Zero-Sum Games and Linear Programming Duality

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John von Neumann (1903–1957)

- set theory
- mathematics of quantum mechanics
- minimax theorem [1928], game theory
- stored-program computer
- self-replicating automata

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from *The Man from the Future (2021):*

“Von Neumann would carry on a conversation with my three-year-old son, and the two of them would talk as equals, and I sometimes wondered if he used the same principle when he talked to the rest of us.” Edward Teller, 1966

3 October 1947: Dantzig meets von Neumann

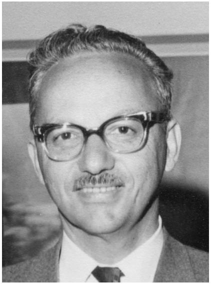
GD: In under one minute I slapped on the blackboard a geometric and algebraic version of the linear programming problem.

Von Neumann stood up and said, “Oh, that!”

[gives eye-popping lecture on LP duality]

JvN: . . . I have recently completed a book with Oscar Morgenstern on the theory of games. I conjecture that the two problems are equivalent.

GD: Thus I learned about Farkas’s Lemma and about duality for the first time.



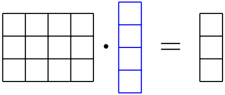
George Dantzig (1914–2005)

Notation, treat vectors and scalars as matrices

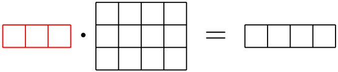
All vectors are column vectors. $\boldsymbol{A}^\top$ = matrix $\boldsymbol{A}$ transposed.

$\mathbf{0} = (\mathbf{0}, \ldots, \mathbf{0})^\top$, $\mathbf{1} = (\mathbf{1}, \ldots, \mathbf{1})^\top$.

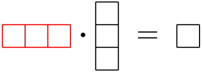
$\boldsymbol{A}\boldsymbol{x}$ = linear combination of columns of $\boldsymbol{A}$



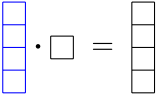
$\boldsymbol{y}^\top\boldsymbol{A}$ = linear combination of rows of $\boldsymbol{A}$



$\boldsymbol{y}^\top\boldsymbol{b}$ = scalar product of $\boldsymbol{y}$ and $\boldsymbol{b}$



$\boldsymbol{x}\alpha$ = (column) vector $\boldsymbol{x}$ scaled by α



$\alpha\boldsymbol{y}^\top$ = row vector $\boldsymbol{y}$ scaled by α

Primal and dual linear programs

Primal LP:

$$\begin{aligned} \textbf{maximize} \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \\ & \boldsymbol{x} \ge \boldsymbol{0}. \end{aligned}$$

Dual LP:

$$\begin{aligned} \textbf{minimize} \quad & \boldsymbol{y}^\top \boldsymbol{b} \\ \text{subject to} \quad & \boldsymbol{y} \ge \boldsymbol{0}, \\ & \boldsymbol{y}^\top \boldsymbol{A} \ge \boldsymbol{c}^\top. \end{aligned}$$

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Weak LP duality: For any **feasible** primal $\boldsymbol{x}$, dual $\boldsymbol{y}$:

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because $\boldsymbol{0} \le (\boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top)\boldsymbol{x}, \quad \boldsymbol{0} \le \boldsymbol{y}^\top(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})\,.$

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because $\boldsymbol{0} \leq (\boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top)\boldsymbol{x}, \quad \boldsymbol{0} \leq \boldsymbol{y}^\top (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})$.

So $\boxed{\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b}} \Rightarrow$ $\boldsymbol{x}$ optimal for primal LP, $\boldsymbol{y}$ optimal for dual LP.

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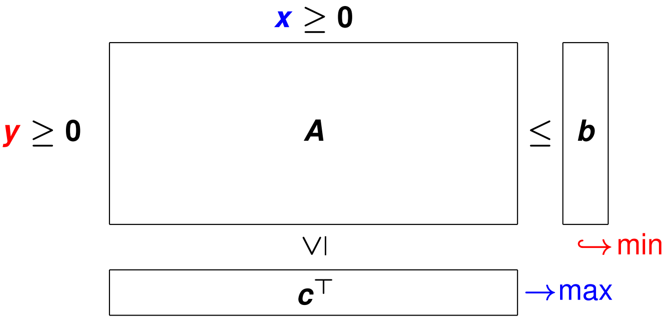
So $\boxed{\boldsymbol{c}^\top \boldsymbol{x} = \boldsymbol{y}^\top \boldsymbol{b}}$ $\Rightarrow$ $\boldsymbol{x}$ optimal for primal LP, $\boldsymbol{y}$ optimal for dual LP.

feasible $\boldsymbol{x}$, $\boldsymbol{y}$ optimal $\Leftrightarrow$ **complementary slackness:**
$\mathbf{0} = (\boldsymbol{y}^\top \boldsymbol{A} - \boldsymbol{c}^\top)\boldsymbol{x}, \quad \mathbf{0} = \boldsymbol{y}^\top(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})$

Tucker diagram

Primal LP: **maximize** $\boldsymbol{c}^\top \boldsymbol{x}$ subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}.$

Dual LP: **minimize** $\boldsymbol{y}^\top \boldsymbol{b}$ subject to $\boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{c}^\top, \quad \boldsymbol{y} \geq \boldsymbol{0}.$



Zero-sum games

Game matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

maximizing row player chooses row $i \in [m] = \{1, \ldots, m\}$

minimizing column player chooses column $j \in [n] = \{1, \ldots, n\}$

payoff a_{ij} to row player = cost to column player

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Mixed-strategy sets

$$Y = \{y \in \mathbb{R}^m \mid y \geq 0,\ \mathbf{1}^\top y = 1\},$$
$$X = \{x \in \mathbb{R}^n \mid x \geq 0,\ \mathbf{1}^\top x = 1\},$$

expected payoff / cost: $\quad y^\top A x$

Best responses

Let $\boldsymbol{x} \in \boldsymbol{X}$. $(\boldsymbol{A}\boldsymbol{x})_i$ = expected payoff in row $\boldsymbol{i}$.

A **best response** $\boldsymbol{y} \in \boldsymbol{Y}$ to $\boldsymbol{x}$ maximizes $\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{x}$.

$$\begin{aligned} & \max\{\boldsymbol{y}^\top(\boldsymbol{A}\boldsymbol{x}) \mid \boldsymbol{y} \in \boldsymbol{Y}\} \\ = \; & \max\{(\boldsymbol{A}\boldsymbol{x})_1, \ldots, (\boldsymbol{A}\boldsymbol{x})_m\} \\ = \; & \min\{\boldsymbol{v} \in \mathbb{R} \mid (\boldsymbol{A}\boldsymbol{x})_1 \leq \boldsymbol{v}, \ldots, (\boldsymbol{A}\boldsymbol{x})_m \leq \boldsymbol{v}\} \\ = \; & \min\{\boldsymbol{v} \in \mathbb{R} \mid \boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}\boldsymbol{v}\} \end{aligned}$$

max-min and min-max strategies

min-max strategy $\hat{\boldsymbol{x}} \in \boldsymbol{X}$:

$$\max_{\boldsymbol{y} \in \boldsymbol{Y}} \boldsymbol{y}^\top \boldsymbol{A} \hat{\boldsymbol{x}} = \min_{\boldsymbol{x} \in \boldsymbol{X}} \max_{\boldsymbol{y} \in \boldsymbol{Y}} \boldsymbol{y}^\top \boldsymbol{A} \boldsymbol{x}$$

$$= \min_{\boldsymbol{x} \in \boldsymbol{X}} \{ \boldsymbol{v} \in \mathbb{R} \mid \boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}\boldsymbol{v} \}$$

max-min strategy $\hat{\boldsymbol{y}} \in \boldsymbol{Y}$:

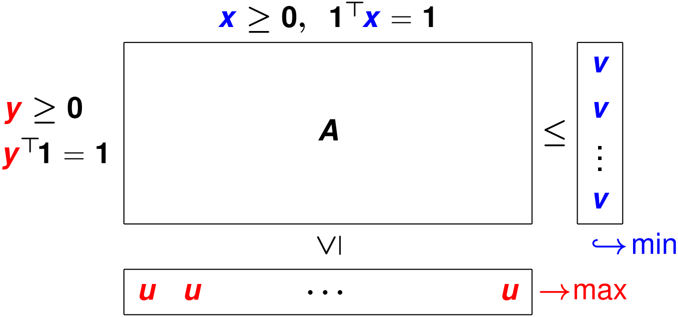
$$\min_{\boldsymbol{y} \in \boldsymbol{Y}} \hat{\boldsymbol{x}}^\top \boldsymbol{A} \boldsymbol{y} = \max_{\boldsymbol{y} \in \boldsymbol{Y}} \min_{\boldsymbol{x} \in \boldsymbol{X}} \boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}$$

$$= \max_{\boldsymbol{y} \in \boldsymbol{Y}} \{ \boldsymbol{u} \in \mathbb{R} \mid \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{u}\mathbf{1}^\top \}$$

Written as general LP

Minimizer: **minimize** v subject to $Ax \le \mathbf{1}v$, $x \in X$.

Maximizer: **maximize** u subject to $y^\top A \ge u\mathbf{1}^\top$, $y \in Y$.



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	$x \ge 0$	v			
$y \ge 0$	A	-1 $\vdots$ -1	$\le$	0 $\vdots$ 0	
u	$-1 \cdots -1$	0	$=$	-1	$\hookrightarrow$ min
	$\ge$	$=$			
	$0 \cdots 0$	-1	$\rightarrow$ max		

von Neumann's minimax theorem

Every zero-sum game $\boldsymbol{A}$ has a **value** $\boldsymbol{v}$:

$$\max_{\boldsymbol{y} \in \boldsymbol{Y}} \min_{\boldsymbol{x} \in \boldsymbol{X}} \boldsymbol{y}^\top \boldsymbol{A} \boldsymbol{x} = \boldsymbol{v} = \min_{\boldsymbol{x} \in \boldsymbol{X}} \max_{\boldsymbol{y} \in \boldsymbol{Y}} \boldsymbol{y}^\top \boldsymbol{A} \boldsymbol{x}$$

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also, with max-min strategy $\hat{\boldsymbol{y}}$ and min-max strategy $\hat{\boldsymbol{x}}$:

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$$\hat{\boldsymbol{y}}^\top \boldsymbol{A} \boldsymbol{x} \geq \hat{\boldsymbol{y}}^\top \boldsymbol{A} \hat{\boldsymbol{x}} \geq \boldsymbol{y}^\top \boldsymbol{A} \hat{\boldsymbol{x}}$$

$\Leftrightarrow$ $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{x}})$ is a **Nash equilibrium** (exists via fixed point theorem).

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The minimax theorem is a consequence of strong LP duality.

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What about the converse?

Dantzig's game [1951]

$$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$$

Dantzig's game [1951]

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$\boldsymbol{B} = -\boldsymbol{B}^\top \;\Rightarrow\;$ symmetric game with value **0** (by minimax theorem),

$\exists$ optimal $\boldsymbol{z} = (\boldsymbol{y}, \boldsymbol{x}, \boldsymbol{t}) \geq \mathbf{0}$ with $\boxed{\boldsymbol{B}\boldsymbol{z} \leq \mathbf{0} \text{ and } \boldsymbol{z}^\top\boldsymbol{B} \geq \mathbf{0}^\top}$:

$$\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\boldsymbol{t} \leq \mathbf{0}\,, \qquad -\boldsymbol{A}^\top\boldsymbol{y} + \boldsymbol{c}\boldsymbol{t} \leq \mathbf{0}\,, \qquad \boldsymbol{b}^\top\boldsymbol{y} - \boldsymbol{c}^\top\boldsymbol{x} \leq \mathbf{0}\,.$$

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If $\boldsymbol{t} > 0$: $\boldsymbol{x}\frac{1}{\boldsymbol{t}}$ primal optimal and $\boldsymbol{y}\frac{1}{\boldsymbol{t}}$ dual optimal.

Dantzig's game [1951]

$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

$B = -B^\top \;\Rightarrow\;$ symmetric game with value **0** (by minimax theorem),

$\exists$ optimal $z = (y, x, t) \geq 0$ with $\boxed{Bz \leq 0 \text{ and } z^\top B \geq 0^\top}$:

$$Ax - bt \leq 0\,, \qquad -A^\top y + ct \leq 0\,, \qquad b^\top y - c^\top x \leq 0\,.$$

If $t > 0$: $x\frac{1}{t}$ primal optimal and $y\frac{1}{t}$ dual optimal.

If $t = 0$ $\boxed{\text{and}}$ $b^\top y < c^\top x$ then $b^\top y < 0$ or $0 < c^\top x$

(otherwise $b^\top y \geq 0 \geq c^\top x$), and $Ax \leq 0$ and $y^\top A \geq 0^\top$.

Unbounded rays

Suppose for some $\bar{\boldsymbol{x}}$:

$$\boldsymbol{A}\bar{\boldsymbol{x}} \leq \boldsymbol{b}\,, \quad \bar{\boldsymbol{x}} \geq \boldsymbol{0}\,,$$

and $\boldsymbol{0} < \boldsymbol{c}^\top \boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{0}$ for some $\boldsymbol{x} \geq \boldsymbol{0}$.

Then $\boldsymbol{A}(\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha) \leq \boldsymbol{b}$, $\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha \geq \boldsymbol{0}$,

$$\boldsymbol{c}^\top(\bar{\boldsymbol{x}} + \boldsymbol{x}\alpha) \; = \; \boldsymbol{c}^\top\bar{\boldsymbol{x}} + (\boldsymbol{c}^\top\boldsymbol{x})\alpha \to \infty$$

as $\alpha \to \infty$.

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as $\alpha \to \infty$. $\Rightarrow$ (by weak duality): dual LP infeasible.

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$\Rightarrow$ **Strong LP duality theorem**

Either primal and dual LP are feasible and then have optimal solutions with equal objective functions,

or at least one LP is infeasible and the other (if feasible) is unbounded (with an unbounded ray).

But what if $t = 0$ $\boxed{\text{and}}$ $b^\top y = c^\top x$?

Dantzig's game gives no information about the LP!

$$B = \begin{bmatrix} 0 & A & -b \\ -A^\top & 0 & c \\ b^\top & -c^\top & 0 \end{bmatrix}$$

This means an unused best response and thus violates
strict complementarity. This only occurs in degenerate cases.

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This means an unused best response and thus violates **strict complementarity.** This only occurs in degenerate cases.

Given $\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{k \times k}$

want $\boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{Bz} \leq \mathbf{0},\ z_k - (\boldsymbol{Bz})_k > 0\,.$

Tucker's Lemma [1956]

For $\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{\boldsymbol{k}\times\boldsymbol{k}}$, $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$:

$$\exists\, \boldsymbol{z} \geq \boldsymbol{0}\,, \quad \boldsymbol{Bz} \leq \boldsymbol{0}\,, \quad \boldsymbol{z_k} - (\boldsymbol{Bz})_{\boldsymbol{k}} > \boldsymbol{0}$$

$$\exists\, \boldsymbol{x} \geq \boldsymbol{0}\,, \ \boldsymbol{y} \geq \boldsymbol{0} \ : \ \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top, \quad \boldsymbol{Ax} \leq \boldsymbol{0}\,, \quad \boldsymbol{x_n} + (\boldsymbol{y}^\top \boldsymbol{A})_{\boldsymbol{n}} > \boldsymbol{0}$$

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$$\Downarrow:\ \ \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{A} \\ -\boldsymbol{A}^\top & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{z} = \binom{\boldsymbol{y}}{\boldsymbol{x}}. \qquad \Uparrow:\ \ \boldsymbol{B} = \boldsymbol{A}\,,\ \ \boldsymbol{z} = \boldsymbol{y} + \boldsymbol{x}$$

$$\exists\, \boldsymbol{x} \geq \boldsymbol{0}\,,\ \boldsymbol{y} \geq \boldsymbol{0}\ \ :\ \ \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top,\ \ \boldsymbol{Ax} \leq \boldsymbol{0}\,,\ \ \boldsymbol{x_n} + (\boldsymbol{y}^\top \boldsymbol{A})_{\boldsymbol{n}} > \boldsymbol{0}$$

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For $\boldsymbol{B} = -\boldsymbol{B}^\top \in \mathbb{R}^{\boldsymbol{k}\times\boldsymbol{k}}$, $\boldsymbol{A} \in \mathbb{R}^{\boldsymbol{m}\times\boldsymbol{n}}$:

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$$\Downarrow:\ \boldsymbol{Ax} \leq \boldsymbol{0}\,,\ -\boldsymbol{Ax} \leq \boldsymbol{0} \qquad \Uparrow:\ \boldsymbol{I_{m\times m}s} + \boldsymbol{Ax} = \boldsymbol{0}$$

$$\exists\, \boldsymbol{x} \geq \boldsymbol{0}\,,\ \boldsymbol{y} \ :\ \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top, \quad \boldsymbol{Ax} = \boldsymbol{0}\,, \quad \boldsymbol{x_n} + (\boldsymbol{y}^\top \boldsymbol{A})_{\boldsymbol{n}} > \boldsymbol{0}$$

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Lemma of Tucker $\Rightarrow$ Lemma of Farkas

Tucker's Lemma :

$$\exists\, \mathbf{x} \geq \mathbf{0}\,,\ \mathbf{y} \ :\ \mathbf{y}^\top \mathbf{A} \geq \mathbf{0}^\top, \quad \mathbf{A}\mathbf{x} = \mathbf{0}\,, \ \boxed{x_n + (\mathbf{y}^\top \mathbf{A})_n > 0}$$

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Apply to $[\,\mathbf{A}\ \ -\mathbf{b}\,]$:

$$\exists\, \mathbf{x} \geq \mathbf{0},\ t \geq 0,\ \mathbf{y} \ :\ \mathbf{A}\mathbf{x} - \mathbf{b}t = \mathbf{0},\ \mathbf{y}^\top \mathbf{A} \geq \mathbf{0}^\top,\ -\mathbf{y}^\top \mathbf{b} \geq 0,$$

$$\boxed{t - \mathbf{y}^\top \mathbf{b} > 0}.$$

Lemma of Tucker $\Rightarrow$ Lemma of Farkas

Tucker's Lemma :

$$\exists\, x \geq 0\,,\; y \;:\; y^\top A \geq 0^\top,\quad Ax = 0\,,\quad \boxed{x_n + (y^\top A)_n > 0}$$

Apply to $[\,A \;\; -b\,]$:

$$\exists\, x \geq 0\,,\; t \geq 0\,,\; y \;:\; Ax - bt = 0\,,\;\; y^\top A \geq 0^\top,\;\; -y^\top b \geq 0\,,$$

$$\boxed{t - y^\top b > 0}\,.$$

$t = 0$: $\quad y^\top b < 0\,.$

$t > 0$: $\quad Ax\frac{1}{t} = b$

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Lemma of Farkas $\Rightarrow$ Lemma of Tucker

Lemma of Farkas :

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$\boldsymbol{A} = [\boldsymbol{A}_1 \cdots \boldsymbol{A}_n]$:

either $\exists\, \boldsymbol{z} \in \mathbb{R}^{n-1} : \boldsymbol{z} \geq \mathbf{0}, \;\; \sum_{j=1}^{n-1} \boldsymbol{A}_j \boldsymbol{z}_j = -\boldsymbol{A}_n$:

let $\boldsymbol{x} = \binom{\boldsymbol{z}}{1}, \quad \boldsymbol{y} = \mathbf{0}$

or $\exists\, \boldsymbol{y} : \boldsymbol{y}^\top \boldsymbol{A}_j \geq 0 \;\; (1 \leq j \leq n-1), \;\; \boldsymbol{y}^\top(-\boldsymbol{A}_n) < 0$:
let $\boldsymbol{x} = \mathbf{0}\,.$

Lemma of Farkas $\Rightarrow$ Lemma of Tucker

Lemma of Farkas :

$$\nexists\, x \geq 0 \; : \; Ax = b \quad \Leftrightarrow \quad \exists\, y \; : \; y^\top A \geq 0^\top, \;\; y^\top b < 0\,.$$

$A = [A_1 \cdots A_n]$:

either $\quad \exists\, z \in \mathbb{R}^{n-1} : z \geq 0, \;\; \sum_{j=1}^{n-1} A_j z_j = -A_n$:

let $\; x = \binom{z}{1}, \;\; y = 0$

or $\quad \exists\, y : y^\top A_j \geq 0 \;\; (1 \leq j \leq n-1), \;\; y^\top(-A_n) < 0$:
let $\; x = 0\,.$

$\Rightarrow \quad x \geq 0, \;\; y^\top A \geq 0^\top, \;\; Ax = 0\,, \;\; x_n + (y^\top A)_n > 0$

= Lemma of Tucker

Dantzig's assumption

... assumes Tucker's Lemma and hence the Lemma of Farkas,

which proves LP duality directly.

The minimax theorem is not of much use here!

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Next: we fix this.

Distilled from Adler [2013].

Tucker's Theorem

Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

Tucker's Lemma: for **any** $j \in \{1, \ldots, n\}$:

$$\exists\, \boldsymbol{x} \geq \boldsymbol{0}\,,\ \boldsymbol{y} \ :\ \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top, \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\,, \ \boxed{x_j + (\boldsymbol{y}^\top \boldsymbol{A})_j > 0}$$

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Summing over all j gives $\boldsymbol{x}$, $\boldsymbol{y}$ with

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= Tucker's Theorem ($\Rightarrow$ Tucker's Lemma)

Also for $\boldsymbol{B} = -\boldsymbol{B}^\top$: $\quad \exists\, \boldsymbol{z} \geq \mathbf{0}\ :\ \boldsymbol{B}\boldsymbol{z} \leq \mathbf{0}, \quad \boldsymbol{z} - \boldsymbol{B}\boldsymbol{z} > \mathbf{0}.$

Stiemke [1915], Gordan [1873]

Stiemke's Theorem

$$\nexists\, \boldsymbol{y} \;:\; \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top, \;\; \boldsymbol{y}^\top \boldsymbol{A} \neq \boldsymbol{0}^\top \quad \Leftrightarrow \quad \exists\, \boldsymbol{x} \;:\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\,, \;\; \boldsymbol{x} > \boldsymbol{0}$$

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$$\exists\, \boldsymbol{x}, \boldsymbol{y} \;:\; \boldsymbol{x} \geq \boldsymbol{0},\;\; \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{0}^\top,\;\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0},\;\; \boldsymbol{x}^\top + \boldsymbol{y}^\top \boldsymbol{A} > \boldsymbol{0}^\top$$

Gordan, Ville [1938], minimax theorem

Gordan's Theorem

$$\nexists\, \boldsymbol{x} \;:\; \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\,,\;\; \boldsymbol{x} \geq \boldsymbol{0}\,,\;\; \boldsymbol{x} \neq \boldsymbol{0} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \;:\; \boldsymbol{y}^\top \boldsymbol{A} > \boldsymbol{0}^\top$$

Ville's Theorem

$$\nexists\, \boldsymbol{x} \;:\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{0}\,,\;\; \boldsymbol{x} \geq \boldsymbol{0}\,,\;\; \boldsymbol{x} \neq \boldsymbol{0} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \geq \boldsymbol{0} \;:\; \boldsymbol{y}^\top \boldsymbol{A} > \boldsymbol{0}^\top$$

minimax theorem

$$\exists\, \boldsymbol{x} \in \boldsymbol{X},\; \boldsymbol{y} \in \boldsymbol{Y},\; \boldsymbol{v} \in \mathbb{R} \;:\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{1}\boldsymbol{v},\quad \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{v}\boldsymbol{1}^\top$$

From Gordan to Tucker

Let $\tilde{\boldsymbol{x}}$ with $\tilde{\boldsymbol{x}} \geq \mathbf{0}$, $\boldsymbol{A}\tilde{\boldsymbol{x}} = \mathbf{0}$ have maximum support

$\boldsymbol{S} = \{\, \boldsymbol{j} \mid \tilde{\boldsymbol{x}}_{\boldsymbol{j}} > \mathbf{0} \,\}$

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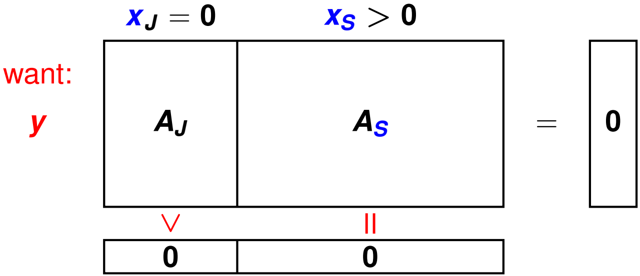
Let $\tilde{\boldsymbol{x}}$ with $\tilde{\boldsymbol{x}} \geq \mathbf{0}$, $\boldsymbol{A}\tilde{\boldsymbol{x}} = \mathbf{0}$ have maximum support

$\boldsymbol{S} = \{\, \boldsymbol{j} \mid \tilde{\boldsymbol{x}}_{\boldsymbol{j}} > \mathbf{0} \,\}$, write $\boldsymbol{x} = (\boldsymbol{x}_{\boldsymbol{J}}, \boldsymbol{x}_{\boldsymbol{S}})$, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}_{\boldsymbol{J}}\boldsymbol{x}_{\boldsymbol{J}} + \boldsymbol{A}_{\boldsymbol{S}}\boldsymbol{x}_{\boldsymbol{S}}$.

From Gordan to Tucker

Let $\tilde{x}$ with $\tilde{x} \geq \mathbf{0}$, $A\tilde{x} = \mathbf{0}$ have maximum support

$S = \{ j \mid \tilde{x}_j > 0 \}$, write $x = (x_J, x_S)$, $Ax = A_J x_J + A_S x_S$.



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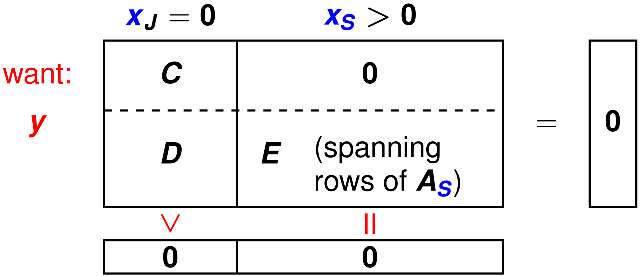

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	$\boldsymbol{x}_J = \mathbf{0}$	$\boldsymbol{x}_S > \mathbf{0}$		
want:	$\boldsymbol{C}$	$\mathbf{0}$	$=$	$\mathbf{0}$
$\boldsymbol{y}$	$\boldsymbol{D}$	$\boldsymbol{E}$ (spanning rows of $\boldsymbol{A}_S$)		
	$\vee$	$\Vert$		
	$\mathbf{0}$	$\mathbf{0}$		

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{x} = \mathbf{0} \quad &\Leftrightarrow \quad \boldsymbol{B}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{A}_J\boldsymbol{x}_J + \boldsymbol{B}\boldsymbol{A}_S\boldsymbol{x}_S = \mathbf{0} \\
&\Leftrightarrow \quad \boldsymbol{C}\boldsymbol{x}_J = \mathbf{0}\,, \\
& \quad \boldsymbol{D}\boldsymbol{x}_J + \boldsymbol{E}\boldsymbol{x}_S = \mathbf{0}\,.
\end{aligned}
$$

From Gordan to Tucker

Let $\tilde{x}$ with $\tilde{x} \geq 0$, $A\tilde{x} = 0$ have maximum support

$S = \{ j \mid \tilde{x}_j > 0 \}$, write $x = (x_J, x_S)$, $Ax = A_J x_J + A_S x_S$.

$$
\begin{array}{c|c|c|}
\text{find:} & x_J = 0 & x_S > 0 \\
\hline
w & C & 0 \\
\hdashline
0 & D & E \ \text{(spanning rows of } A_S\text{)} \\
\hline
 & \vee & \parallel \\
\hline
 & 0 & 0 \\
\hline
\end{array}
\quad = \quad
\begin{array}{|c|}
\hline
0 \\
\hline
\end{array}
$$

$$
\begin{aligned}
Ax = 0 \quad &\Leftrightarrow \quad BAx = BA_J x_J + BA_S x_S = 0 \\
&\Leftrightarrow \quad Cx_J = 0\,, \\
& \quad Dx_J + Ex_S = 0\,.
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Gordan $\Rightarrow$ Tucker

$\boldsymbol{A\tilde{x}} = \mathbf{0}\,,\ \ \boldsymbol{\tilde{x}} \geq \mathbf{0}\,,\ \ \boldsymbol{\tilde{x}_S} > \mathbf{0}$ where $\boldsymbol{\tilde{x}}$ has maximum support $\boldsymbol{S}$.

$$\begin{aligned} \boldsymbol{Ax} = \mathbf{0} \quad &\Leftrightarrow \quad \boldsymbol{BAx} = \boldsymbol{BA_J x_J} + \boldsymbol{BA_S x_S} = \mathbf{0} \\ &\Leftrightarrow \quad \boldsymbol{C x_J} = \mathbf{0}\,, \\ & \quad \boldsymbol{D x_J} + \boldsymbol{E x_S} = \mathbf{0}\,. \end{aligned}$$

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E has full rank $\Rightarrow\ \exists\, x_S\ :\ Dx_J + Ex_S = 0\,.$

$\Rightarrow\ B(A_J x_J + A_S \underbrace{(x_S + \tilde{x}_S \alpha)}_{>0 \text{ as } \alpha \to \infty}) = 0\,,\quad S$ not maximal. ↯

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Gordan $\Rightarrow$

$$\exists\, w\ :\ w^\top C > 0^\top,\quad \left(\binom{w}{0}^\top B\right) A_J > 0\,,\quad \left(\binom{w}{0}^\top B\right) A_S = 0\,. \qquad \square$$

Summary: minimax theorem $\Rightarrow$ LP duality

Recall: Using Dantzig's game $$\boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$$

with $\boldsymbol{B} = -\boldsymbol{B}^\top$ **assumes** Tucker's Lemma

$$\exists\, \boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{B}\boldsymbol{z} \leq \mathbf{0},\ \boldsymbol{z}_k - (\boldsymbol{B}\boldsymbol{z})_k > 0\,.$$

Summary: minimax theorem $\Rightarrow$ LP duality

Recall: Using Dantzig's game $\quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A} & -\boldsymbol{b} \\ -\boldsymbol{A}^\top & \mathbf{0} & \boldsymbol{c} \\ \boldsymbol{b}^\top & -\boldsymbol{c}^\top & 0 \end{bmatrix}$

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---

minimax theorem $\Rightarrow$ Gordan's Theorem, $\Rightarrow$ **Tucker's Theorem**

$$\exists\, \boldsymbol{z} \geq \mathbf{0},\ \boldsymbol{Bz} \leq \mathbf{0},\ \boldsymbol{z} - \boldsymbol{Bz} > \mathbf{0}$$

$\Rightarrow$ LP duality with **strict complementarity**: for feasible LPs

$$\exists\, \boldsymbol{x}, \boldsymbol{y} \ :\ (\boldsymbol{y}^\top\boldsymbol{A} - \boldsymbol{c}^\top)\boldsymbol{x} = 0\,, \qquad \boldsymbol{y}^\top(\boldsymbol{b} - \boldsymbol{Ax}) = 0\,,$$

$$(\boldsymbol{y}^\top\boldsymbol{A} - \boldsymbol{c}^\top) + \boldsymbol{x}^\top > \mathbf{0}^\top, \quad \boldsymbol{y} + (\boldsymbol{b} - \boldsymbol{Ax}) > \mathbf{0}\,.$$

Minimax theorem: Proof by Loomis [1946]

min-max strategy $\boldsymbol{x} \in \boldsymbol{X}$: minimize $\boldsymbol{v}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}\boldsymbol{v}$,

max-min strategy $\boldsymbol{y} \in \boldsymbol{Y}$: maximize $\boldsymbol{u}$ s.t. $\boldsymbol{y}^\top\boldsymbol{A} \geq \boldsymbol{u}\mathbf{1}^\top$,

$$\boldsymbol{u} = \boldsymbol{u}\mathbf{1}^\top\boldsymbol{x} \leq \boldsymbol{y}^\top\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^\top\mathbf{1}\boldsymbol{v} = \boldsymbol{v}.$$

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$\boldsymbol{u}\mathbf{1}^\top = \boldsymbol{y}^\top \boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{x} = \mathbf{1}\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{u} = \boldsymbol{v}$, done.

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$\boldsymbol{u}\mathbf{1}^\top = \boldsymbol{y}^\top\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{x} = \mathbf{1}\boldsymbol{v} \quad \Rightarrow \quad \boldsymbol{u} = \boldsymbol{v}$, done.

Assume $(\boldsymbol{A}\boldsymbol{x})_{\boldsymbol{k}} < \boldsymbol{v}$ for some row $\boldsymbol{k}$, let $\overline{\boldsymbol{A}}$ be $\boldsymbol{A}$ without row $\boldsymbol{k}$.

By **inductive hypothesis**, $\overline{\boldsymbol{A}}$ has game value $\overline{\boldsymbol{v}}$, $\overline{\boldsymbol{A}}\overline{\boldsymbol{x}} \leq \mathbf{1}\overline{\boldsymbol{v}}$.

$$\overline{\boldsymbol{v}} \leq \boldsymbol{u}, \quad \overline{\boldsymbol{v}} \leq \boldsymbol{v}, \quad (\overline{\boldsymbol{A}} \text{ better than } \boldsymbol{A} \text{ for minimizer}).$$

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min-max strategy $\boldsymbol{x} \in \boldsymbol{X}$: minimize $\boldsymbol{v}$ s.t. $\boldsymbol{Ax} \leq \mathbf{1}\boldsymbol{v}$,

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$$\boldsymbol{u} = \boldsymbol{u}\mathbf{1}^\top \boldsymbol{x} \leq \boldsymbol{y}^\top \boldsymbol{Ax} \leq \boldsymbol{y}^\top \mathbf{1}\boldsymbol{v} = \boldsymbol{v}.$$

$\boldsymbol{u}\mathbf{1}^\top = \boldsymbol{y}^\top \boldsymbol{A}$ and $\boldsymbol{Ax} = \mathbf{1}\boldsymbol{v}$ $\Rightarrow$ $\boldsymbol{u} = \boldsymbol{v}$, done.

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By **inductive hypothesis**, $\overline{\boldsymbol{A}}$ has game value $\overline{\boldsymbol{v}}$, $\overline{\boldsymbol{A}}\overline{\boldsymbol{x}} \leq \mathbf{1}\overline{\boldsymbol{v}}$.

$$\overline{\boldsymbol{v}} \leq \boldsymbol{u}, \quad \overline{\boldsymbol{v}} \leq \boldsymbol{v}, \quad (\overline{\boldsymbol{A}} \text{ better than } \boldsymbol{A} \text{ for minimizer}).$$

Claim : $\overline{\boldsymbol{v}} = \boldsymbol{v}$. Intuition: maximizer avoids row $\boldsymbol{k}$ of $\boldsymbol{A}$ anyhow.

Proof that $\overline{\boldsymbol{v}} = \boldsymbol{v}$

minimal $\boldsymbol{v}$ s.t. $\boldsymbol{Ax} \leq \mathbf{1}\boldsymbol{v}$, maximal $\boldsymbol{u}$ s.t. $\boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{u}\mathbf{1}^\top$, $\quad \boldsymbol{u} \leq \boldsymbol{v}$.

$(\boldsymbol{Ax})_{\boldsymbol{k}} < \boldsymbol{v}$, matrix $\overline{\boldsymbol{A}}$ is $\boldsymbol{A}$ without row $\boldsymbol{k}$, value $\overline{\boldsymbol{v}} \leq \boldsymbol{u}$, $\overline{\boldsymbol{v}} \leq \boldsymbol{v}$.

Proof that $\overline{\boldsymbol{v}} = \boldsymbol{v}$

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$\Rightarrow \quad \overline{\boldsymbol{v}} \le \boldsymbol{u} \le \boldsymbol{v} = \overline{\boldsymbol{v}}$, $\boxed{\boldsymbol{u} = \boldsymbol{v}}$. Induction complete. □

"On a theorem of von Neumann"

Theorem Loomis [1946]

Let $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} > \mathbf{0}$.

Then there exist $\boldsymbol{x} \in \boldsymbol{X}$, $\boldsymbol{y} \in \boldsymbol{Y}$, $\boldsymbol{v} \in \mathbb{R}$:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{B}\boldsymbol{x}\boldsymbol{v}\,, \qquad \boldsymbol{y}^\top \boldsymbol{A} \geq \boldsymbol{v}\boldsymbol{y}^\top \boldsymbol{B}\,.$$

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Conversely, theorem is **implied** by the minimax theorem:

value$(\boldsymbol{A} - \alpha \boldsymbol{B}) < \mathbf{0}$ for $\alpha \to \infty$,

value$(\boldsymbol{A} - \alpha \boldsymbol{B}) > \mathbf{0}$ for $\alpha \to -\infty$, continuous in α, hence

value$(\boldsymbol{A} - \alpha \boldsymbol{B}) = \mathbf{0}$ for some $\boldsymbol{v} = \alpha$. □

Conforti, Di Summa, Zambelli [2007]

Theorem

$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ minimally infeasible $\Rightarrow$ $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ minimally infeasible .

Conforti, Di Summa, Zambelli [2007]

Theorem

$Ax \leq b$ minimally infeasible $\Rightarrow$ $Ax = b$ minimally infeasible.

$\Rightarrow$ reversing any inequality $a_i x \leq b_i$ creates feasible system:

$$\forall \text{ row } i \quad \exists\, x \;:\; a_i x > b_i\,, \quad \forall\, k \neq i \;:\; a_k x = b_k\,.$$

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Then apply **linear algebra** (get $\mathbf{0} = -\mathbf{1}$ from infeasible $\boldsymbol{Ax} = \boldsymbol{b}$):

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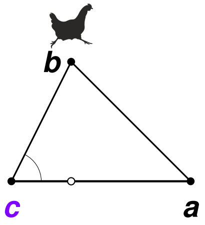
to prove inequality-Farkas (get $\mathbf{0} \leq -\mathbf{1}$ from infeasible $\boldsymbol{Ax} \leq \boldsymbol{b}$):

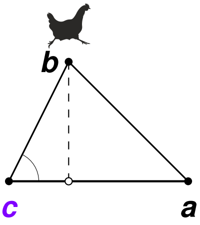
$$\nexists\, \boldsymbol{x} \;:\; \boldsymbol{Ax} \leq \boldsymbol{b} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \geq \mathbf{0} \;:\; \boldsymbol{y}^\top \boldsymbol{A} = \mathbf{0}^\top, \;\; \boldsymbol{y}^\top \boldsymbol{b} < \mathbf{0}\,.$$

How did the chicken cross the triangle?



Consider a triangle with corners ***a***, ***b***, ***c*** and a chicken at ***b*** that wants ???

How did the chicken cross the triangle?



Consider a triangle with corners ***a***, ***b***, ***c*** and a chicken at ***b*** that wants to get to the other side.[citation needed]

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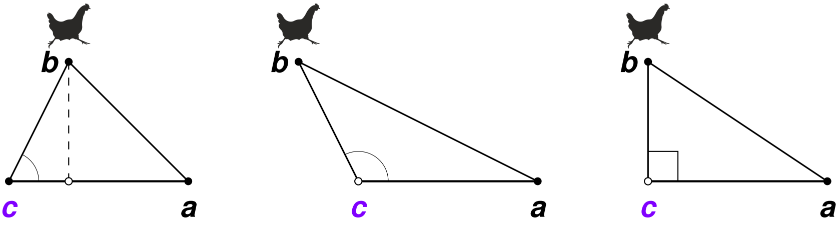

Consider a triangle with corners $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and a chicken at $\boldsymbol{b}$ that wants to get to the other side.

Then the closest point to get there is $\boldsymbol{c}$ if and only if the angle at $\boldsymbol{c}$ is not acute, that is,

$$(\boldsymbol{b}-\boldsymbol{c})^\top(\boldsymbol{a}-\boldsymbol{c}) \leq \mathbf{0}\,.$$

Supporting hyperplane theorem

Theorem

Let $\emptyset \neq \boldsymbol{C} \subset \mathbb{R}^{\boldsymbol{m}}$, closed, convex, $\boldsymbol{b} \notin \boldsymbol{C}$.

Let $\boldsymbol{c} \in \boldsymbol{C}$ with smallest $\|\boldsymbol{b} - \boldsymbol{c}\|$.

Consider hyperplane $\boldsymbol{H}$ with normal vector $\boldsymbol{b} - \boldsymbol{c}$ through $\boldsymbol{c}$:
then all of $\boldsymbol{C}$ on one side, $\boldsymbol{b}$ strictly on the other side of $\boldsymbol{H}$,

$$(\boldsymbol{b} - \boldsymbol{c})^\top(\boldsymbol{b} - \boldsymbol{c}) > \mathbf{0}\,, \quad \forall\, \boldsymbol{a} \in \boldsymbol{C} \;:\; (\boldsymbol{b} - \boldsymbol{c})^\top(\boldsymbol{a} - \boldsymbol{c}) \leq \mathbf{0}\,.$$

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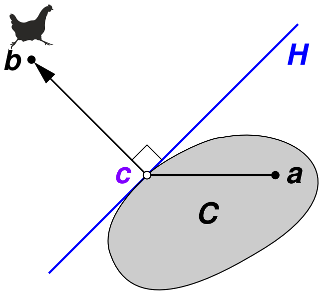
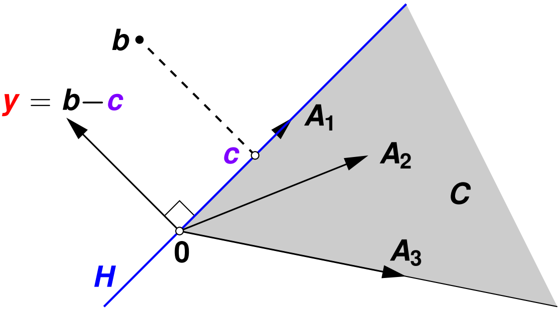


Lemma of Farkas

Cone $\boldsymbol{C} = \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \geq \boldsymbol{0}\}$ and $\boldsymbol{b} \notin \boldsymbol{C}$.

Consider $\boldsymbol{c} \in \boldsymbol{C}$ with smallest $\|\boldsymbol{b} - \boldsymbol{c}\|$, and $\boldsymbol{y} = \boldsymbol{b} - \boldsymbol{c}$. Then

$$\boldsymbol{y}^\top \boldsymbol{b} > 0, \quad (\forall\, \boldsymbol{a} \in \boldsymbol{C} \;:\; \boldsymbol{y}^\top \boldsymbol{a} \leq 0) \quad \boldsymbol{y}^\top \boldsymbol{A} \leq \boldsymbol{0}^\top.$$



Why is the cone $C = \{Ax \mid x \geq 0\}$ closed?

- show: limit a of any sequence of points $a^{(k)}$ in C is in C
- $\forall k$ $\exists$ basis B, $x_B \geq 0$: $a^{(k)} = A_B x_B$
- only finitely many bases B
- restrict to subsequence with one B that occurs infinitely often
- $a = \lim_{k\to\infty} a^{(k)} = A_B \lim_{k\to\infty} \underbrace{A_B^{-1} a^{(k)}}_{\geq 0} \in C$
- need theorem of Carathéodory (and Weierstrass).

Fourier–Motzkin elimination = projection

Lemma (ineq-Farkas, get $\mathbf{0} \leq -\mathbf{1}$ from infeasible $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$):

$\nexists\, \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}} \;:\; \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \quad \Leftrightarrow \quad \exists\, \boldsymbol{y} \geq \mathbf{0} \;:\; \boldsymbol{y}^\top \boldsymbol{A} = \mathbf{0}^\top, \;\; \boldsymbol{y}^\top \boldsymbol{b} < \mathbf{0}\,.$

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Proof By induction on $\boldsymbol{n}$.

Scale rows of $\boldsymbol{Ax} \leq \boldsymbol{b}$ with **affine** $\boldsymbol{a_i}$, $\boldsymbol{b_j}$, $\boldsymbol{c_k}$ as

$$\boldsymbol{a_i}(\boldsymbol{x_2}, \ldots, \boldsymbol{x_n}) \leq \boldsymbol{x_1}\,, \quad \boldsymbol{x_1} \leq \boldsymbol{b_j}(\boldsymbol{x_2}, \ldots, \boldsymbol{x_n})\,, \quad \boldsymbol{c_k}(\boldsymbol{x_2}, \ldots, \boldsymbol{x_n}) \leq \mathbf{0}\,.$$

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Eliminate $\boldsymbol{x_1}$ by writing $\boldsymbol{a_i} \leq \boldsymbol{b_j}$ for all pairs $\boldsymbol{i}, \boldsymbol{j}$.

Fourier–Motzkin elimination = projection

Lemma (ineq-Farkas, get $\mathbf{0} \leq -\mathbf{1}$ from infeasible $Ax \leq b$):

$$\nexists\, x \in \mathbb{R}^n \;:\; Ax \leq b \quad \Leftrightarrow \quad \exists\, y \geq \mathbf{0} \;:\; y^\top A = \mathbf{0}^\top,\;\; y^\top b < \mathbf{0}\,.$$

Proof By induction on n.

Scale rows of $Ax \leq b$ with **affine** a_i, b_j, c_k as

$$a_i(x_2,\ldots,x_n) \leq x_1\,,\quad x_1 \leq b_j(x_2,\ldots,x_n)\,,\quad c_k(x_2,\ldots,x_n) \leq \mathbf{0}\,.$$

Eliminate x_1 by writing $a_i \leq b_j$ for all pairs i, j .

By inductive hypothesis: Either solve in $x_2,\ldots,x_n \geq \mathbf{0}$ and choose any x_1 with $a_i \leq x_1 \leq b_j$ for all i, j, or linearly combine (then also in terms of rows of $Ax \leq b$) to get $\mathbf{0} \leq -\mathbf{1}$. □

Thanks for listening!

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