \\
= E_{s,\pi} \left[ \int_0^{\tau_*} e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau_*} e^{-ru} I(S_u = K) \, d\ell^K_u(S) \right] > 0
\]

(2.14)
holds. Hence, taking any \( s' \) such that \( K < b(\pi) < s' < s \) and using the explicit expression for the process \( S \) through its starting point in (2.1), we obtain from (2.13) that the inequalities:

\[
V_s(s', \pi) - (s' - K)^+
\geq E_{s', \pi} \left[ \int_0^{\tau^*} e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau^*} e^{-ru} I(S_u = K) \, d\ell(K)^{u} (S) \right]
\geq E_{s, \pi} \left[ \int_0^{\tau^*} e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau^*} e^{-ru} I(S_u = K) \, d\ell(K)^{u} (S) \right]
\]

are satisfied. Thus, taking into account the fact that \( 0 < \delta_i < r \), by virtue of the inequality in (2.14) we see that \( (s', \pi) \in C_* \). These arguments, together with the easily proved concavity of the function \( s \mapsto V_s(s, \pi) \) on \((0, \infty)\), show that there exists a function \( b_*(\pi) \) such that \( K < b(\pi) \leq b_*(\pi) \) holds for all \( \pi \in [0, 1] \) and the continuation region in (2.9) for the optimal stopping problem of (2.7) takes the form:

\[
C_* = \{(s, \pi) \in (0, \infty) \times [0, 1] | s < b_*(\pi) \}
\]

and the corresponding stopping region is the closure of the set:

\[
D_* = \{(s, \pi) \in (0, \infty) \times [0, 1] | s > b_*(\pi) \}.
\]

For any \((s, \pi) \in C_*\) fixed, let us now take \( \pi' < \pi \) if \( \delta_0 < \delta_1 \) (or \( \pi < \pi' \) if \( \delta_0 > \delta_1 \)), whenever \( s > K \). Then, using the facts that \((S, \Pi)\) is a time-homogeneous Markov process and \( \tau_\pi = \tau_\pi(s, \pi) \) does not depend on \( \pi' \), taking into account the comparison results for solutions of stochastic differential equations, we obtain from the expression in (2.10) and the structure of the process \( \ell(K)^{u}(S) \) in (2.11) that:

\[
V_s(s, \pi') - (s - K)^+
\geq E_{s, \pi'} \left[ \int_0^{\tau^*} e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau^*} e^{-ru} I(S_u = K) \, d\ell(K)^{u} (S) \right]
\geq E_{s, \pi} \left[ \int_0^{\tau^*} e^{-ru} H(S_u, \Pi_u) I(S_u > K) \, du + \frac{1}{2} \int_0^{\tau^*} e^{-ru} I(S_u = K) \, d\ell(K)^{u} (S) \right]
\]

holds. By virtue of the inequality in (2.14), we may conclude that \((s, \pi') \in C_*\), so that the boundary \( b_*(\pi) \) is decreasing (increasing) on \([0, 1]\) in (2.16)-(2.17) whenever \( \delta_0 < \delta_1 \) (\( \delta_0 > \delta_1 \)), respectively.

Summarizing the facts proved above, we are now ready to formulate the following assertion.
Lemma 2.1 Suppose that $\sigma > 0$ and $0 < \delta_i < r$ for every $i = 0, 1$ in (2.1)-(2.2). Then, in the perpetual American call option pricing problem of (2.3) and (2.7) under partial information, the optimal exercise time has the structure:

$$\tau_* = \inf \{ t \geq 0 \mid S_t \geq b_*(\Pi_t) \} \quad (2.19)$$

where the function $b_*(\pi)$ satisfies the properties:

$$b_*(\pi) : [0, 1] \to (K, \infty) \text{ is decreasing / increasing if } \delta_0 < \delta_1 / \delta_0 > \delta_1 \quad (2.20)$$

$$K < b(\pi) \leq b_*(\pi) \text{ with } b(\pi) = rK / (\delta_0 + (\delta_1 - \delta_0)\pi) \quad (2.21)$$

for all $\pi \in [0, 1]$.

By means of standard arguments based on the application of Itô’s formula, it is shown that the infinitesimal operator $L_{(S, \Pi)}$ of the process $(S, \Pi)$ from (2.4)-(2.5) has the structure:

$$L_{(S, \Pi)} = (r - \delta_0 - (\delta_1 - \delta_0)\pi) s \partial_s + \frac{1}{2} \sigma^2 s^2 \partial_{ss} - (\delta_1 - \delta_0) s \pi (1 - \pi) \partial_{s\pi}$$

$$+ \lambda (1 - 2\pi) \partial_\pi + \frac{1}{2} \left( \frac{\delta_1 - \delta_0}{\sigma} \right)^2 \pi^2 (1 - \pi)^2 \partial_{\pi\pi} \quad (2.22)$$

for all $(s, \pi) \in (0, \infty) \times [0, 1]$.

In order to find analytic expressions for the unknown value function $V_*(s, \pi)$ from (2.7) and the boundary $b_*(\pi)$ from (2.16)-(2.17), we use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [13], [27; Chapter III, Section 8] and [25; Chapter IV, Section 8]) to formulate the associated free-boundary problem:

$$(L_{(S, \Pi)} V)(s, \pi) = r V(s, \pi) \text{ for } (s, \pi) \in C \quad (2.23)$$

$$V(s, \pi) \big|_{s=b(\pi)-} = b(\pi) - K \quad \text{ (instantaneous stopping)} \quad (2.24)$$

$$V(s, \pi) = (s - K)^+ \text{ for } (s, \pi) \in D \quad (2.25)$$

$$V(s, \pi) > (s - K)^+ \text{ for } (s, \pi) \in C \quad (2.26)$$

where $C$ and $D$ are defined as $C_*$ and $D_*$ in (2.16) and (2.17) with $b(\pi)$ instead of $b_*(\pi)$, and the instantaneous-stopping conditions in (2.24) are satisfied for all $\pi \in (0, 1)$.

Note that the superharmonic characterization of the value function (see [27; Chapter III, Section 8] or [25; Chapter IV, Section 9]) implies that $V_*(s, \pi)$ from (2.7) is the smallest function satisfying (2.23)-(2.26) with the boundary $b_*(\pi)$. 

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Remark 2.2 Observe that, since the system in (2.23)-(2.26) admits multiple solutions, we need to find some additional conditions which would specify the appropriate solution providing the value function and the optimal stopping boundary for the initial problem of (2.7). In order to derive such conditions, in the following section, we shall reduce the operator in (2.23) to the normal form. We also note that the fact that the stochastic differential equations for the asset price and the filtering estimate in (2.4) and (2.5) are driven by the same (one-dimensional) innovation Brownian motion yields the property that the infinitesimal operator in (2.22) turns out to be of parabolic type.

2.3 The case of full information

Let us now recall the corresponding problem with full information, which consists of computing the value function:

\[ W^*_s(s,i) = \sup_{\zeta} E_{s,i}[e^{-r\zeta}(S_{\zeta} - K)^+] \]  

(2.27)

where the supremum is taken over all stopping times \( \zeta \) with respect to the filtration \( G_t = \sigma(S_u, \Theta_u \, | \, 0 \leq u \leq t), \, t \geq 0, \) and \( P_{s,i} \) is the measure of the process \( (S, \Theta) \) started at some \( (s,i) \in (0, \infty) \times \{0,1\} \). Since the continuous time Markov chain \( \Theta \) is observable in this formulation, the optimal stopping time for the problem (2.27) should be of the form:

\[ \zeta_* = \inf\{t \geq 0 \mid S_t \geq a_*(\Theta_t)\} \]  

(2.28)

for some points \( a_*(i), \, i = 0,1 \), to be determined.

Following the way of arguments from [14]-[16] and [17], we conclude that the functions \( W^*_s(s,i) \) and the boundaries \( a_*(i), \, i = 0,1 \), solve the coupled system of ordinary differential free-boundary problems:

\[
(r - \delta_i) s W^*_s(s,i) + \frac{1}{2} \sigma^2 s^2 W^*_{ss}(s,i) = \frac{(r + \lambda)}{s} W^*(s,1-i) \quad \text{for} \quad 0 < s < a(i)
\]

(2.29)

\[
W^*_s(s,i) \big|_{s = a(i)^-} = a(i) - K \quad \text{(instantaneous stopping)}
\]

(2.30)

\[
W^*_s(s,i) \big|_{s = a(i)^-} = 1 \quad \text{(smooth fit)}
\]

(2.31)

\[
W^*_s(s,i) = (s - K)^+ \quad \text{for} \quad s > a(i)
\]

(2.32)

\[
W^*_s(s,i) > (s - K)^+ \quad \text{for} \quad s < a(i)
\]

(2.33)

for every \( i = 0,1 \). To simplify the notation, we further assume that \( \delta_0 > \delta_1 \), without loss of generality. By means of straightforward calculations, we obtain that the general solution of the
two-dimensional system of second order ordinary differential equations in (2.29)-(2.33) is given by:

$$W(s, i) = C_1(i) s^{\beta_1} + C_2(i) s^{\beta_2}$$

(2.34)

for $0 < s < a(0)$ and $i = 0, 1$, as well as

$$W(s, 1) = C_3(1) s^{\gamma_1} + \frac{\lambda s}{\delta_1 + \lambda} - \frac{\lambda K}{r + \lambda}$$

(2.35)

for $a(0) < s < a(1)$, where $\beta_2 < \beta_1$ are the two largest roots of the corresponding characteristic equation:

$$\left( r + \lambda - \beta(r - \delta_0) - \frac{1}{2} \beta(\beta - 1)\sigma^2 \right) \left( r + \lambda - \beta(r - \delta_1) - \frac{1}{2} \beta(\beta - 1)\sigma^2 \right) = \lambda^2$$

(2.36)

and $\gamma_2 < 0 < 1 < \gamma_1$ are explicitly given by:

$$\gamma_j = \frac{1}{2} - \frac{r - \delta_1}{\sigma^2} - (-1)^j \sqrt{\left( \frac{1}{2} - \frac{r - \delta_1}{\sigma^2} \right)^2 + 2\left( \frac{r + \lambda}{\sigma^2} \right)}$$

(2.37)

for every $j = 1, 2$. It follows from the conditions in (2.30)-(2.31) as well as from the fact that the function $W_*(s, 1)$ is $C^2$ at the point $a_*(0)$, that the constants $C_j(i)$, $i = 0, 1$, $j = 1, 2$, as well as the boundary $a_*(0)$ are uniquely determined by the equations:

$$C_1(0) a^{\beta_1}(0) + C_2(0) a^{\beta_2}(0) = a(0) - K$$

(2.38)

$$C_1(1) a^{\beta_1}(0) + C_2(1) a^{\beta_2}(0) = C_3(1) a^{\gamma_1}(0) + \frac{\lambda s}{\delta_1 + \lambda} - \frac{\lambda K}{r + \lambda}$$

(2.39)

$$C_1(0) \beta_1 a^{\beta_1}(0) + C_2(0) \beta_2 a^{\beta_2}(0) = a(0)$$

(2.40)

$$C_1(1) \beta_1 a^{\beta_1}(0) + C_2(1) \beta_2 a^{\beta_2}(0) = C_3(1) \gamma_1 a^{\gamma_1}(0) + \frac{\lambda}{\delta_1 + \lambda}$$

(2.41)

$$C_1(1) \beta_1 (\beta_1 - 1) a^{\beta_1}(0) + C_2(1) \beta_2 (\beta_2 - 1) a^{\beta_2}(0) = C_3(1) \gamma_1 (\gamma_1 - 1) a^{\gamma_1}(0).$$

(2.42)

Thus, solving the system in (2.38)-(2.42), by means of straightforward calculations, we obtain that the solution of the free-boundary problem in (2.29)-(2.31) is given by:

$$W(s, 0; a_*(0)) = \sum_{j=1}^{\beta_3} \left( \beta_3 - j \right) a_*(0) - \beta_3 - j K \left( \frac{s}{a_*(0)} \right)^{\beta_3 - j}$$

(2.43)

and

$$W(s, 1; a_*(0)) = \sum_{j=1}^{\beta_3} \beta_3 - j W(a_*(0), 1; a_*(1)) - W_s(a_*(0), 1; a_*(1)) a_*(0) \left( \frac{s}{a_*(0)} \right)^{\beta_3 - j}$$

(2.44)
for $0 < s < a_*(0)$, as well as

$$W(s, 1; a_*(1)) = \left(\frac{\delta_1 a_*(1)}{\delta_1 + \lambda} - \frac{rK}{r + \lambda}\right) \left(\frac{s}{a_*(1)}\right)^{\gamma_1} + \frac{\lambda s}{\delta_1 + \lambda} - \frac{\lambda K}{r + \lambda} \quad (2.45)$$

for $a_*(0) \leq s < a_*(1)$. Here, $a_*(0)$ is determined as a unique root of the equation:

$$\sum_{j=1}^{2} (-1)^j \beta_j (\beta_j - 1)[\beta_{2-j}W(a_*(0), 1; a_*(1)) - W_s(a_*(0), 1; a_*(1))a_*(0)] = (\beta_1 - \beta_2)\gamma_1 rK/(r + \lambda) \quad (2.46)$$

and $a_*(1)$ is explicitly given by:

$$a_*(1) = \frac{\gamma_1 K}{\gamma_1 - 1} \frac{r}{r + \lambda} \frac{\delta_1 + \lambda}{\delta_1} \quad (2.47)$$

Using the appropriate verification assertion that is shown by means of the arguments similar to the ones used in the proof of Lemma 3.1 below, we may therefore conclude that the value functions from (2.27) take the form:

$$W_*(s, 0) = \begin{cases} W(s, 0; a_*(0)), & \text{if } 0 < s < a_*(0) \\ s - K, & \text{if } s \geq a_*(0) \end{cases} \quad (2.48)$$

and

$$W_*(s, 1) = \begin{cases} W(s, 1; a_*(0)), & \text{if } 0 < s < a_*(0) \\ W(s, 1; a_*(1)), & \text{if } a_*(0) \leq s < a_*(1) \\ s - K, & \text{if } s \geq a_*(1) \end{cases} \quad (2.49)$$

where the expressions for $W(s, i; a_*(0))$, $i = 0, 1$, and $W(s, 1; a_*(1))$ are given by (2.43)-(2.44) and (2.45), respectively, under the assumption that $\delta_0 > \delta_1$.

**Remark 2.3** Since the set of stopping times over which the supremum is taken in (2.27) is larger than that in (2.7), it is easily seen that the inequalities $V_*(s, i) \leq W_*(s, i)$ hold, so that $W_*(s, i)$ is an upper bound for $V_*(s, i)$, for every $i = 0, 1$. Furthermore, by virtue of the Markovian structure of the problem in (2.7) for the process $(S, \Pi)$ in (2.4) and (2.5), taking into account the comparison results for solutions of stochastic differential equations, we conclude that the value function $V_*(s, \pi)$ is increasing in $\pi \in [0, 1]$ for each $s > 0$ fixed, under the assumption that $\delta_0 > \delta_1$. The latter fact directly implies that the inequality $V_*(s, \pi) \leq W_*(s, 1)$ is satisfied, so that the standard comparison arguments yield that $b_*(\pi) \leq a_*(1)$ holds for all $(s, \pi) \in (0, \infty) \times [0, 1)$.  

11
3 Main results

In this section, we reduce the operator from (2.22) to the normal form and prove the corresponding verification assertion.

In order to find the normal form of the operator from (2.22) and formulate the associated optimal stopping and free-boundary problem, we follow the arguments of [11] and use the one-to-one correspondence transformation of processes proposed by A.N. Kolmogorov in [19]. For this, let us define the process

\[ Y_t = \frac{S_t^v \Pi_t}{1 - \Pi_t} \]  

for all \( t \geq 0 \), where we set \( \eta = (\delta_0 - \delta_1)/\sigma^2 \). Then, it follows from (2.4)-(2.5) that the process \( S \) takes the form:

\[ dS_t = \left( r - \delta_0 - (\delta_1 - \delta_0) \frac{S_t^v Y_t}{1 + S_t^v Y_t} \right) S_t dt + \sigma S_t dB_t \quad (S_0 = s) \]  

and, by means of Itô’s formula, we get that the process \( Y \) admits the representation:

\[ dY_t = \left( \frac{\lambda (1 - S_t^v Y_t)}{1 + S_t^v Y_t} - \eta (2r - \delta_0 - \delta_1 - \sigma^2) \right) Y_t dt \quad (Y_0 = y \equiv \frac{s^v \pi}{1 - \pi}) \]  

for any \((s, \pi) \in (0, \infty) \times (0, 1)\). It is seen from the equation in (3.3) that the process \( Y \) started at \( y > 0 \) is of bounded variation. It follows from the relation in (3.1) that there exists a one-to-one correspondence between the processes \((S, \Pi)\) and \((S, Y)\). Hence, for any \( \pi \in (0, 1) \) fixed, the value function \( U_*(s, \pi) \) from (2.7) is equal to that of the optimal stopping problem:

\[ U_*(s, y) = \sup_{\tau} E_{s,y} \left[ e^{-r\tau} (S_\tau - K)^+ \right] \]  

where the supremum is taken over all stopping times \( \tau \) with respect to the natural filtration of \((S, Y)\), which clearly coincides with \((F_t)_{t \geq 0}\). Here, \( E_{s,y} \) denotes the expectation under the assumption that the two-dimensional Markov process \((S, Y)\) from (3.2) and (3.3) starts at some \((s, y) \in (0, \infty) \times (0, 1)\). It thus follows from (2.19) that there exist a function \( g_*(y) \geq K \), for \( y > 0 \), such that the optimal stopping time in the problem of (3.4) is of the form:

\[ \tau_* = \inf \{ t \geq 0 | S_t \geq g_*(Y_t) \}. \]  

Standard arguments then show that the infinitesimal operator \( \mathbb{L}_{(S,Y)} \) of the process \((S, Y)\) from (3.2) and (3.3) has the structure:

\[ \mathbb{L}_{(S,Y)} = \left( r - \delta_0 - (\delta_1 - \delta_0) \frac{s^v y}{1 + s^v y} \right) s \partial_s + \frac{1}{2} \sigma^2 s^2 \partial_{ss} \]

\[ + \left( \frac{\lambda (1 - s^v y)}{1 + s^v y} - \eta (2r - \delta_0 - \delta_1 - \sigma^2) \right) y \partial_y \]  

\[ + \left( (r - \delta_0) \frac{s^v y}{1 + s^v y} \right) \sum_{i=1}^d \partial_{s_i} \]  

\[ + \left( \frac{\lambda (1 - s^v y)}{1 + s^v y} - \eta (2r - \delta_0 - \delta_1 - \sigma^2) \right) \sum_{i=1}^d \partial_{y_i} \]  

[12]
for all \((s, y) \in (0, \infty)^2\). We are now ready to formulate the associated free-boundary problem for the unknown value function \(U_*(s, y)\) from (3.4) and the boundary \(g_*(y)\) from (3.5):

\[
\begin{align*}
\mathbb{L}_{(S,Y)}U(s, y) &= rU(s, y) \quad \text{for} \quad 0 < s < g(y) \\
U(s, y) \big|_{s=g(y)-} &= g(y) - K \quad \text{(instantaneous stopping)} \\
U(s, y) &= (s - K)^+ \quad \text{for} \quad s > g(y) \\
U(s, y) > (s - K)^+ \quad \text{for} \quad s < g(y)
\end{align*}
\] (3.7)

where the instantaneous stopping condition in (3.8) is satisfied for all \(y > 0\). Moreover, we assume that the following conditions:

\[
\begin{align*}
U(s, y) \big|_{s=0+} &= 0 \quad \text{(natural boundary)} \\
U_s(s, y) \big|_{s=g(y)-} &= 1 \quad \text{(smooth fit)}
\end{align*}
\] (3.11) (3.12)

hold and the one-sided derivative:

\[
U_y(s, y) \big|_{s=g(y)-} \quad \text{exists}
\] (3.13)

for all \(y > 0\). Note that the condition of (3.11) can be used because 0 is a natural boundary for the process \(S\) in the model of (3.2)-(3.3).

We further search for solutions of the parabolic-type free-boundary problem in (3.7)-(3.10) satisfying the conditions in (3.11)-(3.13) and such that the resulting boundary is continuous and of bounded variation. Since such free-boundary problems cannot, in general, be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the related variational inequalities and their connection with the optimal stopping problems have been extensively studied in the literature (see, e.g. [9], [4], [20] or [23]).

We continue with the following verification lemma related to the free-boundary problem in (3.7)-(3.13).

**Lemma 3.1** Assume that the function \(U(s, y; g_*(y))\) and the continuous monotone boundary \(g_*(y)\) such that \(K < g(y) \leq g_*(y)\) form a unique solution of the free-boundary problem in (3.7)-(3.10) and satisfying the conditions of (3.11)-(3.13). Here, the function \(g(y)\) is defined as the inverse to \(g^{-1}(s) = (\delta_0 s - rK)s^{-\eta}/(rK - \delta_1 s)\) for each \(s > rK/(\delta_0 \vee \delta_1)\) with \(\eta = (\delta_0 - \delta_1)/\sigma^2\). Then, the value function of the optimal stopping problem in (3.4) takes the form:

\[
U_*(s, y) = \begin{cases} 
U(s, y; g_*(y)), & \text{if } 0 < s < g_*(y) \\
 s - K, & \text{if } s \geq g_*(y)
\end{cases}
\] (3.14)
and \( g_*(y) \) provides the optimal stopping boundary for (3.5), for all \((s, y) \in (0, \infty)^2\).

**Proof:** Let us denote by \( U(s, y) \) the right-hand side of the expression in (3.14). Hence, applying the change-of-variable formula with local time on surfaces from [24] to \( e^{-rt}U(s, y) \) and \( g_*(y) \), and taking into account the smooth-fit condition in (3.12), we obtain:

\[
e^{-rt} U(S_t, Y_t) = U(s, y) + \int_0^t e^{-ru} (L_{(S,Y)U} - rU)(S_u, Y_u) I(S_u \neq g_*(Y_u)) du + M_t
\]

where the process \( M = (M_t)_{t \geq 0} \) defined by:

\[
M_t = \int_0^t e^{-ru} U_s(s, Y_u) \sigma_s d\overline{B}_u
\]

is a continuous square integrable martingale under \( P_{s,y} \) with respect to \((\mathcal{F}_t)_{t \geq 0}\).

It follows from the equation in (3.7) and the conditions of (3.8)-(3.10) that the inequality \( (L_{(S,Y)U} - rU)(s, y) \leq 0 \) holds for any \((s, y) \in (0, \infty)^2\) such that \( s \neq g_*(y) \), as well as \( U(s, y) \geq (s - K)^+ \) is satisfied for all \((s, y) \in (0, \infty)^2\). Note that the former inequality is equivalent to the assumption that \( g(y) \leq g_*(y) \) holds for all \( y > 0 \). Recall the assumption that the boundary \( g_*(y) \) is continuous and monotone and the fact that \( g_*(y) \) and the process \( Y \) from (3.1) are of bounded variation. We thus conclude that the time spent by the process \( S \) at the boundary \( g_*(Y) \) is of Lebesgue measure zero, so that the indicator which appears in (3.15) can be ignored. Hence, the expression in (3.15) with the structure of the stopping time in (3.5) yields that the inequalities:

\[
e^{-r\tau} (S_\tau - K)^+ \leq e^{-r\tau} U(S_\tau, Y_\tau) \leq U(s, y) + M_\tau\]

hold for any stopping time \( \tau \) of the process \( (S, Y) \) started at \((s, y) \in (0, \infty)^2\).

Let \((\tau_n)_{n \in \mathbb{N}}\) be an arbitrary localizing sequence of stopping times for the processes \( M \).

Taking in (3.17) the expectation with respect to the measure \( P_{s,y} \), by means of Doob’s optional sampling theorem, we get that the inequalities:

\[
E_{s,y} [e^{-r(\tau \wedge \tau_n)} (S_{\tau \wedge \tau_n} - K)^+] \leq E_{s,y} [e^{-r(\tau \wedge \tau_n)} U(S_{\tau \wedge \tau_n}, Y_{\tau \wedge \tau_n})] \leq U(s, y) + E_{s,y}[M_{\tau \wedge \tau_n}] = U(s, y)
\]

hold for all \((s, y) \in (0, \infty)^2\). Hence, letting \( n \) go to infinity and using Fatou’s lemma, we obtain:

\[
E_{s,y} [e^{-r\tau} (S_\tau - K)^+] \leq E_{s,y} [e^{-r\tau} U(S_\tau, Y_\tau)] \leq U(s, y)
\]
for any stopping time $\tau$ and all $(s, y) \in (0, \infty)^2$. By virtue of the structure of the stopping time in (3.5), it is readily seen that the inequalities in (3.19) hold with $\tau_*$ instead of $\tau$ when $s \geq g_*(y)$.

It remains to show that the equalities are attained in (3.19) when $\tau_*$ replaces $\tau$ for $(s, y) \in (0, \infty)^2$ such that $s < g_*(y)$. By virtue of the assumption that the function $U(s, y)$ and the boundary $g_*(y)$ satisfy the system in (3.7)-(3.9), it follows from the expression in (3.15) and the structure of the stopping time in (3.5) that the equalities:

$$e^{-r(\tau_* \wedge \tau_n)} U(S_{\tau_* \wedge \tau_n}, Y_{\tau_* \wedge \tau_n}) = U(s, y) + M_{\tau_* \wedge \tau_n}$$

(3.20)

hold for all $(s, y) \in (0, \infty)^2$ and any localizing sequence $(\tau_n)_{n \in \mathbb{N}}$ of $M$. Observe that it follows from the structure of the process $S$ in (2.1) that $e^{-r\tau_*} S_{\tau_*}$ and thus $e^{-r\tau_*} U(S_{\tau_*}, Y_{\tau_*})$ are equal to zero on the event $\{\tau_* = \infty\}$ ($P_{s,y}$-a.s.). Therefore, taking the expectations in (3.20) and letting $n$ go to infinity, we use the fact that the process $(M_{\tau_* \wedge t})_{t \geq 0}$ is a uniformly integrable martingale and apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$E_{s,y} [e^{-r\tau_*} (S_{\tau_*} - K)^+] = E_{s,y} [e^{-r\tau_*} U(S_{\tau_*}, Y_{\tau_*})] = U(s, y)$$

(3.21)

for all $(s, y) \in (0, \infty)^2$, which together with the inequalities in (3.19) directly imply the desired assertion. \hfill \Box

We are now in a position to formulate the main assertion of the paper, which follows from a straightforward combination of Lemmas 2.1 and 3.1 above and the standard change-of-variable arguments. More precisely, after obtaining the solution $U_*(s, y)$ with $g_*(y)$ of the free-boundary problem in (3.7)-(3.10), which satisfies the conditions of (3.11)-(3.13), we put $y = s^{-\eta} \pi/(1 - \pi)$, in order to get the solution of the initial perpetual American call option pricing problem stated in (2.3).

**Theorem 3.2** Suppose that the assumptions of Lemma 3.1 hold with $\delta_0 > \delta_1$. Then, in the perpetual American call option pricing problem of (2.3) and (2.7) for the underlying asset price process $S$ from (2.1)-(2.2), the value function takes the form:

$$V_*(s, \pi) = \begin{cases} U_*(s, s^{-\eta} \pi/(1 - \pi)), & \text{if } 0 < s < g_*(s^{-\eta} \pi/(1 - \pi)) \\ s - K, & \text{if } s \geq g_*(s^{-\eta} \pi/(1 - \pi)) \end{cases}$$

(3.22)

and the optimal exercise boundary $b_*(\pi)$ in (2.19) satisfying (2.20)-(2.21) is uniquely determined as the inverse to $b_*^{-1}(s) = s^{\eta} g_*^{-1}(s)/(1 + s^{\eta} g_*^{-1}(s))$ for each $s > rK/(\delta_0 \lor \delta_1)$ with
\( \eta = (\delta_0 - \delta_1)/\sigma^2 \). Moreover, the functions \( V_*(s, \pi) \) and \( b_*(\pi) \) are monotone (increasing) and \( K < b(\pi) \leq b_*(\pi) \) holds with \( b(\pi) = rK/(\delta_0 + (\delta_1 - \delta_0)\pi) \) for all \( \pi \in [0,1] \).

**Remark 3.3** In order to construct a uniformly lower estimate for the value \( V_*(s, \pi) \) from (3.22), let us introduce the function:

\[
\tilde{W}(s, \pi) = \begin{cases} 
W(s, 0; a_*(0)) (1 - \pi) + W(s, 1; a_*(0)) \pi, & \text{if } 0 < s < \tilde{a}(\pi) \\
 s - K, & \text{if } s \geq \tilde{a}(\pi)
\end{cases}
\]  

(3.23)

where the expressions for \( W(s, i; a_*(0)) \), \( i = 0, 1 \), are given by (2.43)-(2.44). Here, the boundary \( \tilde{a}(\pi) \) is uniquely determined as the smallest root of the equation:

\[
W(a(\pi), 0; a_*(0)) (1 - \pi) + W(a(\pi), 1; a_*(0)) \pi = a(\pi) - K
\]  

(3.24)

for each \( \pi \in (0,1) \). Then, by means of straightforward computations, it is shown that the function \( \tilde{W}(s, \pi) \) from (3.23) surprisingly solves the partial differential equation in (2.23) above for \( 0 < s < \tilde{a}(\pi) \), and the inequality:

\[
\tilde{W}_s(s, \pi) \bigg|_{s=\tilde{a}(\pi)-} < 1
\]  

(3.25)

holds for \( \pi \in (0,1) \). Taking into account the condition of (3.25), by means of the arguments of Lemma 3.1, we may therefore conclude from Theorem 3.1 that the function \( \tilde{W}(s, \pi) \) from (3.23) represents a lower bound for the initial value function \( V_*(s, \pi) \) from (2.7). Hence, standard comparison arguments yield that the boundary \( \tilde{a}(\pi) \), which is uniquely determined by (3.24), turns out to be a lower bound for the initial exercise boundary \( b_*(\pi) \). More precisely, it follows that the inequality \( \tilde{W}(s, \pi) \leq V_*(s, \pi) \) and thus \( \tilde{a}(\pi) \leq b_*(\pi) \) hold for all \( (s, \pi) \in (0, \infty) \times [0,1] \).

**Remark 3.4** We can now combine the assertions of Remarks 2.2 and 3.1 and use the obvious facts that the function \( \tilde{W}(s, \pi) \) is increasing in \( \pi \in (0,1) \) for each \( s > 0 \) fixed, as well as \( \tilde{W}(s, 0+) = W_*(s, 0) \), whenever \( \delta_0 > \delta_1 \). Namely, we conclude that the inequalities \( W_*(s, 0) \leq V_*(s, \pi) \leq W_*(s, 1) \) and thus \( a_*(0) \leq b_*(\pi) \leq a_*(1) \) are satisfied for all \( (s, \pi) \in (0, \infty) \times [0,1] \), under the assumption that \( \delta_0 > \delta_1 \). In other words, the corresponding problem of (2.27) in a model with full information provides both lower and upper bounds for the value function and the exercise boundary of the initial American option problem of (2.7) in a model with partial information.
4 Conclusions

In this section, we provide closed form estimates for the value function and the exercise boundary, and make some concluding remarks concerning analytic properties of the solution of the free-boundary problem, under certain relations between the parameters of the model.

Let us now introduce a function \( \hat{U}(s, y) \) and a boundary \( \hat{g}(y) \) as a solution of the free-boundary problem consisting of the differential equation:

\[
(L(S, Y)U - rU)(s, y) = \left( \frac{\lambda(1 - s^n y)}{1 + s^n y} - \frac{\eta}{2} \frac{(2r - \delta_0 - \delta_1 - \sigma^2)}{y} \right) y U_y(s, y) \quad \text{for} \quad 0 < s < g(y)
\]

instead of that in (3.7), for each \( y > 0 \) fixed, and the boundary conditions of (3.8)-(3.10) as well as (3.11)-(3.12). The general solution of the resulting homogeneous ordinary differential equation in (4.1) takes the form:

\[
U(s, y) = \sum_{j=1}^{2} C_j(y) s^{\alpha_j} F(\psi_{j1}, \psi_{j2}; 1 + \varphi_0; s^n y)
\]

(4.2)

where \( C_j(y) \), \( j = 1, 2 \), are some arbitrary continuously differentiable functions, so that the condition in (3.13) holds, as well as

\[
\alpha_j = \frac{1}{2} - \frac{r - \delta_0}{\sigma^2} - (-1)^j \varphi_0 \eta \quad \text{and} \quad \psi_{kl} = 1 - (-1)^k \varphi_k - (-1)^l \varphi_l
\]

(4.3)

with

\[
\varphi_i = \frac{1}{\eta} \sqrt{\frac{\delta_i^2}{\sigma^4} + \delta_i \left( 1 - \frac{2r}{\sigma^4} \right) + \left( \frac{r}{\sigma^2} + \frac{1}{2} \right)^2}
\]

(4.4)

for every \( i = 0, 1 \) and \( j, k, l = 1, 2 \). Here, \( F(a, b; c; x) \) denotes Gauss’ hypergeometric function, which admits the integral representation:

\[
F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} \, dt
\]

(4.5)

for \( c > b > 0 \) and has the series expansion:

\[
F(a, b; c; x) = 1 + \sum_{k=1}^{\infty} \frac{(a_k)(b_k)}{(c_k)} \frac{x^k}{k!}
\]

(4.6)

for \( c \neq 0, -1, -2, \ldots \) and \( (c)_k = c(c + 1) \cdots (c + k - 1) \), \( k \in \mathbb{N} \), where \( \Gamma \) denotes Euler’s Gamma function and the series converges under all \( |x| < 1 \) (see, e.g. [1; Chapter XV] and [2; Chapter II]). Observe that we should have \( C_2(y) = 0 \) in (4.2), since otherwise \( U(s, y) \to \pm \infty \)
as $s \downarrow 0$, that must be excluded by virtue of the obvious fact that the value function in (3.4) is bounded under $s \downarrow 0$, for any $y > 0$ fixed. Then, applying the conditions of (3.8) and (3.12) to the function in (4.2) with $C_2(y) = 0$, we get that the following equalities:

$$C_1(y) g^{\alpha_1}(y) F(\psi_{11}, \psi_{12}; 1 + \varphi_0; g^\eta(y)y) = g(y) - K$$

(4.7)

$$\alpha_1 C_1(y) g^{\alpha_1}(y) F(\psi_{11}, \psi_{12}; 1 + \varphi_0; g^\eta(y)y)$$

(4.8)

$$+ \eta C_1(y) g^{\alpha_1+\eta}(y)y \frac{(1 + \varphi_0)^2 - \varphi_1^2}{1 + \varphi_0} F(1 + \psi_{11}, 1 + \psi_{12}; 2 + \varphi_0; g^\eta(y)y) = g(y)$$

hold for each $y > 0$ fixed. Hence, solving the system of (4.7)-(4.8), we get that the solution of the system of (4.1) with (3.8) and (3.11)-(3.12) is given by:

$$U(s, y; \hat{g}(y)) = (\hat{g}(y) - K) \left( \frac{s}{\hat{g}(y)} \right)^{\alpha_1} F\left(\psi_{11}, \psi_{12}; 1 + \varphi_0; s^\eta y\right)$$

(4.9)

for all $0 < s < \hat{g}(y)$, where $\hat{g}(y)$ satisfies the equation:

$$\frac{(1 + \varphi_0)^2 - \varphi_1^2}{1 + \varphi_0} F(1 + \psi_{11}, 1 + \psi_{12}; 2 + \varphi_0; g^\eta(y)y) = \frac{\alpha_1 K + (1 - \alpha_1)g(y)}{(g(y) - K)\eta g^\eta(y)y}$$

(4.10)

for any $y > 0$ fixed. The uniqueness of the solution of (4.10) as well as the condition of (3.10) are verified using the properties of Gauss’ hypergeometric function.

Taking into account the facts proved above, let us now formulate the following assertion.

**Corollary 4.1** Using the same arguments as in the proof of Lemma 3.1 above, it is shown that the function:

$$\hat{U}(s, y) = \begin{cases} U(s, y; \hat{g}(y)), & \text{if } 0 < s < \hat{g}(y) \\ s - K, & \text{if } s \geq \hat{g}(y) \end{cases}$$

(4.11)

with $U(s, y; \hat{g}(y))$ from (4.9) coincides with the value function of the optimal stopping problem:

$$\hat{U}(s, y) = \sup_{\tau} E_{s,y} \left[ e^{-\tau}(S_\tau - K)^+ + \int_0^\tau e^{-rt} \left( \frac{\lambda(1 - S_\tau^\eta Y_t)}{1 + \delta_0 + \sigma^2} - \frac{\eta}{2} (2r - \delta_0 - \delta_1 - \sigma^2) \right) Y_t \hat{g}(S_t, Y_t) I(S_t < \hat{g}(Y_t)) dt \right]$$

(4.12)

and $\hat{g}(y)$ from (4.10) provides the hitting boundary for the stopping time:

$$\hat{\tau} = \inf\{t \geq 0 \mid S_t \geq \hat{g}(Y_t)\}$$

(4.13)

which turns out to be optimal in (4.12). Note that the functions $\hat{U}(s, y)$ and $\hat{g}(y)$ can also be considered as estimates for the true value function $U_*(s, y)$ and the boundary $g_*(y)$ from (2.7) and (2.19), for all $(s, y) \in (0, \infty)^2$. 
Remark 4.2 Let us finally consider the case in which \( \lambda = 0 \). This means that \( \Theta_t \equiv \theta \) for all \( t \geq 0 \), where \( \theta \) is a random variable taking two values 1 and 0 with probabilities \( P(\theta = 1) = \pi \) and \( P(\theta = 0) = 1 - \pi \) for \( \pi \in [0, 1] \). This assumption reflects the situation in which the issuing firm does not change the dividend policy which is inaccessible to a usual small investor during the whole infinite time interval. The same model appears by solving the Bayesian sequential testing problem for two simple hypotheses about the unknown local drift of an observed Wiener process (see, e.g. [27; Chapter IV, Section 2] or [25; Chapter VI, Section 1]). In that case, \( \pi \) and \( 1 - \pi \) play the role of a priori probabilities of the statistical hypotheses \( H_1 : \theta = 1 \) and \( H_0 : \theta = 0 \), respectively. It follows directly from the structure of the expressions in (4.12) and (4.13) that the functions \( \hat{U}(s, y) \) and \( \hat{g}(y) \) coincide with the value function \( U_*(s, y) \) and the boundary \( g_*(y) \) of the initial optimal stopping problem of (3.4) and (3.5), under the additional assumption that \( \delta_0 + \delta_1 = 2r - \sigma^2 \) is satisfied.

Remark 4.3 It follows from the assumptions of Corollary 4.1 that the boundary \( \hat{g}(y) \) from the equation in (4.10) is continuously differentiable on \((0, \infty)\). Hence, under the assumptions that \( \lambda = 0 \) and \( \delta_0 + \delta_1 = 2r - \sigma^2 \) holds, by means of straightforward computations applied to the expression in (4.9), it is shown using (4.10) that the smooth-fit condition:

\[
\left. (\partial_y U_*)(s, y) \right|_{s = g_*(y)} = 0
\]

is satisfied for all \( y > 0 \). Thus, taking into account the expression in (3.12), which holds for \( U_*(s, y) \) at \( g_*(y) \) in this case, and the one-to-one correspondence given by (3.1), we conclude from (4.14) that the smooth-fit condition:

\[
\left. (\partial_\pi V_*)(s, \pi) \right|_{s = b_*(\pi)} = 0
\]

with \( b_*(\pi) = g_*(s^{-\pi/\pi}(1 - \pi)) \) is satisfied for all \( \pi \in (0, 1) \). This property can be explained by the fact that the continuous process \( S \) intersects the continuous boundary \( g_*(Y) \) and thus \( b_*(\Pi) \) with a positive probability.

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